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PROPOSITIONAL SEQUENCE-CALCULI FOR INCONSISTENT SYSTEMS

ANDRÉS R. RAGGIO

Contradictions arise mostly at the beginning and at the end of a theoretical construction. If the meaning of words is not fixed a proposition and its negation can both be true. And, as it is well known, a theory in spite of the axiomatic fixation of its concepts must contain inconsistencies if its means of expression are sufficiently rich. If classical—but also intuitionistic—propositional logic is used as the basic logical frame of a theory the deduction of a contradiction produces its complete trivialization: every proposition is deducible in it, *ex falso sequitur quodlibet.* The minimal logic avoids this logical principle but from a contradiction we can deduce in it the negation of every proposition. By weakening the classical logic S. Jaśkowski (*cf.* [6]) and specially Newton C. A. da Costa have built propositional calculi which enable us to overcome this difficulty. By the way these systems contain logical laws which Hegelians in spite of their famous rejection of the principle of contradiction must acknowledge as valid.

Da Costa has built a hierarchy of propositional calculi $\mathbf{C}_n(1 \leq n \leq \omega)$ whose decidability has not been settled yet. We tried first to solve this problem constructing equivalent sequence-calculi and proving the corresponding cut-theorems. But this new hierarchy which we called $\mathbf{CG}_n(1 \leq n \leq \omega)$ showed some restrictions which are only justified from an intuitionistic point of view. By dropping these restrictions we have constructed a new hierarchy $\mathbf{WG}_n(1 \leq n \leq \omega)$ of decidable calculi with the same essential properties of the $\mathbf{C}_n$.*

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§1. *Equivalence of* C_n *and* CG_n. Da Costa uses the following axiom system for his C_n (*cf.* [1])

$$
\begin{array}{ll}
(1) & \mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{A}) \\
(2) & (\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{C})) \supset (\mathfrak{A} \supset \mathfrak{C})) \\
(3) & \dfrac{\mathfrak{A},\ \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}} \\
(4) & \mathfrak{A} \,\&\, \mathfrak{B} \supset \mathfrak{A} \\
(5) & \mathfrak{A} \,\&\, \mathfrak{B} \supset \mathfrak{B} \\
(6) & \mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{A} \,\&\, \mathfrak{B}) \\
(7) & \mathfrak{A} \supset \mathfrak{A} \vee \mathfrak{B} \\
(8) & \mathfrak{B} \supset \mathfrak{A} \vee \mathfrak{B} \\
(9) & (\mathfrak{A} \supset \mathfrak{C}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \vee \mathfrak{B} \supset \mathfrak{C})) \\
(10) & \mathfrak{A} \vee \neg \mathfrak{A} \\
(11) & \neg\neg \mathfrak{A} \supset \mathfrak{A} \\
(12) & \mathfrak{B}^{(n)} \supset ((\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \neg \mathfrak{B}) \supset \neg \mathfrak{A})) \\
(13) & \mathfrak{A}^{(n)} \,\&\, \mathfrak{B}^{(n)} \supset (\mathfrak{A} \,\&\, \mathfrak{B})^{(n)} \\
(14) & \mathfrak{A}^{(n)} \,\&\, \mathfrak{B}^{(n)} \supset (\mathfrak{A} \vee \mathfrak{B})^{(n)} \\
(15) & \mathfrak{A}^{(n)} \,\&\, \mathfrak{B}^{(n)} \supset (\mathfrak{A} \supset \mathfrak{B})^{(n)} \\
(16) & \mathfrak{A}^{(n)} \supset (\neg \mathfrak{A})^{(n)}
\end{array}
$$

$\mathfrak{B}^0$ is short for $\neg(\mathfrak{B} \,\&\, \neg \mathfrak{B})$ and $\mathfrak{B}^{(n)}$ is defined by the following recursion

$$
\mathfrak{B}^{(1)} = \mathfrak{B}^0 \qquad \mathfrak{B}^{(n+1)} = \mathfrak{B}^{(n)} \,\&\, \mathfrak{B}^{\overbrace{00\,0\ldots0}^{n+1\text{ times}}}
$$

C_ω contains only the first eleven axiom schemata and rules; $\mathsf{C}_n (1 \leq n < \omega)$ has all sixteen with the corresponding value of n.

