Notre Dame Journal of Formal Logic
Volume X, Number 1, January 1969

REMARKS ON CLASSIFICATION OF THEORIES BY THEIR COMPLETE EXTENSIONS[1]

KAREL L. de BOUVÈRE

As far back as in 1928 A. Tarski introduced the concept of *degree of completeness* of a theory, [2]. Subsequently he distinguished between cardinal and ordinal degree of completeness, [3], and refined the former to the so-called *characteristic pair* of a theory, [4]. In 1936, 1937, A. Mostowski made a thorough study of the relationship between the characteristic numbers and the *structural type* of a theory, [1]. Closely related with the work of Tarski and Mostowski is A. Lindenbaum, who recognized, independently, the importance of the concept of structural type and whose theorem, stating that every consistent theory has a consistent and complete extension, is basic to the studies involved.

This paper intends to introduce the student of Logic to the metamathematics mentioned above, but by a method different from those used by the original authors. Tarski's contributions to the subject matter arise, more or less as corollaries, within a mighty axiomatic framework intended to cover the whole of the theory of theories, too heavy, it seems, for a student who as a start wants to study, say, classification of theories. Mostowski's paper on the matter requires more knowledge of topology than the average logician, who is not a mathematician, possesses.

The method employed in this paper could be called "naive". However, starting "from scratch", with structures established in the form of trees, it arrives at the desired results in a simple and rather self-contained way (only some fragmentary acquaintance with cardinal and ordinal arithmetic is required). Results coinciding with results of Lindenbaum, Mostowski, Tarski are mentioned as theorems. Side-results, due to the particular method, are mentioned as remarks. The terminology, in so far as it is new, is intended *ad hoc*, i.e., to promote readability of this paper.

§1. Introductory Remarks. Our considerations are confined to a first-order language with identity containing no other than non-logical constants

Received July 5, 1968

of finitary rank. The cardinal number of the set of non-logical constants is assumed to be $\leq \aleph_0$. Sentences are assumed to be of finite length. By S we denote the set of all sentences of the language. Speaking of sentences, we refer to elements of S, speaking of theories we refer to subsets of S which are closed under logical deduction with respect to S. Obviously, the cardinal number of the set of all sentences is $\leq \aleph_0$, the cardinal number of the set of all theories is $\leq 2^{\aleph_0}$. A theory T is said to be consistent if, and only if, $T \subset S$, i.e., T is a proper subset of S. A theory T is said to be complete if, and only if, for every sentence s either $s \in T$ or $\overline{s} \in T$, but not both $s \in T$ and $\overline{s} \in T$, where $\overline{s}$ is the negation of s. In this sense, every inconsistent theory is incomplete and every complete theory is consistent. If T' and T'' are theories, then by their logical union, $T' \uplus T''$, we understand $\mathbf{Cn}(T' \cup T'')$, i.e., the closure under logical deduction with respect to S of $T' \cup T''$. We say that the theories T', T'' are compatible if, and only if, $T' \uplus T''$ is consistent. We say that a sentence s is compatible with a theory T if, and only if, theory $\mathbf{Cn}(\{s\})$ is compatible with T. Notice, $T \uplus \mathbf{Cn}(\{s\}) = T \uplus \{s\}$. Obviously, if there is a sentence s such that as well s as $\overline{s}$ is compatible with a theory T, then not only T is consistent, but also neither $s \in T$ nor $\overline{s} \in T$ and T is incomplete. Conversely, if a theory T is consistent and incomplete, then there is a sentence, say s, such that neither $s \in T$ nor $\overline{s} \in T$; considering $T \uplus \{s\}$ we notice that $\overline{s} \in T \uplus \{s\}$ would imply (by the deduction theorem) $s \rightarrow \overline{s} \in T$ or $\overline{s} \in T$ contrary to the assumption; hence, $T \uplus \{s\}$ is consistent, and so is $T \uplus \{\overline{s}\}$, or, as well s as $\overline{s}$ is compatible with T. Thus we can state:

A theory T is consistent and incomplete if, and only if, there is a sentence, say s, such that as well s as $\overline{s}$ is compatible with T.

A theory T' is said to be an extension of theory T, or equivalently, T is said to be a subtheory of T', if, and only if, $T \subseteq T' \subseteq S$. We are interested in the set of complete extensions of a theory T, i.e., of those extensions of T which are complete theories. We shall say that two complete extensions T' and T'' of a theory T are different if, and only if, they are incompatible. We shall say that a sentence s completes a theory T, or that T is completed by s, if, and only if, $T \uplus \{s\}$ is a complete extension of T. Theory T is said to be finitely completable if, and only if, there is a sentence which completes T. If $T \uplus \{s\}$ is a complete extension of T, then we call $T \uplus \{s\}$ a finite completion of T. If, with Tarski and Mostowski we introduce the characteristic numbers $\alpha(T)$ and $\pi(T)$ of a theory T, then $\alpha(T)$ is the cardinal number of the set of all finite completions of T and $\pi(T)$ the cardinal number of the set of all other complete extensions of T.[2] The following theorem and its proof play an essential part in the subsequent considerations:

Theorem 1 (Lindenbaum): *Every consistent theory has a complete extension.*

Proof: Let T be a consistent theory. If T is complete, then T itself is a complete extension of T. If T is incomplete, then we consider an enumeration $s_1, s_2, \ldots, s_n, \ldots$ of all members of S and we define:

$$T_1 = \begin{cases} T \uplus \{s_1\}, & \text{if } s_1 \text{ is compatible with } T; \\ T \uplus \{\overline{s}_1\}, & (=T), \text{ otherwise;} \end{cases}$$

$$T_2 = \begin{cases} T_1 \uplus \{s_2\}, & \text{if } s_2 \text{ is compatible with } T_1; \\ T_1 \uplus \{\overline{s}_2\}, & (=T_1), \text{ otherwise;} \end{cases}$$

. . . .
. . . .
. . . .

$$T_n = \begin{cases} T_{n-1} \uplus \{s_n\}, & \text{if } s_n \text{ is compatible with } T_{n-1}; \\ T_{n-1} \uplus \{\overline{s}_n\}, & (=T_{n-1}), \text{ otherwise;} \end{cases}$$

. . . .
. . . .
. . . .

Finally we define $T_\infty = \bigcup_{i=1}^{\infty} T_i$. Obviously T_∞ *is a complete extension of* T.

