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THE SHORTEST AXIOMS OF THE IMPLICATIONAL CALCULUS

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Jan Łukasiewicz has proved that all theses of the Implicational Calculus of Propositions are derivable from the axiom

(I) *CCCpqrCCrpCsp*

and has outlined a proof to the effect that (I) is the shortest single axiom from which all theses are derivable.[1] Whether or not there are 13 letter axioms other than (I) which could serve as single axioms of the Calculus, Łukasiewicz noted,[2] is not known.

In this article it is proved that (I) is the only 13 letter single axiom with the possible exception of

(II) *CCrpCCCpqrCsp.*

This article leaves unanswered the question of whether or not (II) does serve as a single axiom. If it does, then there are a total of two 13 letter single axioms; if it does not, then (I) is the only 13 letter single axiom.

Since there are 132 forms of punctuation for these 13 letter wff's, and since there are 877 unique combinations and permutations of seven variables, there are 115,764 unique 13 letter wff's. Each of these which is analytic is, initially, a candidate for single axiom status. The first practical problem is to determine the analytic wff's *without* writing out 115,764 128 row truth tables.

The easiest way to determine the analytic wff's is to use three charts:

Chart I is a list of the 132 forms of punctuation.
Chart II is a list of the 877 unique combinations and permutations of seven variables.
Chart III is a list of which of these 877 unique combinations and permutations can be assigned to each of 64 symmetrical couplets. (These couplets result from reducing 128 standard assignments of two values, involving seven variables, to 64; the second member of each couplet is superfluous, since anything ending in a '1' is analytic.) (What follows is only part of Chart III.)