CG_ω differs from the propositional part of Gentzen's **LK** (*cf.* [4], p. 191) only in the following rules

$$
\text{instead of } \mathbf{NEA},\ \mathsf{CG}_\omega \text{ has } \mathbf{NEA'} \quad \frac{\mathfrak{A}, \Gamma \rightarrow \theta}{\neg\neg \mathfrak{A}, \Gamma \rightarrow \theta}
$$

$$
\text{instead of } \mathbf{FES},\ \mathsf{CG}_\omega \text{ has } \mathbf{FES'} \quad \frac{\mathfrak{A}, \Gamma \rightarrow \mathfrak{B}}{\Gamma \rightarrow \mathfrak{A} \supset \mathfrak{B}}
$$

The $\mathsf{CG}_n (1 \leq n < \omega)$ have all rules of CG_ω and also the following rule **NEA''** with the corresponding value of n

$$
\frac{\mathfrak{A}_1^{(n)}, \mathfrak{A}_2^{(n)}, \ldots, \mathfrak{A}_p^{(n)}, \Gamma \rightarrow \theta, \beta(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p)}{\neg\beta(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p), \mathfrak{A}_1^{(n)}, \mathfrak{A}_2^{(n)}, \ldots, \mathfrak{A}_p^{(n)}, \Gamma \rightarrow \theta} \quad \mathbf{NEA''}
$$

where $\beta(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p)$ is any schema built from $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p$ using only $\neg$, $\vee$, & and $\supset$. In the sequel we use the terminology of [4].

The equivalence of C_ω and CG_ω means that

a) If $\vdash_{\mathsf{C}_\omega} \mathfrak{A}$, then $\vdash_{\mathsf{CG}_\omega} \rightarrow \mathfrak{A}$ The proof is trivial.

b1) If $\vdash_{\mathsf{CG}_\omega} \mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \rightarrow \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_p$,
then $\vdash_{\mathsf{C}_\omega} \mathfrak{A}_1 \,\&\, \mathfrak{A}_2 \,\&\, \ldots \,\&\, \mathfrak{A}_n \supset \mathfrak{B}_1 \vee \mathfrak{B}_2 \vee \ldots \mathfrak{B}_p$

b2) If $\vdash_{\mathbf{CG}_\omega} \to \mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_p$, then $\vdash_{\mathbf{C}_\omega} \mathfrak{B}_1 \vee \mathfrak{B}_2 \vee \ldots \vee \mathfrak{B}_p$

Cases b1) and b2) exhaust all possibilities because no sequence with empty succedent is deducible in $\mathbf{CG}_\omega$ (proof by induction over the axioms and rules of $\mathbf{CG}_\omega$). We must prove first that the formulas of $\mathbf{C}_\omega$ which according to b1) and b2) correspond to the axioms of $\mathbf{CG}_\omega$ are deducible in $\mathbf{C}_\omega$, and second that the rules of $\mathbf{C}_\omega$ which correspond to the rules of $\mathbf{CG}_\omega$ are admissible in $\mathbf{C}_\omega$ (*cf.* in [7] p. 40 the concept of "zulaessig"). These proofs use only elementary deductive properties of $\mathbf{C}_\omega$; by the way all deductive properties contained in Theorem 2 from [5] except the *reductio ad absurdum* are valid in $\mathbf{C}_\omega$ (*cf.* Théorème 1 from [1]).