Corollary 1.1: *Every consistent and incomplete theory has at least two different complete extensions.*

Corollary 1.2: *If a theory can be completed by a sentence and also by the negation of that sentence, then it has not a different third complete extension.*

If a theory T is complete, then the only complete extension of T is T itself. This complete extension may be considered to be a finite completion of T for various reasons. One can say that the empty sentence completes T, and also that every sentence of T completes T. Conversely, if a theory T has a complete extension, then certainly T is consistent; therefore, if T has only one complete extension, then T has to be complete because of corollary 1.1. And if T is complete, its only complete extension is a finite completion as argued above. Hence, we can state:

A theory T is complete if, and only if, $\mathfrak{a}(T) = 1$ *and* $\mathfrak{n}(T) = 0$.

If a theory T is incomplete we distinguish two cases: T is inconsistent, T is consistent. If T is inconsistent, then obviously it has no complete extension. Conversely, by theorem 1, if T has no complete extension, then T is inconsistent. Hence, we can state:

A theory T is inconsistent if, and only if, $\mathfrak{a}(T) = 0$ *and* $\mathfrak{n}(T) = 0$.

If T is consistent and incomplete, then we subdistinguish two cases:

T is not finitely completable and T is finitely completable.

If the consistent and incomplete theory T is not finitely completable, then there is no sentence which completes T and there are no finite completions of T, or $\alpha(T) = 0$. In this case we say that T is *essentially incomplete*. It amounts to saying that not only T, but every extension of T with a finite number of sentences is incomplete.

If the consistent and incomplete theory T is finitely completable we distinguish further:

We say that T is finitely and unambiguously completable if, and only if, the incompleteness can be removed in one finite way only. There is a sentence, say s, such that $T \uplus \{s\}$ is a complete extension of T, and if there is another sentence, say s', such that $T \uplus \{s'\}$ is complete, then $T \uplus \{s\} = T \uplus \{s'\}$, or what amounts to the same thing, $s \leftrightarrow s' \in T$. In this case $\alpha(T) = 1$.

We say that T is finitely and ambiguously completable if, and only if, the incompleteness can be removed in more than one finite way. There are at least two sentences, say s_1 and s_2, such that $T \uplus \{s_1\}$ and $T \uplus \{s_2\}$ are complete extensions of T, but at the same time $T \uplus \{s_1\}$ and $T \uplus \{s_2\}$ are incompatible, or $T \uplus \{s_1\} \uplus \{s_2\}$ is inconsistent. In other words, $T \uplus \{s_1\}$ and $T \uplus \{s_2\}$ are different complete extensions of T. In this case $\alpha(T) > 1$.

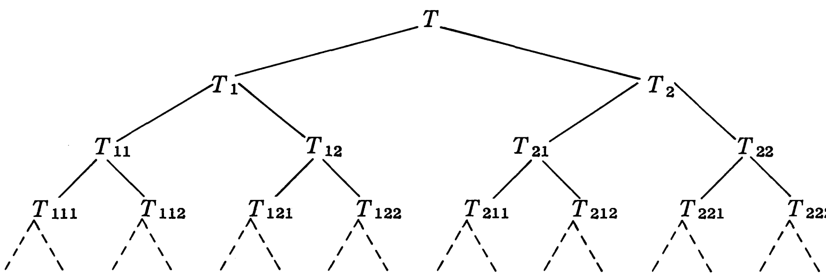
Notice that no harm is done if an inconsistent theory is called essentially incomplete. Hence we can say that a theory T is essentially incomplete if, and only if, $\alpha(T) = 0$. Similarly no harm is done if a complete theory is called finitely and unambiguously completable, so that we can say that a theory T is finitely and unambiguously completable if, and only if, $\alpha(T) = 1$. Adding that a theory T is finitely and ambiguously completable if, and only if, $\alpha(T) > 1$, we have an exhaustive classification of theories, be it a rather rough one (and therefore provisional).

§2. Trees. We construct what we like to call a *tree of complete extensions* for a theory T. Let, e.g., theory T be consistent and essentially incomplete ($\alpha(T) = 0$, $n(T) \neq 0$). Since there are only denumerably many sentences in the language of T, we can think of some enumeration of these sentences, say $s_1', s_2', s_3', \ldots$. Let s_1 be the first sentence in this enumeration which is compatible with T without being valid in T, and let s_2 be $\bar{s}_1$, the negation of s_1. There certainly is such a sentence s_1, because T is consistent and incomplete. Consider $T_1 = T \uplus \{s_1\}$ and $T_2 = T \uplus \{s_2\}$. Both T_1 and T_2 are consistent proper extensions of T. Since in our example T is essentially incomplete, T_1 and T_2 are essentially incomplete also. Hence, we can continue and we take the first sentence in the enumeration after s_1 which is compatible with T_1 without being valid in T_1 and we call this sentence s_{11}, its negation s_{12}. We consider $T_{11} = T_1 \uplus \{s_{11}\}$ and $T_{12} = T_1 \uplus \{s_{12}\}$, which are both consistent and essentially incomplete theories, proper extensions of T and T_1. Similarly we take the first sentence in the enumeration $s_1', s_2', s_3', \ldots$ after s_1 which is compatible with T_2 without being valid in T_2 and we call this sentence s_{21}, its negation s_{22}; the sentence s_{21} might be the same as s_{11}, but this is not necessary. We form then $T_{21} = T_2 \uplus \{s_{21}\}$ and $T_{22} = T_2 \uplus \{s_{22}\}$, which are both consistent and essentially incomplete theories, proper extensions of T and T_2. Next we take the first sentence in the enumeration after s_{11} which is compatible with T_{11} without being valid in it, and we call this sentence s_{111}, its negation s_{112};

likewise we obtain the couples (s_{121}, s_{122}), (s_{211}, s_{212}), (s_{221}, s_{222}) and we form the theories:

$$\begin{aligned} T_{111} &= T_{11} \uplus \{s_{111}\}; & T_{112} &= T_{11} \uplus \{s_{112}\}; \\ T_{121} &= T_{12} \uplus \{s_{121}\}; & T_{122} &= T_{12} \uplus \{s_{122}\}; \\ T_{211} &= T_{21} \uplus \{s_{211}\}; & T_{212} &= T_{21} \uplus \{s_{212}\}; \\ T_{221} &= T_{22} \uplus \{s_{221}\}; & T_{222} &= T_{22} \uplus \{s_{222}\}; \end{aligned}$$

This way we go on ad infinitum. We can do so in our example, because in the case of a consistent and essentially incomplete theory T every extension of T which appears as $T_{a_1 \ldots a_n}$ with $a_i \in \{1,2\}$ for $1 \leqslant i \leqslant n$ and $n \in N$, is again consistent and essentially incomplete. We obtain the following tree:



It is obvious that in the case of a consistent and essentially incomplete theory the tree goes on growing in all directions, i.e., every node gives birth to a splitting into two directions and every new direction produces a new node. In the usual sense we speak of a *branch* of this tree, in this case a branch being a sequence of consistent theories of increasing strength:

$$T \subset T_{a_1} \subset T_{a_1 a_2} \subset \ldots \subset T_{a_1 a_2 \ldots a_{n-1}} \subset T_{a_1 a_2 \ldots a_{n-1} a_n} \subset \ldots$$

with $a_i \in \{1,2\}$, $i \in N$.