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CHART I

1. *CxCxCxCxCxCxx*
2. *CxCxCxCxCCxxx*
3. *CxCxCxCCxxCxx*
4. *CxCxCxCCxCxxx*
5. *CxCxCxCCCxxxx*
6. *CxCxCCxxCxCxx*
7. *CxCxCCxxCCxxx*
8. *CxCxCCxCxxCxx*
9. *CxCxCCCxxxCxx*
10. *CxCxCCxCxCxxx*
11. *CxCxCCxCCxxxx*
12. *CxCxCCCxxCxxx*
13. *CxCxCCCxCxxxx*
14. *CxCxCCCCxxxxx*
15. *CxCCxxCxCxCxx*
16. *CxCCxxCxCCxxx*
17. *CxCCxxCCxxCxx*
18. *CxCCxxCCxCxxx*
19. *CxCCxxCCCxxxx*
20. *CxCCxCxxCxCxx*
21. *CxCCxCxxCCxxx*
22. *CxCCCxxxCxCxx*
23. *CxCCCxxxCCxxx*
24. *CxCCxCxCxxCxx*
25. *CxCCxCCxxxCxx*
26. *CxCCCxxCxxCxx*
27. *CxCCCxCxxxCxx*
28. *CxCCCCxxxxCxx*
29. *CxCCxCxCxCxxx*
30. *CxCCxCxCCxxxx*
31. *CxCCxCCxxCxxx*
32. *CxCCxCCxCxxxx*
33. *CxCCxCCCxxxxx*
34. *CxCCCxxCxCxxx*
35. *CxCCCxxCCxxxx*
36. *CxCCCxCxxCxxx*
37. *CxCCCCxxxCxxx*
38. *CxCCCxCxCxxxx*
39. *CxCCCxCCxxxxx*
40. *CxCCCCxxCxxxx*
41. *CxCCCCxCxxxxx*
42. *CxCCCCCxxxxxx*
43. *CCxxCxCxCxCxx*
44. *CCxxCxCxCCxxx*
45. *CCxxCxCCxxCxx*
46. *CCxxCxCCxCxxx*
47. *CCxxCxCCCxxxx*
48. *CCxxCCxxCxCxx*
49. *CCxxCCxxCCxxx*
50. *CCxxCCxCxxCxx*
51. *CCxxCCCxxxCxx*
52. *CCxxCCxCxCxxx*
53. *CCxxCCxCCxxxx*
54. *CCxxCCCxxCxxx*
55. *CCxxCCCxCxxxx*
56. *CCxxCCCCxxxxx*
57. *CCxCxxCxCxCxx*
58. *CCxCxxCxCCxxx*
59. *CCxCxxCCxxCxx*
60. *CCxCxxCCxCxxx*
61. *CCxCxxCCCxxxx*
62. *CCCxxxCxCxCxx*
63. *CCCxxxCxCCxxx*
64. *CCCxxxCCxxCxx*
65. *CCCxxxCCxCxxx*
66. *CCCxxxCCCxxxx*
67. *CCxCxCxxCxCxx*
68. *CCxCxCxxCCxxx*
69. *CCxCCxxxCxCxx*
70. *CCxCCxxxCCxxx*
71. *CCCxxCxxCxCxx*
72. *CCCxxCxxCCxxx*
73. *CCCxCxxxCxCxx*
74. *CCCxCxxxCCxxx*
75. *CCCCxxxxCxCxx*
76. *CCCCxxxxCCxxx*
77. *CCxCxCxCxxCxx*
78. *CCxCxCCxxxCxx*
79. *CCxCCxxCxxCxx*
80. *CCxCCxCxxxCxx*
81. *CCxCCCxxxxCxx*
82. *CCCxxCxCxxCxx*
83. *CCCxxCCxxxCxx*
84. *CCCxCxxCxxCxx*
85. *CCCCxxxCxxCxx*
86. *CCCxCxCxxxCxx*
87. *CCCxCCxxxxCxx*
88. *CCCCxxCxxxCxx*
89. *CCCCxCxxxxCxx*
90. *CCCCCxxxxxCxx*
91. *CCxCxCxCxCxxx*
92. *CCxCxCxCCxxxx*
93. *CCxCxCCxxCxxx*
94. *CCxCxCCxCxxxx*
95. *CCxCxCCCxxxxx*
96. *CCxCCxxCxCxxx*
97. *CCxCCxxCCxxxx*
98. *CCxCCxCxxCxxx*
99. *CCxCCCxxxCxxx*
100. *CCxCCxCxCxxxx*
101. *CCxCCxCCxxxxx*
102. *CCxCCCxxCxxxx*
103. *CCxCCCxCxxxxx*
104. *CCxCCCCxxxxxx*
105. *CCCxxCxCxCxxx*
106. *CCCxxCxCCxxxx*
107. *CCCxxCCxxCxxx*
108. *CCCxxCCxCxxxx*
109. *CCCxxCCCxxxxx*
110. *CCCxCxxCxCxxx*
111. *CCCxCxxCCxxxx*
112. *CCCCxxxCxCxxx*
113. *CCCCxxxCCxxxx*
114. *CCCxCxCxxCxxx*
115. *CCCxCCxxxCxxx*
116. *CCCCxxCxxCxxx*
117. *CCCCxCxxxCxxx*
118. *CCCCCxxxxCxxx*
119. *CCCxCxCxCxxxx*
120. *CCCxCxCCxxxxx*
121. *CCCxCCxxCxxxx*
122. *CCCxCCxCxxxxx*
123. *CCCxCCCxxxxxx*
124. *CCCCxxCxCxxxx*
125. *CCCCxxCCxxxxx*
126. *CCCCxCxxCxxxx*
127. *CCCCCxxxCxxxx*
128. *CCCCxCxCxxxxx*
129. *CCCCxCCxxxxxx*
130. *CCCCCxxCxxxxx*
131. *CCCCCxCxxxxxx*
132. *CCCCCCxxxxxxx*

CHART II

1. *ppppppp*
2. *ppppppr*
3. *pppppro*
4. *pppprpp*
5. *ppprppp*
6. *pprpppp*
7. *prppppp*
8. *rpppppp*
9. *ppppprr*
10. *pppprpr*
11. *ppppro*
12. *ppprppr*
13. *ppprpro*
14. *pppro*
15. *pprpppr*