The equivalence of $\mathbf{C}_n$ and $\mathbf{CG}_n$ ($1 \leq n < \omega$) means that

a), b1) and b2) are the same as in the former case.

b3) If $\vdash_{\mathbf{CG}_n} \mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_n \to$, then $\vdash_{\mathbf{C}_n} \mathfrak{A}_1 \,\&\, \mathfrak{A}_2 \,\&\, \ldots \,\&\, \mathfrak{A}_n \supset \mathfrak{B}_1^{(n)} \,\&\, \mathfrak{B}_1 \,\&\, \neg \mathfrak{B}_1$

b4) If $\vdash_{\mathbf{CG}_n} \to$, then $\vdash_{\mathbf{C}_n} \mathfrak{B}_1^{(n)} \,\&\, \mathfrak{B}_1 \,\&\, \neg \mathfrak{B}_1$

where $\mathfrak{B}_1^{(n)} \,\&\, \mathfrak{B}_1 \,\&\, \neg \mathfrak{B}_1$ is a formula (not a schema) from which all formulas of $\mathbf{C}_n (1 \leq n < \omega)$ are deducible (*cf.* [1] Théorème 6).

$$\dfrac{\dfrac{\mathfrak{A} \to \mathfrak{A} \qquad \dfrac{\dfrac{\dfrac{\mathfrak{A} \to \mathfrak{A} \qquad \mathfrak{B} \to \mathfrak{B}}{\mathfrak{A}, \mathfrak{A} \supset \mathfrak{B} \to \mathfrak{B}}}{\mathfrak{B}^{(n)}, \mathfrak{A} \supset \mathfrak{B} \to \neg \mathfrak{A}, \mathfrak{B}}}{\neg \mathfrak{B}, \mathfrak{B}^{(n)}, \mathfrak{A} \supset \mathfrak{B} \to \neg \mathfrak{A}}}{\mathfrak{B}^{(n)}, \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \supset \neg \mathfrak{B} \to \neg \mathfrak{A}}}{\to (12)} \qquad \mathbf{NEA}''$$

This deduction of $\to (12)$ in $\mathbf{CG}_n (1 \leq n < \omega)$ and the corresponding deductions of the axioms (13)–(16) prove a) of the equivalence. The other b1)–b4) can be proved using elementary deductive properties of $\mathbf{C}_n (1 \leq n < \omega)$. In the analysis of **NEA**'' it is important that

$$\beta(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p) \vdash_{\mathbf{C}_n} \neg \beta(\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p) \,\&\, \mathfrak{A}_1^{(n)} \,\&\, \mathfrak{A}_2^{(n)} \,\&\, \ldots \,\&\, \mathfrak{A}_p^{(n)} \supset \mathfrak{C}$$

§2. *The rule* **FES** *is not admissible in* $\mathbf{CG}_\omega$. If **FES** were admissible in $\mathbf{CG}_\omega$, then we could make the following deduction

$$\dfrac{\dfrac{\dfrac{\mathfrak{C} \to \mathfrak{C} \qquad \dfrac{\dfrac{\mathfrak{A} \to \mathfrak{A}}{\mathfrak{A} \to \mathfrak{A}, \mathfrak{B}} \qquad \dfrac{\mathfrak{B} \to \mathfrak{B}}{\mathfrak{B} \to \mathfrak{A}, \mathfrak{B}}}{\mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A}, \mathfrak{B}}}{\mathfrak{C}, \mathfrak{C} \supset \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A}, \mathfrak{B}}}{\dfrac{\mathfrak{C} \supset \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A}, \mathfrak{C} \supset \mathfrak{B}}{\dfrac{\mathfrak{C} \supset \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A} \vee (\mathfrak{C} \supset \mathfrak{B}), \mathfrak{A} \vee (\mathfrak{C} \supset \mathfrak{B})}{\to (\mathfrak{C} \supset \mathfrak{A} \vee \mathfrak{B}) \supset \mathfrak{A} \vee (\mathfrak{C} \supset \mathfrak{B})}}} \qquad \mathbf{FES} \text{ but not } \mathbf{FES}'$$

But as $\mathbf{CG}_\omega$ and $\mathbf{C}_\omega$ are equivalent we could also deduce $(\mathfrak{C} \supset \mathfrak{A} \vee \mathfrak{B}) \supset \mathfrak{A} \vee (\mathfrak{C} \supset \mathfrak{B})$ in $\mathbf{C}_\omega$. The following normal matrix with the distinguished elements 1 and 2 shows, choosing 3 for $\mathfrak{C}$, 3 for $\mathfrak{A}$ and 4 for $\mathfrak{B}$, the opposite.

