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THE APPLICATION OF TERNARY SEMIGROUPS TO THE STUDY OF n-VALUED SHEFFER FUNCTIONS

JAMES ROSENBERG

If we consider a set of one-variable truth functions we can define on this set a product operation, namely composition. If we assume that this set is closed under composition then the algebraic structure which results is that of a semigroup. In this paper we extend this notion to consider sets of binary truth functions by introducing the concept of a ternary semigroup, and prove a theorem concerning n-valued Sheffer functions. (For one of the most recent papers on this subject with an excellent bibliography see [1].) The methods presented are entirely algebraic, but then it may be argued that problems involving the characterization of n-valued Sheffer functions belong more properly to abstract algebra than symbolic logic.

1. *Definition:* A *ternary semigroup* is a set G with a closed ternary product operation fgh such that for any $f, g, h, x, y \varepsilon G$,

$$(fgh)xy = f(gxy)\ (hxy)^1$$

For the best example of a ternary semigroup consider a set F of binary functions on a set T—i.e. a set of functions which map $T \times T \to T$. Define a ternary product on F by the *ternary composition map*:

$$fgh(x,y) = f(g(x,y), h(x,y))\ x, y\ \varepsilon\ T,\ f, g, h\ \varepsilon\ F$$

The similarity between this definition and the condition of Definition 1 will readily be seen. In fact, if we assume that for $f,g,h\ \varepsilon\ F\ fgh\ \varepsilon\ F$ then F is a ternary semigroup under composition. In this case F will be said to be a ternary semigroup *acting on T*. We will define isomorphism in the natural way, namely two ternary semigroups G and H will be said to be *isomorphic* iff there exists a 1-1 onto map $\phi: G \to H$ such that $\phi(abc) = \phi(a)\phi(b)\phi(c)$ for any $a, b, c\ \varepsilon\ G$.

2. *Theorem: For any ternary semigroup G there is a set T and a ternary semigroup H acting on T such that G is isomorphic to H.*

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1. If a system of notation were used in which function arguments were placed on the *left* of the function symbol the condition would be written $xy(fgh) = (xyf)(xyg)h$.

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Proof: Let $\infty \notin G$, and define $T = G \cup \{\infty\}$. For each $x \varepsilon G$ define $\xi_x : T \times T \to T$ as follows. For $(a,b) \varepsilon G \times G$ $\xi_x(a,b) = xab$, and for $(a,b) \varepsilon T \times T - G \times G$ $\xi_x = x$. Let $H = \{\xi_x | x \varepsilon G\}$, and define $\phi : G \to H$ by $\phi(x) = \xi_x$. Clearly ϕ is onto. Now if $(a,b) \varepsilon T \times T - G \times G$ then $\xi_x \xi_y \xi_z(a,b) = \xi_x(\xi_y(a,b), \xi_z(a,b)) = \xi_x(y,z) = xyz$ and $\xi_{xyz}(a,b) = xyz$. If $(a,b) \varepsilon G \times G$ then $\xi_x \xi_y \xi_z(a,b) = \xi_x(yab, zab) = x(yab)(zab)$, and $\xi_{xyz}(a,b) = (xyz)ab$; but by Definition 1 $x(yab)(zab) = (xyz)ab$. Thus we have shown $\xi_x \xi_y \xi_z = \xi_{xyz}$. The fact that ϕ is 1-1 follows immediately from the fact that $\xi_x(\infty, \infty) = x$.

We will now restrict our attention solely to ternary semigroups which act on a set. If G is a ternary semigroup acting on T and $f \varepsilon G$ then the expression $f(a,a)$, $a \varepsilon T$ defines a one-variable function on T. In the following section we consider these one-variable functions and their interraction with the two variable functions in G.

3. *Definition:* Let G be a ternary semigroup acting on T, $f \varepsilon G$. By $f^\wedge$ we will mean the function mapping $T \to T$ by

$$f^\wedge(a) = f(a,a).$$

The set $G^\wedge = \{f^\wedge | f \varepsilon G\}$ will be called the *companion* of G.