If, as in this example, theory T is consistent and essentially incomplete, then all branches go on indefinitely and each branch of the tree is a sequence of increasing strength matching the construction of the proof of theorem 1 (theorem of Lindenbaum). Therefore, the union of all theories of a branch constitutes a complete extension of the theory T, or what amounts to the same thing, each branch of the tree generates a complete extension of T. Moreover, there are no other complete extensions of T than those generated in this arbitrary tree (arbitrary, because based on an arbitrary enumeration). We argue: Let T^* be a complete extension of T. Then either s_1 or s_2 is valid in T^*, say s_1; consider next s_{11}: either s_{11} or s_{12} is valid in T^*, say s_{12}; continue with s_{121}: either s_{121} or s_{122} is valid in T^*, say s_{121}; and so on. Obviously, there is a branch in the tree which is completely in T^*, or, the complete extension of T generated by this branch coincides with T^*.

If T is finitely and unambiguously completable ($\alpha(T) = 1$), then there is a sentence, say s, such that $T \uplus \{s\}$ is complete; if there are other sentences with this property, then they are equivalent in T with s. Again we take an arbitrary enumeration of all sentences in the symbolism of T

and we construct a tree of complete extensions according to the procedure described above. As soon as we hit the sentence s, or a sentence equivalent in T with s, or a sentence $s_{a_1 \ldots a_n}$ such that the sentences s_{a_1} & $s_{a_1 a_2}$ &. . . & $s_{a_1 a_2 \ldots a_{n-1}}$ & $s_{a_1 a_2 \ldots a_{n-1} a_n}$ is equivalent in T with s (where $a_i \in \{1, 2\}$ for $1 \leqslant i \leqslant n$ and $n \in N$), then the branch breaks off at this sentence. This particular branch is fully described by $T \uplus \{s\}$ and we remark that any other branch of the tree contains at least one sentence which is the negation of a valid sentence of $T \uplus \{s\}$. This implies that any other complete extension of T generated by a branch of the tree is incompatible with $T \uplus \{s\}$. Therefore, if T is finitely and unambiguously completable, there is one and only one branch of the tree that breaks off. It can be shown in a similar way as in the case of essential incompleteness that the tree generates all complete extensions of T.

If T is finitely and ambiguously completable ($\alpha(T) > 1$), then again we can take some enumeration of all sentences in the language of T and construct a tree of complete extensions, according to the procedure described above for the case of essential incompleteness. In a similar way as in the case of unambiguous completability we can prove that there are exactly as many branches that break off as there are different finite completions of T.

It may be clear now what we understand by a tree of complete extensions for a theory. Let T be a theory and let $\mathbf{E}$ be any enumeration of all sentences in the language of T. A binary tree based on T and $\mathbf{E}$ constructed as described above is called a tree of complete extensions for T. If T is inconsistent, then the empty set can be considered to be its tree of complete extensions. If T is complete, the whole tree consists of T itself. We can summarize the pertinent properties in the following remarks:

Remark 1: Every branch of a tree of complete extensions for a theory T generates one complete extension of T, and every complete extension of T is generated by one branch of this tree.

Remark 2: Branches of a tree of complete extensions for a theory T that break off generate finite completions of T, and finite completions of T are generated by branches of this tree that break off.

With help of the tree of complete extensions it is easy to prove certain theorems concerning characteristic numbers. As first examples we can take some theorems, which are specifications of the results of Tarski[3] and Mostowski.[4]

Theorem 2: *If T is consistent and essentially incomplete, then the cardinal number of the set of complete extensions of T is $2^{\aleph_0}$. Or equivalently, if T is consistent and $\alpha(T) = 0$, then $\mathrm{n}(T) = 2^{\aleph_0}$.*

Proof: The cardinality of the set of complete extensions of T is that of the set of branches of a tree of complete extensions for T (remark 1). If T is essentially incomplete, then none of the branches in such a tree breaks off (remark 2). Obviously the cardinality in question is the same as that of the set of the different infinite sequences whose elements are the digits 1 and 2, vis. $2^{\aleph_0}$.

Theorem 3: If T is incomplete but finitely and unambiguously completable, then the cardinality of the set of complete extensions of T is $2^{\aleph_0}$. Or equivalently, if T is incomplete and $\alpha(T) = 1$, then $n(T) = 2^{\aleph_0}$.

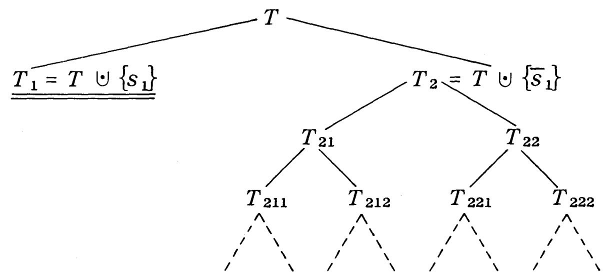
Proof: The cardinality of the set of complete extensions of T is that of the set of branches of a tree of complete extensions for T (remark 1). From the proof of the foregoing theorem it is obvious that the cardinality of the set of branches is at most $2^{\aleph_0}$. Further, there are at least two different branches (corollary 1.1 and remark 1), one of which does not break off (remark 2). Considering a tree of complete extensions for T we remark that this implies that certainly not both T_1 and T_2 are complete, and moreover, (again by remark 2), that either T_1 or T_2 is essentially incomplete. Hence, $n(T)$ equals $2^{\aleph_0}$ (theorem 2).

If one attempts an analysis, similar to that in the proof of theorem 3, for the case where there are more finite completions than one, there appears a complication. The enumeration **E** can be such, that the construction of the tree breaks off completely. In order to clarify this case we construct another kind of tree, which we like to call *a tree of finite completions*. Let T be an incomplete theory and let **C** be such an enumeration of all sentences which complete T, that, if several completing sentences are equivalent in T, only one of them is listed. Let further **E** be again an enumeration of all sentences in the language of T. We construct a tree of finite completions for T proceeding in the same way as in the construction of a tree of complete extensions for T, but instead of using the enumeration **E** we start with using **C**. If we do not exhaust **C**, then we call the tree based on T and **C** a tree of finite completions for T. If we do exhaust **C**, then there are two possibilities:

(i) The last theory which appears in the tree is complete. In this case all the branches of the tree break off and we call the thus established finite tree based on T and **C** a tree of finite completions for T.