16. *pprpprp*
17. *pprprpp*
18. *pprrppp*
19. *prppppr*
20. *prppprp*
21. *prpprpp*
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25. *rpppprp*
26. *rppprpp*
27. *rpprppp*
28. *rprpppp*
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37. *pprrppr*
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47. *prrpprp*
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50. *rpppprr*
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53. *rpprppr*
54. *rpprprp*
55. *rpprrpp*
56. *rprpppr*
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59. *rprrppp*
60. *rrppppr*
61. *rrppprp*
62. *rrpprpp*
63. *rrprppp*
64. *rrrpppp*
65. *ppppprq*
66. *pppprpq*
67. *pppprqp*
68. *ppprppq*
69. *ppprpqp*
70. *ppprqpp*
71. *pprpppq*
72. *pprppqp*
73. *pprpqpp*
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607. *qsrpppr*
608. *qsrpprp*
609. *qsrprpp*
610. *qsrrppp*
611. *pprrqqs*
612. *pprrqsq*
613. *pprrsqq*
614. *pprqrqs*
615. *pprqrsq*
616. *pprqqrs*
617. *pprqqsr*
618. *pprqsrq*
619. *pprqsqr*
620. *pprsrqq*
621. *pprsqrq*
622. *pprsqqr*
623. *ppsrrqq*
624. *ppsrqrq*
625. *ppsrqqr*
626. *prprqqs*
627. *prprqsq*
628. *prprsqq*
629. *prpqrqs*
630. *prpqrsq*
631. *prpqqrs*
632. *prpqqsr*
633. *prpqsrq*
634. *prpqsqr*
635. *prpsrqq*
636. *prpsqrq*
637. *prpsqqr*
638. *prrpqqs*
639. *prrpqsq*
640. *prrpsqq*
641. *prrqpqs*
642. *prrqpsq*
643. *prrqqps*
644. *prrqqsp*
645. *prrqspq*
646. *prrqsqp*
647. *prrspqq*
648. *prrsqpq*
649. *prrsqqp*
650. *prqprqs*
651. *prqprsq*
652. *prqpqrs*
653. *prqpqsr*
654. *prqpsrq*
655. *prqpsqr*
656. *prqrpqs*
657. *prqrpsq*
658. *prqrqps*
659. *prqrqsp*
660. *prqrspq*
661. *prqrsqp*
662. *prqqprs*
663. *prqqpsr*
664. *prqqrps*
665. *prqqrsp*
666. *prqqspr*
667. *prqqsrp*
668. *prqsprq*
669. *prqspqr*
670. *prqsrpq*
671. *prqsrqp*
672. *prqsqpr*
673. *prqsqrp*
674. *prsprqq*
675. *prspqrq*
676. *prspqqr*
677. *prsrpqq*
678. *prsrqpq*
679. *prsrqqp*
680. *prsqprq*
681. *prsqpqr*
682. *prsqrpq*
683. *prsqrqp*
684. *prsqqpr*
685. *prsqqrp*
686. *psprrqq*
687. *psprqrq*
688. *psprqqr*
689. *psrprqq*
690. *psrpqrq*
291. *psrpqqr*
692. *psrrpqq*
693. *psrrqpq*
694. *psrrqqp*
695. *psrqprq*
696. *psrqpqr*
697. *psrqrpq*
698. *psrqrqp*
699. *psrqqpr*
700. *psrqqrp*
701. *sprrqq*
702. *spprqrq*
703. *spprqqr*
704. *sprprqq*
705. *sprpqrq*
706. *sprpqqr*
707. *sprrpqq*
708. *sprrqpq*
709. *sprrqqp*
710. *sprqprq*
711. *sprqpqr*
712. *sprqrpq*
713. *sprqrqp*
714. *sprqqpr*
715. *sprqqrp*
716. *pppprqst*
717. *pprpqst*
718. *pprqpst*
719. *pprqspt*
720. *pprqstp*
721. *prppqst*
722. *prpqpst*
723. *prpqspt*
724. *prpqstp*
725. *prqppst*
726. *prqpspt*
727. *prqpstp*
728. *prqsppt*
729. *prqsptp*
730. *prqstpp*
731. *rpppqst*
732. *rpppqst*
733. *rpppqspt*
734. *rpppqstp*
735. *rpqppst*
736. *rpqpspt*
737. *rpqpstp*
738. *rpqsppt*
739. *rpqsptp*
740. *rpqstpp*
741. *rqpppst*
742. *rqppspt*
743. *rqppstp*
744. *rqpsppt*
745. *rqpsptp*
746. *rqpstpp*
747. *rqsppot*
748. *rqspptp*
749. *rqsptpp*
750. *rqstppp*
751. *pprrqst*
752. *pprqrst*
753. *pprqsrt*
754. *pprqstr*
755. *ppqrrst*
756. *ppqrsrt*
757. *ppqrstr*
758. *ppqsrrt*
759. *ppqsrtr*
760. *ppqstrr*
761. *prprqst*
762. *prpqrst*
763. *prpqsrt*
764. *prpqstr*
765. *prrpqst*
766. *prrqpst*
767. *prrqspt*
768. *prrqstp*
769. *prqprst*
770. *prqpsrt*
771. *prqpstr*
772. *prqrpst*
773. *prqrspt*
774. *prqrstp*
775. *prqsprt*
776. *prqsptr*
777. *prqsrpt*
778. *prqsrtp*
779. *prqstpr*
780. *prqstrp*
781. *pqprrst*
782. *pqprsrt*
783. *pqprstr*
784. *pqpsrrt*
785. *pqpsrtr*
786. *pqpstrr*
787. *pqrprst*
788. *pqrpsrt*
789. *pqrpstr*
790. *pqrrpst*
791. *pqrrspt*
792. *pqrrstp*
793. *pqrsprt*
794. *pqrsptr*
795. *pqrsrpt*