$\mathfrak{A}$	$\neg\mathfrak{A}$
1	4
2	2
3	1
4	1

		$\mathfrak{B}$			
$\mathfrak{A} \supset \mathfrak{B}$		1	2	3	4
	1	1	2	3	4
$\mathfrak{A}$	2	1	1	3	4
	3	1	2	1	4
	4	1	1	1	1

		$\mathfrak{B}$			
$\mathfrak{A} \vee \mathfrak{B}$		1	2	3	4
	1	1	1	1	1
$\mathfrak{A}$	2	1	2	2	2
	3	1	2	3	3
	4	1	2	3	4

		$\mathfrak{B}$			
$\mathfrak{A} \,\&\, \mathfrak{B}$		1	2	3	4
	1	1	2	3	4
$\mathfrak{A}$	2	2	2	3	4
	3	3	3	3	4
	4	4	4	4	4

§3. Trying to apply Gentzen's proof of his Hauptsatz to the **CG**$_n$($1 \leq n \leq \omega$) we encounter great difficulties in handling the rules **FES'** and **NEA''**. The first contains the typical intuitionistic restriction that the succedentia should have at most one formula. But the other rules of the **CG**$_n$ are free from this restriction and this prevents the application of Gentzen's proof (*cf.* [4] 3.232.1). Also the rule **NEA''** contains only a conditioned and not as **NEA** an unconditioned introduction of negation in the antecedent. Precisely this difference raises difficulties to the application of Gentzen's proof. By the way, the **C**$_n$($1 \leq n \leq \omega$) have no finite characteristic matrices (*cf.* Théorème 6 from [1]) and this also suggests the conjecture that both the **C**$_n$ and **CG**$_n$ are undecidable.

But if we start from classical logic—and this is the standpoint of da Costa—there is no justification of **FES'** so far its substitution by **FES** produces no unpleasant consequences. On the other hand we can try to express the special conditions for the negation introduction contained in **NEA''** by a more general rule like **NEA** and some supplementary restriction on the formula to be negated. According to these suggestions we have built a new hierarchy **WG**$_n$($1 \leq n \leq \omega$) as follows

> **WG**$_\omega$ differs from the propositional part of Gentzen's **LK** only in the following rule:
>
> instead of **NEA**, **WG**$_\omega$ has **NEA'**: $\dfrac{\mathfrak{A},\, \Gamma \rightarrow \theta}{\neg\neg\mathfrak{A},\, \Gamma \rightarrow \theta}$
>
> **WG**$_n$($1 \leq n \leq \omega$) contains all rules of **WG**$_\omega$ and **NEA'''**: $\dfrac{\Gamma \rightarrow \theta,\, \mathfrak{A}^{[n]}}{\neg\mathfrak{A}^{[n]},\, \Gamma \rightarrow \theta}$

with the corresponding value of n and where $\mathfrak{A}^{[1]} = \mathfrak{A} \,\&\, \neg\mathfrak{A}$ and $\mathfrak{A}^{[n+1]} = \mathfrak{A}^{[n]} \,\&\, \neg\mathfrak{A}^{[n]}$.

§4. *Gentzen's Hauptsatz for* **WG**$_n$($1 \leq n \leq \omega$). We only need to change the original proof contained in [4] page 196 ff. in those parts where negation plays an essential role. We adopt the numbering of [4] and make in the deductive steps the same suppositions. Let us analyze first the case $n = \omega$.

3.113.35 The principal sign of $\mathfrak{M}$ (*mix* formula) is $\neg$. But then the end of the deduction is

$$\dfrac{\dfrac{\neg\mathfrak{A},\ \Gamma_1 \to \theta_1}{\Gamma_1 \to \theta_1,\ \neg\neg\mathfrak{A}}\ \textbf{NES} \qquad \dfrac{\mathfrak{A},\ \Gamma_2 \to \theta_2}{\neg\neg\mathfrak{A},\ \Gamma_2 \to \theta_2}\ \textbf{NEA}'}{\Gamma_1,\ \Gamma_2 \to \theta_1,\ \theta_2}\ \textit{Mix}$$