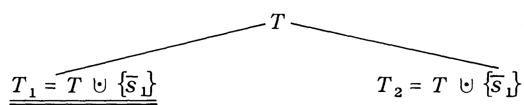
(ii) The last theory which appears in the tree, say T', is incomplete In this case T' is not only incomplete, but essentially incomplete, since there is no sentence left which could complete it. We then continue constructing for this essentially incomplete theory T' a tree of consistent and complete extensions based on T' and **E**. The whole of the infinite tree thus appearing, based on T, **C** and **E**, we call a tree of finite completions for T.

As a first example we take the case of theorem 3, i.e., let T be incomplete but finitely and unambiguously completable, or what amounts to the same thing, let T be incomplete and let **C** consist out of one single sentence, say s_1. We obtain a tree of finite completions for this theory T according to the procedure described above and already after one step we are in the situation of case (ii).



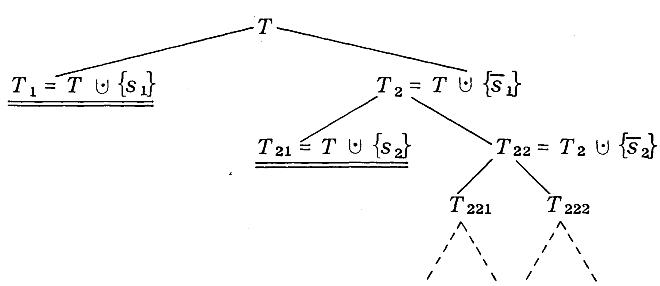
In this tree theory T_1 is complete, the branch breaks off (indicated by double underlining). Theory T_2 is essentially incomplete since T_1 is the only finite completion of T. Obviously, this tree of finite completions based on T, **C** and **E** is at the same time a tree of complete extensions for T. As such the tree illustrates theorem 3.

For a second example we pass to the case of an incomplete theory T which has exactly two different finite completions, or $\mathbf{C} = \langle s_1, s_2 \rangle$ with $T \uplus \{s_1\}$ and $T \uplus \{s_2\}$ being the only finite completions of T and $T \uplus \{s_1\} \uplus \{s_2\}$ being inconsistent. The first step in the construction of the tree provides us with:



If T_2 is complete, then s_1 is equivalent in T with s_2 and we have $T_2 = T \uplus \{\bar{s}_1\} = T \uplus \{s_2\}$; we have exhausted **C** and the last theory appearing in the tree is complete; we are in the situation described above in (i), we can put double lines under T_2, all branches have come to an end and a tree of finite completions is established; as is seen easily from corollary 1.2 this tree of finite completions for T is again a tree of complete extensions for T.

If T_2 is incomplete, then we argue as follows: $T \uplus \{s_2\}$ is complete, the sentence s_1 is not valid in $T \uplus \{s_2\}$ (because $T \uplus \{s_1\} \uplus \{s_2\}$ is inconsistent), therefore $\bar{s}_1$ is valid in $T \uplus \{s_2\}$, or, s_2 is compatible with $T \uplus \{\bar{s}_1\}$; hence it is possible to continue the construction of the tree as follows:



Since T_1 and T_{21} are the only finite completions and $T_{21} = T_2 \uplus \{s_2\}$ is a proper extension of T_2, theory T_{22} has to be essentially incomplete and in the same way as in the first example we end up in the situation described above in (ii). It can be seen readily that in this case again the tree of finite completions thus constructed is at the same time a tree of complete extensions for T.

If there are more finite completions than two, the same type of construction as in the foregoing example can be carried out. We obtain left-hand branches that break off immediately until **C** is exhausted; if **C** is infinite, then only the extreme right-hand branch goes on growing ad infinitum; if **C** is finite, then either the whole tree is finite or after a finite number of steps we hit an essentially incomplete theory and from there the tree grows out as sketched above.

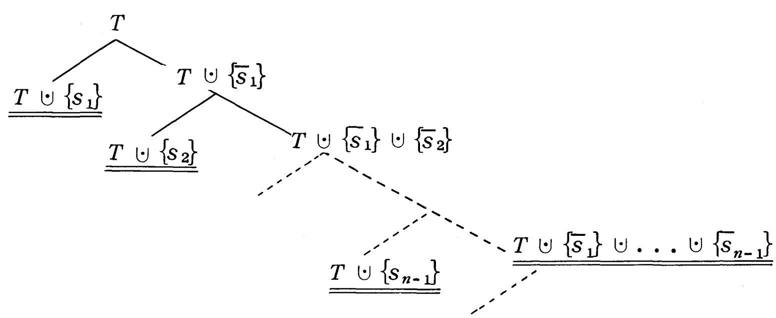
With help of the tree of finite completions it is easy again to prove certain theorems about characteristic numbers. We mentioned already how theorem 3 is illustrated by the first tree of finite completions we sketched. Another example is the following of Mostowski's results[5] which follows immediately from the considerations expounded above:

Theorem 4: *If T is incomplete but finitely and ambiguously completable, and T has a finite number of finite completions, then the cardinal number of the set of all other complete extensions of T is either* 0 *or* $2^{\aleph_0}$. *Or equivalently said, if* $1 < \alpha(T) < \aleph_0$, *then either* $\pi(T) = 0$ *or* $\pi(T) = 2^{\aleph_0}$.

A remark of a somewhat different nature which can be proved easily with help of the picture of the tree of finite completions is the following:

Remark 3: If T is incomplete but finitely and ambiguously completable, and the finite completions $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, . . . , $T \uplus \{s_n\}$ are the only different complete extensions of T, then for each i such that $1 \leqslant i \leqslant n$, the sentence s_i is equivalent in T with the sentence $\overline{s}_1 \& \overline{s}_2 \& \ldots \& \overline{s}_{i-1} \& \overline{s}_{i+1} \& \ldots \& \overline{s}_n$. Or equivalently, if $\alpha(T) = n$, where $1 < n < \aleph_0$, and if $\pi(T) = 0$, then there are sentences $s_1, s_2, \ldots, s_n$, such that $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, . . . , $T \uplus \{s_n\}$ are different finite completions of T and for each i, where $1 \leqslant i \leqslant n$, the sentence s_i is equivalent in T with the sentence $\overline{s}_1 \& \overline{s}_2 \& \ldots \& \overline{s}_{i-1} \& \overline{s}_{i+1} \& \ldots \& \overline{s}_n$.

Proof: We construct a tree of finite completions based on T and $\mathbf{C} = \langle s_1, s_2, \ldots, s_n \rangle$. It is the lower end that interests us.