4. *Theorem: If G is a ternary semigroup acting on T then the companion of G is a (binary) semigroup under composition.*

Proof: Since $G^\wedge$ is a set of one-variable functions the composition is automatically associative, so in order to prove $G^\wedge$ is a binary semigroup it is merely necessary to show that for $f,g \varepsilon G$, $f^\wedge g^\wedge \varepsilon G^\wedge$. But $f^\wedge g^\wedge(a) = f^\wedge(g^\wedge(a)) = f^\wedge(g(a,a)) = f(g(a,a),g(a,a)) = fgg(a,a) = (fgg)^\wedge(a)$ and $fgg \varepsilon G$.

Now that we have seen that these one-variable truth functions are closed under composition we must examine "mixed composition" between two-variable and one-variable functions. In particular, if G is a ternary semigroup acting on T, $f,g,h \varepsilon G$, $a,b \varepsilon T$, then the expression $f(g^\wedge(a),h^\wedge(b))$ defines a function of a and b. Does this function belong to G? This question leads to the following notion.

5. *Definition:* A ternary semigroup G acting on T will be said to be *realized* iff for every $f,g,h \varepsilon G$ there exists $z \varepsilon G$ such that for any $a,b \varepsilon T$,

$$f(g^\wedge(a),h^\wedge(b)) = z(a,b).$$

The *realization* of G, denoted G^*, is the intersection of all realized ternary semigroups acting on T which contain G.

Examples: (1) The ternary semigroup $B(T)$ of all functions mapping $T \times T \to T$ is realized. (2) Let $T = \{true, false\}$ and let $f : T \times T \to T$ be the Sheffer Stroke—i.e. $f(a,b) = \sim(a \wedge b)$. Define G to be the ternary semigroup *generated by* $\{f\}$, i.e. the intersection of all ternary semigroups acting on T which contain $\{f\}$. It can be verified by direct computation that $G = \{f,g,h,i\}$ where $g(a,b) = a \wedge b$, $h(a,b) = true$, and $i(a,b) = false$. Define $t(a,b) =$

$f(f\hat{}(a), f\hat{}(b))$. $t(a,b) = (a|a)|(b|b) = \sim a|\sim b = \sim(\sim a \wedge \sim b) = a \vee b$. Thus $t \notin G$ and G is not realized.

We now give an equivalent definition of realization which is easier to work with.

6. *Theorem: Let G be a ternary semigroup acting on T. Define* $G_0 = G$, G_i = *the ternary semigroup generated by* $G_{i-1} \bigcup \{<f,g,h> | f,g,h \varepsilon G_{i-1}\}$ *where* $<f,g,h>$: $T \times T \to T$ *is defined by* $<f,g,h>(a,b) = f(g\hat{}(a), h\hat{}(b))$. *Let* $G^+ = \bigcup_{i=0}^{\infty} G_i$. *Then* $G^+ = G^*$.

The proof is very straightforward, and consists essentially of observing that G^+ is a realized ternary semigroup acting on T and that $G_i \subseteq G^*$ implies $G_{i+1} \subseteq G^*$. Now G^* is a ternary semigroup, and has companion $G^{*\hat{}}$. It is natural to inquire about the relation between $G\hat{}$ and $G^{*\hat{}}$.

7. *Theorem: If G is a ternary semigroup acting on T then* $G\hat{} = G^{*\hat{}}$.