As the rank of the *mix* is 2, $\neg\neg\mathfrak{A}$ is not contained in θ_1 or Γ_2. The necessary transformation is

$$\dfrac{\dfrac{\dfrac{\mathfrak{A},\ \Gamma_2 \to \theta_2}{\Gamma_2 \to \theta_2,\ \neg\mathfrak{A}}\ \textbf{NES} \qquad \neg\mathfrak{A},\ \Gamma_1 \to \theta_1}{\Gamma_2,\ \Gamma_1^* \to \theta_2^*,\ \theta_1}\ \textit{Mix}}{\Gamma_1,\ \Gamma_2 \to \theta_1,\ \theta_2}$$

$\neg\mathfrak{A}$ can occur in Γ_1 or θ_2. But as the grade of $\neg\mathfrak{A}$ is less than the grade of $\neg\neg\mathfrak{A}$ the new *mix* can be eliminated according to the inductive hypothesis.

3.121.22 In this case Gentzen supposes that the right rank is greater than 1 (3.121) and that the *mix* formula $\mathfrak{M}$ occurs in Σ and Γ (3.121) but not in Π (3.121.2). The rule above the right upper sequence of the *mix* is a **NEA'**.

a) $\mathfrak{M}$ is the same as $\neg\neg\mathfrak{A}$. The end of the deduction is

$$\dfrac{\Pi \to \Sigma \qquad \dfrac{\mathfrak{A},\ \Gamma \to \Omega}{\neg\neg\mathfrak{A},\ \Gamma \to \Omega}\ \textbf{NEA}'}{\Pi,\ \Gamma^* \to \Sigma^*,\ \Omega}\ \textit{Mix}\ \text{I}$$

The necessary change is

$$\dfrac{\Pi \to \Sigma \qquad \dfrac{\dfrac{\Pi \to \Sigma \qquad \mathfrak{A},\ \Gamma \to \Omega}{\Pi,\ \mathfrak{A},\ \Gamma^* \to \Sigma^*,\ \Omega}\ \textit{Mix}\ \text{II}}{\neg\neg\mathfrak{A},\ \Pi,\ \Gamma^* \to \Sigma^*,\ \Omega}\ \text{eventually interchanges and } \textbf{NEA}'}{\Pi,\ \Gamma^* \to \Sigma^*,\ \Omega}\ \textit{Mix}\ \text{III}$$

The *mixes* II and III have the same left rank as the *mix* I; the right rank of II is less than that of I. Because $\neg\neg\mathfrak{A}$ can not occur in $\mathfrak{A}$, Π or Γ^* the right rank of III is 1. The new *mixes* II and III can be eliminated according to the inductive hypothesis.

b) $\mathfrak{M}$ differs from $\neg\neg\mathfrak{A}$. The end of the deduction is

$$\dfrac{\Pi \to \Sigma \qquad \dfrac{\mathfrak{A},\ \Gamma \to \Omega}{\neg\neg\mathfrak{A},\ \Gamma \to \Omega}\ \textbf{NEA}'}{\Pi,\ \neg\neg\mathfrak{A},\ \Gamma^* \to \Sigma^*,\ \Omega}\ \textit{Mix}$$

The necessary transformation is

$$\dfrac{\dfrac{\Pi \to \Sigma \qquad \mathfrak{A},\ \Gamma \to \Omega}{\Pi,\ \mathfrak{A}^*,\ \Gamma^* \to \Sigma^*,\ \Omega}\ \textit{Mix}}{\Pi,\ \neg\neg\mathfrak{A},\ \Gamma^* \to \Sigma^*,\ \Omega}\ \text{eventually interchanges and } \textbf{NEA}'$$

The new *mix* has a smaller right rank than the original *mix* and the same left rank. But then we can apply the inductive hypothesis. This completes the proof of Gentzen's Hauptsatz for $\mathbf{WG}_\omega$. $\mathbf{WG}_n(1 \leq n \leq \omega)$ also has the rule **NEA'''**. But this rule is a special case of **NEA**, and we can apply without change Gentzen's original proof.