Theory $T \uplus \{\overline{s}_1\} \uplus \ldots \uplus \{\overline{s}_{n-1}\}$ has to be complete, otherwise it would give birth to a new splitting and there would be at least $n+1$ different complete extensions, contrary to the assumption. Therefore, $T \uplus \{s_n\} = T \uplus \{\overline{s}_1\} \uplus \ldots \uplus \{\overline{s}_{n-1}\}$, *or* $\overline{s}_1 \& \overline{s}_2 \& \ldots \& \overline{s}_{n-1} \leftrightarrow \overline{s} \in T$. Since we can apply this argument for any permutation of $s_1, \ldots, s_n$, the remark is proved.

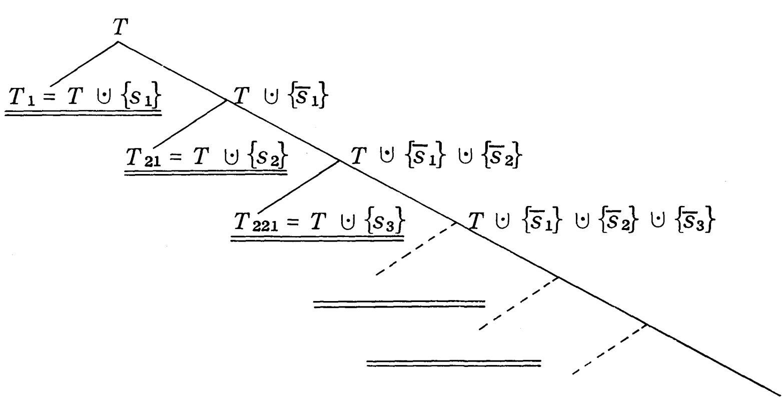
The following remark, which is an extension of corollary 1.2, falls into the same category.

Remark 4: Let T be incomplete but finitely and ambiguously completable and let $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, . . . , $T \uplus \{s_n\}$ be different finite completions of T. If for some i such that $1 \leqslant i \leqslant n$ the sentence s_i is equivalent in T with the sentence $\bar{s}_1 \,\&\, \bar{s}_2 \,\&\, \ldots \,\&\, \bar{s}_{i-1} \,\&\, \bar{s}_{i+1} \,\&\, \ldots \,\&\, \bar{s}_n$, then $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, . . . , $T \uplus \{s_n\}$ are the only different complete extensions of T. Or equivalently, let $\alpha(T) > 1$ and let $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, . . . , $T \uplus \{s_n\}$ be different finite completions of T. If for some i such that $1 \leqslant i \leqslant n$ the sentence s_i is equivalent in T with $\bar{s}_1 \,\&\, \ldots \,\&\, \bar{s}_{i-1} \,\&\, \bar{s}_{i+1} \ldots \,\&\, \bar{s}_n$, then $\alpha(T) = n$ and $n(T) = 0$.

Proof: Let s_n be equivalent in T with $\bar{s}_1 \,\&\, \bar{s}_2 \,\&\, \ldots \,\&\, \bar{s}_{n-1}$. If we start constructing a tree of finite completions based on T and such an enumeration C that $s_1, \ldots, s_n$ forms the initial segment of C, then the tree breaks off at $T \uplus \{\bar{s}_1\} \uplus \{\bar{s}_2\} \uplus \ldots \uplus \{\bar{s}_{n-1}\}$ as illustrated in the diagram of the foregoing proof. In an analogous manner as in the case of corollary 1.2 it can be shown that there is not a different $(n+1)^{\text{th}}$ complete extension of T. Since the argument applies to any permutation of $s_1, \ldots, s_n$, the remark is proved.

Remark 5 (a corollary of remarks 3 and 4): Let T be incomplete but finitely and ambiguously completable and let $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, . . . , $T \uplus \{s_n\}$ be different finite completions of T. If for some i such that $1 \leqslant i \leqslant n$ the sentence s_i is equivalent in T with the sentence $\bar{s}_1 \,\&\, \ldots \,\&\, \bar{s}_{i-1} \,\&\, \bar{s}_{i+1} \,\&\, \ldots \,\&\, \bar{s}_n$, then for each j such that $1 \leqslant j \leqslant n$ the sentence s_j is equivalent in T with the sentence $\bar{s}_1 \,\&\, \ldots \,\&\, \bar{s}_{j-1} \,\&\, \bar{s}_{j+1} \,\&\, \ldots \,\&\, \bar{s}_n$.

At this point we can pay some more attention to the case where C is infinite, i.e., the case where there are denumerably many different finite completions of a theory T, or what amounts to the same thing, where there are denumerably many completing sentences for the incomplete theory T not two of which are equivalent in T. We remarked already that in this case the tree of finite completions goes on growing at the extreme right-hand side, while the branches to the left break off immediately. We obtain the following picture for $\mathsf{C} = \langle s_1, s_2, s_3, \ldots \rangle$:



Considering the extreme right-hand branch of this tree we remark that it generates theory $T \uplus \bigcup_{n=1}^{\infty} \{\bar{s}_n\}$. The consistency of this theory is established in the following theorem, which is implied already in the results of Tarski[6].

Theorem 5: If T *is incomplete but there are denumerably many completing sentences* $s_1, s_2, s_3, \ldots$ *for* T, *not two of which are equivalent in* T, *then* $T \uplus \bigcup_{n=1}^{\infty} \{\bar{s}_n\}$ *is consistent.*

Proof: Theory $T \uplus \{s_1\}$ is complete and as such it contains either s_2 or $\bar{s}_2$. It is given that $T \uplus \{s_1\}$ and $T \uplus \{s_2\}$ are different, or, that $T \uplus \{s_1\} \uplus \{s_2\}$ is inconsistent. Hence, $\bar{s}_2 \epsilon T \uplus \{s_1\}$. Similarly, all $\bar{s}_3, \bar{s}_4, \bar{s}_5, \ldots, \epsilon T \uplus \{s_1\}$. Consequently, $T \uplus \bigcup_{n=2}^{\infty} \{\bar{s}_n\} \subseteq T \uplus \{s_1\}$. Hence, $T \uplus \bigcup_{n=2}^{\infty} \{\bar{s}_n\}$ is consistent and s_1 is at least compatible with this theory. We know further, that if a theory is complete, all sentences compatible with it belong to it. However, if $s_1 \epsilon T \uplus \bigcup_{n=2}^{\infty} \{\bar{s}_n\}$, then it follows from the deduction theorem that there is a finite number of sentences among $\bar{s}_2, \bar{s}_3, \bar{s}_4, \ldots$, say $\bar{s}_1^*, \bar{s}_2^*, \ldots, \bar{s}_{m-1}^*$, $\bar{s}_m^*$, such that $\bar{s}_1 \longleftrightarrow \bar{s}_1^* \,\&\, \bar{s}_2^* \,\&\, \ldots \,\&\, \bar{s}_{m-1}^* \,\&\, \bar{s}_m^* \epsilon T$, or, by remark 4, there is only a finite number of different completing sentences for T, contrary to the hypothesis. It follows that $T \uplus \bigcup_{n=2}^{\infty} \{\bar{s}_n\}$ is incomplete and hence that s_1 is not valid in it. Therefore, both s_1 and $\bar{s}_1$ are compatible with $T \uplus \bigcup_{n=2}^{\infty} \{\bar{s}_n\}$ and $T \uplus \bigcup_{n=1}^{\infty} \{s_n\}$ is consistent.