Proof: In view of Theorem 6 it is sufficient to show that $G_i\hat{} = G_{i+1}\hat{}$, since each G_i, by construction, is a ternary semigroup. Let $f_j, g_j, h_j \varepsilon G_i$, $j = 1,2,3$, and let $z = <f_1, g_1, h_1> <f_2, g_2, h_2> <f_3, g_3, h_3>$. We wish to show that $z\hat{} \varepsilon G_i\hat{}$. For $a \varepsilon T$,

$$\begin{aligned}
&z\hat{}(a) = <f_1,g_1,h_1> (<f_2,g_2,h_2> (a,a), <f_3,g_3,h_3> (a,a)) = \\
&<f_1,g_1,h_1> (f_2 [g_2\hat{}(a), h_2\hat{}(a)], f_3 [g_3\hat{}(a), h_3\hat{}(a)]) = \\
&<f_1,g_1,h_1> (f_2g_2h_2(a,a), f_3g_3h_3(a,a)) = \\
&f_1 [g_1\hat{}(f_2g_2h_2(a,a)), h_1\hat{}(f_3g_3h_3(a,a))] = \\
&f_1([g_1(f_2g_2h_2)(f_2g_2h_2)](a,a), [h_1(f_3g_3h_3)(f_3g_3h_3)](a,a)) = \\
&f_1[g_1(f_2g_2h_2)(f_2g_2h_2)][h_1(f_3g_3h_3)(f_3g_3h_3)](a,a)
\end{aligned}$$

and since G_i is closed under ternary products it follows that indeed $z\hat{} \varepsilon G_i\hat{}$. From the mechanics of the above expansion it can easily be seen that if z is any finite ternary product of elements in $G \cup \{<f,g,h> | f,g,h \varepsilon G_i\}$ then $z\hat{} \varepsilon G_i\hat{}$. Thus $G_{i+1}\hat{} \subseteq G_i\hat{}$ and hence $G_{i+1}\hat{} = G_i\hat{}$.

We now turn our attention to ternary semigroups generated by a single binary operator. If f is a unary operator on a set T and G is the binary semigroup generated by $\{f\}$ then it is evident that $G = \{f^i | i = 0, 1, \ldots\}$ where the exponent has its usual meaning. In other words G has a more or less "cyclic" structure. If however f is a binary operator on T and G is the ternary semigroup generated by $\{f\}$ unless we place a restriction on f we cannot be sure of such a simple structure. This leads to the following notion.

8. *Definition:* A binary operator f on a set T will be called *slightly associative* iff for each $x \varepsilon T$ the groupoid generated by $\{x\}$ under f is a semigroup—i.e. iff for each $x \varepsilon T$ in any finite product by f in which only x appears the parentheses may be placed in any order.

The meaning of this definition is simply that for any $x \varepsilon T$ the expression x^i (using product f) is unambiguously defined for any natural number i.

Note that if $f(x,x) = x$ for each $x \varepsilon T$ then f is slightly associative but may not be associative. We shall now use the symbol $\mathbf{gen}\{f\}$ for the ternary semigroup generated by $\{f\}$.

9. *Theorem: Let f be a slightly associative binary operator on T, and define $f_1 = f, f_i = fff_{i-1}$, $i = 2,3, \ldots$. Then* $\mathbf{gen}\{f\} = \{f_i\}$

Proof: $f_2(a,b) = fff(a,b) = f(f(a,b), f(a,b)) = (a \cdot b)^2$ where $a \cdot b = f(a,b)$, since f is slightly associative and we may use non-negative exponents. By induction it is evident that $f_i(a,b) = (a \cdot b)^i$. Thus $f_i f_j f_k(a,b) = f_i((a \cdot b)^j, (a \cdot b)^k) = [(a \cdot b)^j \cdot (a \cdot b)^k]^i = (a \cdot b)^{i(j+k)} = f_{i(j+k)}(a,b)$. Thus $\{f_i\}$ is closed under ternary products and hence is a ternary semigroup, so $\mathbf{gen}\{f\} \subseteq \{f_i\}$. But clearly for each i $f_i \varepsilon \mathbf{gen}\{f\}$. Hence $\{f_i\} = \mathbf{gen}\{f\}$.