§5. *Deductive properties of* $\mathbf{WG}_n(1 \leq n \leq \omega)$.

a) $\mathbf{WG}_n(1 \leq n \leq \omega)$ is decidable and has the subformula property (*cf.* [4] page 195).

b) $\mathbf{WG}_\omega$ is not finitely trivializable. A calculus is finitely trivializable (*cf.* [1] page 3791) if there is a formula (not a schema) from which all formulas are deducible. If $\mathbf{WG}_\omega$ were finitely trivializable, then there is a cut-free deduction of $\mathfrak{A}_1 \rightarrow \mathfrak{B}_1$, where $\mathfrak{A}_1$ is a formula with the aforementioned property and $\mathfrak{B}_1$ is a propositional variable not occurring in $\mathfrak{A}_1$. But then the cut-free deduction of $\mathfrak{A}_1 \rightarrow \mathfrak{B}_1$ has a deductive thread with the following properties:

1) The rules **FES, NES, OES, UES** and the cut-rule do not occur in this thread;

2) The succedentia of the sequences belonging to this thread are not empty and contain only formulas $\mathfrak{B}_1$;

3) This deductive thread ends with the sequence $\mathfrak{B}_1 \rightarrow \mathfrak{B}_1$.

But then if we follow this deductive thread in the inverse direction we could prove that $\mathfrak{B}_1$ is a subformula of $\mathfrak{A}_1$. But this contradicts our hypothesis (this proof is also valid in $\mathbf{CG}_\omega$).

c) $\mathbf{WG}_n(1 \leq n < \omega)$ is finitely trivializable

$$\dfrac{\dfrac{\mathfrak{A}^{[n]} \rightarrow \mathfrak{A}^{[n]}}{\mathfrak{A}^{[n]}, \mathfrak{A}^{[n]} \rightarrow}}{\mathfrak{A}^{[n+1]} \rightarrow \mathfrak{B}}$$

d) In $\mathbf{WG}_\omega \rightarrow \neg(\mathfrak{A} \,\&\, \neg\mathfrak{A})$ and $\rightarrow \mathfrak{A} \supset (\neg\mathfrak{A} \supset \mathfrak{B})$ are not deducible.

e) In $\mathbf{WG}_n(1 \leq n < \omega) \rightarrow \neg\mathfrak{A}^{[n+1]}$ and $\rightarrow \mathfrak{A}^{[n]} \supset (\neg\mathfrak{A}^{[n]} \supset \mathfrak{B})$ are deducible, but not the corresponding schemata with a smaller value of n.

If we use the $\mathbf{WG}_n(1 \leq n \leq \omega)$ as basic propositional calculi of an axiom system, then by smaller values of n increases the danger of trivialization.

f) $\mathbf{WG}_\omega$ is a proper extension of $\mathbf{CG}_\omega$. In $\mathbf{WG}_\omega$ $\neg(\mathfrak{B} \,\&\, \neg\mathfrak{B})$, $\mathfrak{A} \supset \mathfrak{B}$, $\mathfrak{A} \supset \neg\mathfrak{B} \rightarrow \neg\mathfrak{A}$ and $\neg(\mathfrak{A} \,\&\, \neg\mathfrak{A}) \rightarrow \neg(\neg\mathfrak{A} \,\&\, \neg\neg\mathfrak{A})$ are not deducible.

g) In $\mathbf{WG}_n(1 \leq n < \omega)$ $\neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n]}, \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \supset \neg\mathfrak{B} \rightarrow \neg\mathfrak{A}$ and $\neg\mathfrak{A}^{[1]}, \neg\mathfrak{A}^{[2]}, \ldots, \neg\mathfrak{A}^{[n]} \rightarrow \neg(\neg\mathfrak{A})^{[m]}$ $(m \geq 1)$ are deducible, but not the corresponding schemata with smaller values of n.