The consistency of the right-hand branch being settled we turn to the question of its completeness. We can state, that this theory may as well be complete as incomplete. Concrete examples of either case can be constructed easily. The case where there are denumerably many finite completions $T \uplus \{s_1\}$, $T \uplus \{s_2\}, \ldots$ but where the right-hand branch theory $T \uplus \bigcup_{n=1}^{\infty} \{\bar{s}_n\}$ is not complete, is the interesting one. This possibility implies that in the case where $\mathbf{C}$ is infinite the tree of finite completions based on T and $\mathbf{C}$ is not necessarily a tree of complete extensions for T. If $T \uplus \bigcup_{n=1}^{\infty} \{\bar{s}_n\}$ is incomplete, then the extreme right-hand branch of the tree of finite completions does not generate a complete extension of T and there are at least two different complete extensions of T, vis. complete extensions of $T \uplus \bigcup_{n=1}^{\infty} \{\bar{s}_n\}$ (by corollary 1.1), which are not generated by this tree.

For shortness sake one might call theory T a *lone tree theory* if, and only if, a tree of finite completions for T is at the same time a tree of complete extensions for T. It follows from the definitions of the trees and the considerations expounded above that theory T is always a lone tree theory except in the case where there are infinitely many different finite completions $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, ... and the right-hand branch theory $T \uplus \bigcup_{n=1}^{\infty} \{\overline{s}_n\}$ is incomplete.

§3. Classification. Let us call a theory T *virtually complete* if, and only if, T is consistent and, if T^* is a complete extension of T, then there exists a sentence, say s^*, such that $T^* = T \uplus \{s^*\}$. In other words, theory T is virtually complete if, and only if, it is consistent and all complete extensions of T are finite completions of T. Or in terms of characteristic numbers, theory T is virtually complete if, and only if $\alpha(T) > 0$, $n(T) = 0$. It does no harm that according to this definition a complete theory is automatically virtually complete. If a theory T is incomplete but virtually complete, then it can be seen readily that T is ambiguously completable, or $\alpha(T) > 1$, by corollary 1.1. However, the converse of the last statement is not true; let theory T be finitely but ambiguously completable in such a way that there are denumerably many sentences, say $s_1, s_2, \ldots$ such that $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, ... are all different complete extensions of T. As stated before, there exist such theories. As we have seen, the theory $T \uplus \bigcup_{n=1}^{\infty} \{s_n\}$ is consistent and as such, by theorem 1, it has a complete extension, say T^*. But there is no sentence s such that $T^* = T \uplus \{s\}$, because, if this were the case, then s (or a sentence equivalent in T with s) would appear somewhere in the enumeration $s_1, s_2, \ldots$ and T^* would contain both s and $\overline{s}$ and thus be inconsistent contrary to the assumption that T^* is a complete extension. Hence, in this case, although T is ambiguously completable ($\alpha(T) > 1$), this theory is not virtually complete. In a more general way virtual completeness is excluded from this kind of theories by the following result of Tarski[7]:

Theorem 6: If a theory T is virtually complete, then it has only a finite number of complete extensions; conversely, if a theory T is consistent and has only a finite number of complete extensions, then it is virtually complete. Or equivalently, if $\alpha(T) > 0$ and $n(T) = 0$, then $\alpha(T) < \aleph_0$; conversely, if $0 < \alpha(T) + n(T) < \aleph_0$, then $n(T) = 0$.

Proof: If T is complete, then it has only one complete extension; if a theory has only one complete extension; then it is complete and hence virtually complete. So the theorem is proved for T being complete and we can assume from now on that T is incomplete.

(i) If T is incomplete but virtually complete, then $\alpha(T) > 1$, as remarked above. Assume $\alpha(T) = \aleph_0$. Then there are sentences $s_1, s_2, \ldots$ such that $T \uplus \{s_1\}$, $T \uplus \{s_2\}$, ... are different complete extensions of T. As we have seen previously, $T \uplus \bigcup_{n=1}^{\infty} \{\overline{s}\}$ has a consistent and complete extension, say T_*, for which there is no sentence s such that $T_* = T \uplus \{s\}$, contrary to the hypothesis that T is virtually complete. Hence, $\alpha(T) < \aleph_0$.

(ii) Let there be only a finite number of different complete extensions of the incomplete theory T, say $T_1, \ldots, T_n$. Assume that there is no sentence s_1 such that $T_1 = T \dot\cup \{s_1\}$. It is given that T_1 is different from all $T_2, \ldots, T_n$ and therefore T_1 has to contain a sentence, say t_2, which is not valid in T_2; similarly a sentence, say t_3, which is not valid in T_3; ... finally a sentence, say t_n, which is not valid in T_n. Consider $T' = T \dot\cup \{t_2\} \dot\cup \{t_3\} \dot\cup \ldots \dot\cup \{t_n\}$. Obviously, $T \subset T' \subseteq T_1$. If $T' = T_1$, then for s_1 we can take the sentence $t_2 \mathbin{\&} t_3 \mathbin{\&} \ldots \mathbin{\&} t_n$ and $T_1 = T \dot\cup \{s_1\}$ contrary to the assumption. However, if T' is different from T_1, then T_1 has to contain a sentence, say t_1, which is not valid in T'. But then $T' \dot\cup \{\bar{t}_1\}$ would be consistent and as such it would have a complete extension (by theorem 1) which is different from all $T_1, \ldots, T_n$ contrary to the hypothesis that these are the only complete extension of T. It follows that there is a sentence s_1 such that $T_1 = T \dot\cup \{s_1\}$. Similarly, there are sentences $s_2, \ldots, s_n$ such that $T_2 = T \dot\cup \{s_2\}, \ldots, T_n = T \dot\cup \{s_n\}$. Hence T is virtually complete.

Having introduced the concept of virtual completeness we can define, in a more or less analogous way, the concept of almost essential incompleteness. We shall say that a theory T is *almost essentially incomplete* if, and only if, T is not essentially incomplete but has a consistent extension, say $T*$, which is essentially incomplete. Or, what amounts to the same thing, theory T is almost essentially incomplete if, and only if, $\alpha(T) > 0$ and $n(T) = 2^{\aleph_0}$. It follows immediately from theorem 3 that incomplete theories, which are finitely and unambiguously completable, are almost essentially incomplete. As a re-formulation of theorem 4 we have:

If a theory T is finitely and ambiguously completable and has only a finite number of finite completions, then T is either virtually complete or almost essentially incomplete.