We are now about ready to use the machinery developed above to obtain a result concerning n-valued Sheffer functions. Let $N = \{1,2, \ldots, n\}$ and as above let $B(N)$ be the ternary semigroup of all functions mapping $N \times N \to N$ (i.e. the set of all n-valued binary truth functions.)

10. *Theorem: Let $f{:}N \times N \to N$ be a slightly associative binary operator on N, where $n > 2$. Then* $(\mathbf{gen}\{f\})^* \neq B(N)$.

Proof: Let $G = \mathbf{gen}\{f\}$. We wish to show that $G^* \neq B(N)$. If F_n is the set of all unary operators on N then it is sufficient to show that $G^{*\hat{}} \neq F_n$, since $B(N)^{\hat{}} = F_n$. But by Theorem 7 $G^{\hat{}} = G^{*\hat{}}$ so it is sufficient to show that $G^{\hat{}} \neq F_n$. Since f is slightly associative $G = \{f_i\}$ where $f_i(a,b) = (a \cdot b)^i$, $(a \cdot b) = f(a,b)$, by Theorem 9. Now for each $a \varepsilon N$ define α_a as the smallest natural number k s.t. for some natural number $j \neq 0$ $a^k = a^{k+j}$, and define β_a as the smallest natural number j s.t. $a^{\alpha_a} = a^{\alpha_a + j}$. It is clear that $\beta_a, \alpha_a \leq n$. Choose $a_0 \varepsilon N$ s.t. $\alpha_{a_0} = \mathbf{max}_{a \varepsilon N}\{\alpha_a\}$, and define $\delta = \text{l.c.m.}_{a \varepsilon N}\{\beta_a\}$. Now for any $a \varepsilon N$ $a^{\alpha_a} = a^{\alpha_a + k\beta_a} k = 1,2, \ldots$ so in particular it follows that for $i \geq \alpha_a$ $a^i = a^{i+\delta}$ and hence if $i \geq \alpha_{a_0}$ $a^i = a^{i+\delta}$ so that $f_i = f_{i+\delta}$. From this we see that $\# G \leq \alpha_{a_0} + \delta$. But $\alpha_{a_0} \leq n$ and δ is the l.c.m. of a set of numbers all $\leq n$, so certainly $\delta \leq n!$ Thus we have $\# G \leq n + n!$ and if $n > 2$, $n + n! < n^n$. Since $\# G^{\hat{}} \leq \# G$, $\# G^{\hat{}} < n^n$, but $\# F_n = n^n$ so obviously $G^{\hat{}} \neq F_n$.

11. *Definition:* A binary operator f on T will be said to *produce the identity* iff the identity map on T, 1_T is a member of $(\mathbf{gen}\{f\})^{\hat{}}$.

We see immediately from Theorem 9 that a slightly associative binary operator on T produces the identity iff there exists a natural number $n > 0$ s.t. $a^{2n} = a$ for all $a \varepsilon T$. We are now ready for the main result.

12. *Main Theorem: Let f be slightly associative binary operator on N which produces the identity $n > 2$. Then f is not an n-valued Sheffer function—i.e. $\{f\}$ is not functionally complete.*

Proof: Let f be slightly associative binary operator on N which produces the identity. We see from the argument in the proof of Theorem 10 that

$(\text{gen}\{f\}^*)^\wedge \neq F_n$. Now it can be easily seen that if g is any binary operator on T which can be defined by means of composition in terms of $\{f\}$, then $g^\wedge \varepsilon (\text{gen}\{f\}^*)^\wedge$. The conclusion follows immediately.

The above results indicate the manner in which the theory of ternary semigroups can be applied to the study of n-valued Sheffer functions. A more extensive treatment of this theory will be found in [2].

REFERENCES

[1] Graham, R. L., "On n-Valued Functionally Complete Truth Functions," *The Journal of Symbolic Logic,* (June 1967), pp. 190-195.

[2] Rosenberg, J., *Introduction to the Theory of Ternary Semigroups*, Mathematics Department, Pomona College, in preparation..

Pomona College
Claremont, California