$$\dfrac{\dfrac{\mathfrak{A} \rightarrow \mathfrak{A} \qquad \dfrac{\mathfrak{A} \rightarrow \mathfrak{A} \qquad \mathfrak{B}, \neg\mathfrak{B}, \neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n-1]} \rightarrow \mathfrak{B}^{[n]}}{\mathfrak{A}, \mathfrak{A} \supset \mathfrak{B}, \neg\mathfrak{B}, \neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n-1]} \rightarrow \mathfrak{B}^{[n]}}}{\mathfrak{A}, \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \supset \neg\mathfrak{B}, \neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n-1]} \rightarrow \mathfrak{B}^{[n]}}}{\neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n]}, \mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \supset \neg\mathfrak{B} \rightarrow \neg\mathfrak{A}}$$

FEA and other rules
FEA and other rules
NES, NEA''' and other rules

$$\dfrac{\dfrac{\mathfrak{A}, \neg\mathfrak{A}, \neg\mathfrak{A}^{[1]}, \neg\mathfrak{A}^{[2]}, \ldots, \neg\mathfrak{A}^{[n-1]} \to \mathfrak{A}^{[n]}}{\neg\mathfrak{A}, \neg\neg\mathfrak{A}, \neg(\neg\mathfrak{A})^{[1]}, \neg(\neg\mathfrak{A})^{[2]}, \ldots, \neg(\neg\mathfrak{A})^{[n-1]}, \neg\mathfrak{A}^{[1]}, \neg\mathfrak{A}^{[2]}, \ldots, \neg\mathfrak{A}^{[n-1]} \to \mathfrak{A}^{[n]}}\ \textbf{NEA}' \text{ and other rules}}{\dfrac{(\neg\mathfrak{A})^{[n]}, \neg\mathfrak{A}^{[1]}, \neg\mathfrak{A}^{[2]}, \ldots, \neg\mathfrak{A}^{[n-1]} \to \mathfrak{A}^{[n]}}{\neg\mathfrak{A}^{[1]}, \neg\mathfrak{A}^{[2]}, \ldots, \neg\mathfrak{A}^{[n]} \to \neg(\neg\mathfrak{A})^{[n]}}\ \textbf{NEA}''' \text{ and } \textbf{NES}}$$

The right initial sequence of the former and the initial sequence of this deduction are deducible sequences of the form

$$\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_p \to \mathfrak{A}_1 \,\&\, \mathfrak{A}_2 \,\&\, \ldots \,\&\, \mathfrak{A}_p$$

h) In $\mathbf{WG}_1$ $\neg\mathfrak{A}^{[1]} \,\&\, \neg\mathfrak{B}^{[1]} \to \neg(\mathfrak{A} \,\&\, \mathfrak{B})^{[1]}$ is not deducible. Therefore (13) is independent from (1)–(12) in $\mathbf{C}_1$. Axioms (14) and (15) of $\mathbf{C}_1$ are also independent.

i) If $\vdash_{\mathbf{WG}_s} \to \mathfrak{A}$, then $\vdash_{\mathbf{WG}_t} \to \mathfrak{A}$ $(1 \le t < s \le \omega)$
On the other hand there are sequences which are deducible in $\mathbf{WG}_t$ but not in $\mathbf{WG}_s$. Therefore the $\mathbf{WG}_n$ build a genuine heirarchy. The Théorèmes 2 and 3 from [1] are valid in $\mathbf{WG}_1$.

j) Supposing that $\neg\mathfrak{B}_1, \neg\mathfrak{B}_2, \ldots, \neg\mathfrak{B}_p$, are all the negations which

1) are proper subformulas of $\mathfrak{A}$
and
2) do not have the form $\neg\neg\mathfrak{C}$ or $\neg\mathfrak{C}^{[n]}$,
then for $1 \le n < \omega$

$$\vdash_{\mathbf{LK}} \to \mathfrak{A} \text{ iff } \vdash_{\mathbf{WG}_n} \mathfrak{B}_1^{[1]}, \neg\mathfrak{B}_1^{[2]}, \ldots, \neg\mathfrak{B}_1^{[n]}, \neg\mathfrak{B}_2^{[1]}, \neg\mathfrak{B}_2^{[2]}, \ldots, \neg\mathfrak{B}_2^{[n]}, \ldots, \neg\mathfrak{B}_p^{[1]}, \neg\mathfrak{B}_p^{[2]}, \ldots, \neg\mathfrak{B}_p^{[n]} \to \mathfrak{A}$$