The additional condition in the last statement, that T has only a finite number of finite completions, stresses the fact that not every finitely but ambiguously completable theory T which is not virtually complete is necessarily almost essentially incomplete. A finitely but ambiguously completable theory T with denumerably many different finite completions $(\alpha(T) = \aleph_0)$ is as we have seen (in theorem 6) not virtually complete, but it needs not to be almost essentially incomplete either. It is true that we can bring all finite completions $T \dot\cup \{s_1\}, T \dot\cup \{s_2\}, \ldots$ in one tree of finite completions but at the same time this procedure generates the consistent theory $T \dot\cup \dot{\bigcup}_{n=1}^{\infty} \{\bar{s}_n\}$ (cf. theorem 5). It is possible that $T \dot\cup \dot{\bigcup}_{n=1}^{\infty} \{\bar{s}_n\}$ is complete and hence that there are no more complete extensions than $T \dot\cup \{s_1\}$, $T \dot\cup \{s_2\}, \ldots$ and $T \dot\cup \dot{\bigcup}_{n=1}^{\infty} \{\bar{s}_n\}$; in this case $(\alpha(T) = \aleph_0,\ n(T) = 1)$ theory T is not almost essentially incomplete. Apparently, there are incomplete theories which are neither essentially incomplete, nor almost essentially incomplete, nor virtually complete. The following remark states a common property of these theories:

Remark 6: If a theory T is neither essentially incomplete, nor almost essentially incomplete, nor virtually complete, then there are denumerably many different finite completions of T.

Proof: Theory T is not finitely and unambiguously completable because this would imply that T is almost essentially incomplete contrary to the hypothesis. Since T is not essentially incomplete either, the theory has to be finitely and ambiguously completable (cf. the exhaustive classification on p. 4). However, for such a finitely and ambiguously completable theory we obtain by the above re-formulation of theorem 4:

$$\textit{If } \alpha(T) > 1,\ \alpha(T) < \aleph_0,\ n(T) \neq 0, \textit{ then } \alpha(T) = 2^{\aleph_0}$$

Hence we deduce a contrapositive:

$$\textit{If } \alpha(T) > 1,\ \alpha(T) < 2^{\aleph_0},\ n(T) \neq 0, \textit{ then } \alpha(T) = \aleph_0,$$

which proves the remark.

Among the theories whose finite completions have cardinality $\aleph_0$ we choose those for which in addition the cardinality of the other complete extensions is >0 and $\leqslant \aleph_0$. We shall say that a theory T is $\aleph_0$*-incomplete* if, and only if, $\alpha(T) = \aleph_0$ and $0 < n(T) \leqslant \aleph_0$. The next two remarks justify this choice. They state that the four categories (virtually complete, essentially incomplete, almost essentially incomplete, $\aleph_0$-incomplete theories) partition the set of all theories.

Remark 7: The four categories:

(i) virtually complete theories
(ii) essentially incomplete theories
(iii) almost essentially incomplete theories
(iv) $\aleph_0$-incomplete theories

are pairwise disjoint.

Proof: By simple inspection of the characteristic numbers we learn that overlapping is excluded:

Categories	Characteristic Numbers	
Essentially incomplete	$\alpha(T) = 0$	$n(T) = \begin{cases} 0 \\ 2^{\aleph_0} \end{cases}$
Almost essentially incomplete	$\alpha(T) > 0$	$n(T) = 2^{\aleph_0}$
Virtually complete	$0 < \alpha(T) < \aleph_0$	$n(T) = 0$
$\aleph_0$-incomplete	$\alpha(T) = \aleph_0$	$0 < n(T) \leqslant \aleph_0$

If T is a lone tree theory and $\alpha(T) < \aleph_0$, then either $n(T) = 0$ or $n(T) = 2^{\aleph_0}$. If T is a lone tree theory and $\alpha(T) = \aleph_0$, then $n(T) = 1$. In other words, every lone tree theory belongs to one of the four categories mentioned above.

If T is not a lone tree theory, then $\alpha(T) = \aleph_0$ and the right hand branch theory is incomplete. It suits the arguments to use upper and lower indices. Let $T_0^0 = T$ and let $s_1^0, s_2^0, \ldots$ be an enumeration of all sentences (not two of which are equivalent in T_0^0) completing T_0^0. Let $T_i^0 = T_{i-1}^0 \cup \{\overline{s_i^0}\}$, for $i \in N$. Thus we have defined the sequence of increasing strength $T_1^0, T_2^0, \ldots$, extensions of T_0^0. Next we define $T_0^1 = \bigcup_{i=1}^{\infty} T_i^0$. Theory T_0^1, formerly called the right hand branch theory, is consistent by theorem 5. If T_0^1 is a lone tree theory, we obtain the following scheme, where $k \in N$:

$\alpha(T_0^1)$	$n(T_0^1)$	$\alpha(T_0^0)$	$n(T_0^0)$
0	$2^{\aleph_0}$	$\aleph_0$	$2^{\aleph_0}$
k	$2^{\aleph_0}$	$\aleph_0$	$2^{\aleph_0} + k = 2^{\aleph_0}$
k	0	$\aleph_0$	k
$\aleph_0$	1	$\aleph_0$	$\aleph_0 + 1 = \aleph_0$

We read from the scheme that in this case also theory T belongs to one of the four categories. If T_0^1 is not a lone tree theory, we continue and define $T_0^2 = \bigcup_{i=1}^{\infty} T_i^1$, where $T_i^1 = T_{i-1}^1 \cup \{\overline{s_i^1}\}$, $i \in N$ and where $s_1^1, s_2^1, \ldots$ is the enumeration of all sentences (not two of which are equivalent in T_0^1) completing T_0^1. If T_0^2 is a lone tree theory, again we can conclude easily that the original theory T belongs to one of the four categories. If T_0^2 is not a lone tree theory we continue and define T_0^3.

Apparently the above sketch tries to describe a procedure where new, stronger extensions of T are defined until one hits a lone tree theory. More precisely and more in general we define a collection $\mathcal{T}$ of extensions of T, well ordered by inclusion, as follows: $\mathcal{T}$ contains as elements all those and only those theories T_0^μ, where

(i) μ is an ordinal number and $\mu > 0$;

(ii) If $T_0^\kappa, T_0^\lambda \in \mathcal{T}$ and $\kappa < \lambda$ then T_0^κ is not a lone tree theory;

(iii) If μ is a limit number, then $T_0^\mu = \bigcup_{\lambda<\mu} T_0^\lambda$;

(iv) If μ is not a limit number then $T_0^\mu = \bigcup_{i=1}^{\infty} T_i^{\mu-1}$, where $T_i^{\mu-1} = T_{i-1}^{\mu-1} \cup \{\overline{s_i^{\mu-1}}\}$, $i \in N$ and $s_1^{\mu-1}, s_2^{\mu-1}, \ldots$ is an enumeration of all sentences (not two of which are equivalent in $T_0^{\mu-1}$) completing $T_0^{\mu-1}$.

Obviously, for $\mathcal{T}$ to contain a greatest element, say T_0^τ it is necessary and sufficient that T_0^τ is a lone tree theory. With T_0^0 there is associated the set of sentences $S^0 = \{s_i^0\}$, $i \in N$, and with every $T_0^\lambda \in \mathcal{T}$ which is not a lone tree theory there is associated a set of sentences $S^\lambda = \{s_i^\lambda\}$, $i \in N$, the sets of sentences completing T_0^0 and T_0^λ respectively. Not only within S^0 and within every such S^λ no two sentences are equivalent in T, but the sets are pairwise disjoint in such a way even that no two sentences belonging to different sets are equivalent in T. Since the cardinal number of the set of all sentences is $\leqslant \aleph_0$, the union of all sets S^0, S^λ has to have a cardinal number $\leqslant \aleph_0$. This implies that $\mathcal{T}$ has to have a greatest element, say T_0^τ, and τ has to be $< \Omega$ in order to keep the cardinality down to $\aleph_0$. This greatest theory T_0^τ is a lone tree theory. The above considerations lead to:

Remark 8: A theory T is either essentially incomplete, or almost essentially incomplete, or virtually complete, or $\aleph_0$-incomplete.

Proof: As remarked above, if T is a lone tree theory, then T belongs to one of the four categories. If T is not a lone tree theory, then we can define the collection $\mathcal{T}$ as above with greatest element T_0^τ and $\tau < \Omega$. For each T_0^μ with $\mu < \tau$, the finite completions of T_0^μ are not finite complete extensions of T. Hence, each T_0^μ, $\mu < \tau$, contributes $\aleph_0$ complete extensions to $n(T)$. Since $\tau < \Omega$, the total contributions of the theories T_0^{μ} with $\mu < \tau$ to $n(T)$ amount to $\aleph_0$. Since T_0^τ is a lone tree theory it contributes to $n(T)$ either $2^{\aleph_0}$ or $k + 2^{\aleph_0}$, or k, or $\aleph_0 + 1$, so that in any case either $n(T) = 2^{\aleph_0}$ or $n(T) \leqslant \aleph_0$, with $n(T) < \aleph_0$ only possible for $\tau = 1$. It follows that also a not lone tree theory belongs to one of the four categories.

From the above considerations it is easy to obtain the "characteristic pairs" as established by Tarski and Mostowski, i.e., the only possible ordered pairs $< a(T), n(T)>$ for a theory T. They are:

$$\begin{array}{lll} <0, 0>, & <k, 0>, & <\aleph_0, k>, \\ <0, 2^{\aleph_0}>, & <k, 2^{\aleph_0}>, & <\aleph_0, \aleph_0>, \\ & & <\aleph_0, 2^{\aleph_0}>, \end{array}$$

where $k \in N$. However, for the cases that $a(T) = \aleph_0$ and $n(T) = \aleph_0$ or $n(T) = 2^{\aleph_0}$ they do not quite characterize the theory concerned, as observed already by Tarski[8]. The characterization of a theory to which this context refers concerns a certain *structural type*. Let T', T'' be theories, let $\mathcal{T}'$, $\mathcal{T}''$ be the sets of all extensions of T', T'' respectively and let $<\mathcal{T}', \subseteq>$, $<\mathcal{T}'', \subseteq>$ be the corresponding structures partially ordered by inclusion. In this context T' and T'' are said to have the same structural type if, and only if $<\mathcal{T}', \subseteq>$ and $<\mathcal{T}'', \subseteq>$ are isomorphic. The structural type of a theory T is then the class of all theories which have the same structural type as T. The treatment in this paper makes it very plausible that indeed the pairs $<\aleph_0, \aleph_0>$ and $<\aleph_0, 2^{\aleph_0}>$ do not determine the structural type of T. Consider, e.g., the case of a theory T with $a(T) = \aleph_0$, and T_0^1 being essentially incomplete as compared to a theory U with $a(U) = \aleph_0$ and $a(U_0^1) = \aleph_0$ and U_0^2 being essentially incomplete. However, a more precise and positive approach to this characterization of theories with help of the tree pictures as sketched above, although seemingly plausible, is beyond the scope of this paper.

NOTES

1. Part of this paper was prepared while the author was with the University of Amsterdam under Euratom Contract No. 010-60-12.

2. *Cf.*, e.g., [4], p. 370 of the English translation. Tarski speaks of an *axiomatizable extension* where we use *finite completion*. Notice that, if s completes T, theory $T \uplus \{s\}$ is not necessarily an axiomatizable theory in the usual sense.

3. *Cf.* [4], theorems 37, 39 and context, pp. 367 ff. of the English translation. See also [3], p. 109 of the English translation.

4. *Cf.* [1], pp. 46-48.

5. *Cf.* [1], p. 48, Satz 8. See also [4], p. 370 of the English translation.

6. *Cf.* [4], pp. 367-370 of the English translation (the exclusion of the case $\alpha = \aleph_0$, $n = 0$).

7. *Cf.* [4], pp. 367-370 of the English translation.

8. *Cf.* [4], p. 371 of the English translation.

REFERENCES

[1] Mostowski, A., "Abzaehlbare Boolesche Koerper und ihre Anwendung auf die allgemeine Metamathematik", *Fundamenta Mathematicae*, **29** (1937), pp. 34 ff.

[2] Tarski, A., "Ueber einige fundamentale Begriffe der Metamathematik", *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, **23** (1930), iii, pp. 22 ff. (English translation in *Logic, Semantics, Metamathematics, papers from 1923 to 1938 by Alfred Tarski, translated by J. H. Woodger*, Oxford, 1956, pp. 30 ff.)

[3] Tarski, A., "Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften I", *Monatshefte fuer Mathematik und Physik*, **37**, (1930), pp. 361 ff. (English translation as in [2], pp. 60 ff.)

[4] Tarski, A., "Grundzuege des Systemenkalkuels, II", *Fundamenta Mathematicae*, **26** (1936), pp. 283 ff. (English translation of this paper together with the translation of its first part as in [2], pp. 342 ff.)

The University of Santa Clara
Santa Clara, California