The implication from right to left is trivial; that from left to right can be proved in the following way. We can suppose that there is a cutfree deduction of $\to \mathfrak{A}$ in **LK**. We choose in it a **NEA** rule—if there are no such rules the **LK** deduction is already a $\mathbf{WG}_n$ deduction:

$$\textbf{NEA}\ \frac{\Gamma \to \theta, \mathfrak{B}}{\neg\mathfrak{B}, \Gamma \to \theta}$$

If $\mathfrak{B}$ has the form $\mathfrak{C}^{[n]}$, then this **NEA** is already a **NEA'''**. In the opposite case we make the following change

$$\dfrac{\dfrac{\Gamma \to \theta, \mathfrak{B} \qquad \mathfrak{B}, \neg\mathfrak{B}, \neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n-1]} \to \mathfrak{B}^{[n]}}{\Gamma, \neg\mathfrak{B}, \neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n-1]} \to \theta^*, \mathfrak{B}^{[n]}}\ \textit{Cut}}{\neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n]}, \neg\mathfrak{B}, \Gamma \to \theta}\ \textbf{NEA}''' \text{ and other rules}$$

The right upper sequence of the cut is deducible in $\mathbf{WG}_n$. Let us suppose that $\mathfrak{B}$ has not the form $\neg\mathfrak{C}$. In all sequences of the deductive thread starting in the lower sequence of **NEA'''** and ending in the end sequence we add—if necessary—in their antecedentia the formulas $\neg\mathfrak{B}^{[1]}, \neg\mathfrak{B}^{[2]}, \ldots, \neg\mathfrak{B}^{[n]}$. In this way we obtain a new deduction with a **NEA** less and an end sequence corresponding to the conditions of our theorem. On the other hand if $\mathfrak{B}$ has the form $\neg\mathfrak{C}$, then according to the second part of g) we can add instead of the formulas $\neg\mathfrak{B}^{[d]}$ the corresponding $\neg\mathfrak{C}^{[d]}$. By repeating this procedure we obtain a deduction in $\mathbf{WG}_n$ of the sequence we need (*cf.* Théorème 4 from [1]).

There is no principal difficulty in extending the $\mathbf{WG}_n$ into quantificational calculi. Only we must take care of the restrictions in the use of free variables; for example, in j) the added formulas $\neg\mathfrak{B}^{[d]}$ must be the corresponding universal quantifications. By the way and considering j) and Church's theorem these quantificational extensions are not decidable (*cf.* [2]).

§6. Adding the rule

$$\mathbf{NES'}\ \frac{\Gamma \rightarrow \theta, \mathfrak{A}}{\Gamma \rightarrow \theta, \neg\neg\mathfrak{A}}$$

we can extend the $\mathbf{WG}_n$ $(1 \leq n \leq \omega)$. In the so defined $\mathbf{WG}'_n$ $(1 \leq n \leq \omega)$ hierarchy all theorems of the preceding section except the last part of i) are valid. The Théorèmes 2 and 3 from [1] are also valid except the undeducibility of $\rightarrow \mathfrak{A} \supset \neg\neg\mathfrak{A}$. Furthermore all odd and all even iterations of negation are equivalent in $\mathbf{WG}'_n$; but this is not very important because there is no law of replacement for equivalent schemata either in $\mathbf{WG}_n$ or in $\mathbf{WG}'_n$.

Choosing the intuitionistic and not the classical logic as starting point we can build a hierarchy $\mathbf{WGI}_n$ $(1 \leq n \leq \omega)$ in the following way:

$\mathbf{WGI}_n$ differs from $\mathbf{WG}_n$ by having **NES'** instead of **NEA'**. Moreover the succedentia can have at most one formula.

The theorems a), b), c), d), e), j) and i)—except the undeducibility of $\rightarrow \mathfrak{A} \supset \neg\neg\mathfrak{A}$—are also valid. On the other hand the second sequence of g) is not deducible.

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Universidad Nacional de Córdoba
Córdoba, Argentina