796. *pqrsrtp*	812. *qpprsrt*	828. *qprstrp*	844. *qsprprt*	861. *prqstup*
797. *pqrstpr*	813. *qpprstr*	829. *qpsprrt*	845. *qsprptr*	862. *rppqstu*
798. *pqrstrp*	814. *qppsrrt*	830. *qpsprtr*	846. *qsprrpt*	863. *rpqpstu*
799. *pqsprrt*	815. *qppsrtr*	831. *qpsptrr*	847. *qsprrtp*	864. *rpqsptu*
800. *pqsprtr*	816. *qppstrr*	832. *qpsrprt*	848. *qsprtpr*	865. *rpqstpu*
801. *pqsptrr*	817. *qprprst*	833. *qpsrptr*	849. *qsprtrp*	866. *rpqstup*
802. *pqsrprt*	818. *qprpsrt*	834. *qpsrrpt*	850. *qsptprr*	867. *rqppstu*
803. *pqsrptr*	819. *qprpstr*	835. *qpsrrtp*	851. *qsptrpr*	868. *rqpsptu*
804. *pqsrrpt*	820. *qprrpst*	836. *qpsrtpr*	852. *qsptrrp*	869. *rqpstpu*
805. *pqsrrtp*	821. *qprrspt*	837. *qpsrtrp*	853. *qstpprr*	870. *rqpstup*
806. *pqsrtpr*	822. *qprrstp*	838. *qpstprr*	854. *qstprpr*	871. *rqspptu*
807. *pqsrtrp*	823. *qprsprt*	839. *qpstrpr*	855. *qstprrp*	872. *rqsptpu*
808. *pqstprr*	824. *qprsptr*	840. *qpstrrp*	856. *pprqstu*	873. *rqsptup*
809. *pqstrpr*	825. *qprsrpt*	841. *qspprrt*	857. *prpqstu*	874. *rqstppu*
810. *pqstrrp*	826. *qprsrtp*	842. *qspprtr*	858. *prqpstu*	875. *rqstpup*
811. *qpprrst*	827. *qprstpr*	843. *qspptrr*	859. *prqsptu*	876. *rqstupp*
			860. *prqstpu*	877. *prqstuv*

CHART III

Couplet number **2**

1. 1111112
2222221

2, 65, 66, 68, 71, 75, 80, 86, 89, 91, 98, 100, 104, 116, 118, 122, 128, 146, 148, 152, 158, 166, 191, 195, 198, 200, 211, 214, 216, 223, 225, 229, 366, 367, 368, 370, 371, 373, 376, 377, 379, 382, 386, 387, 389, 392, 396, 401, 402, 404, 407, 408, 410, 411, 413, 415, 419, 422, 424, 431, 432, 434, 435, 437, 439, 443, 444, 446, 449, 451, 455, 461, 464, 466, 473, 475, 479, 491, 492, 494, 495, 497, 499, 503, 504, 506, 509, 511, 515, 521, 522, 524, 527, 531, 533, 537, 543, 551, 554, 556, 563, 565, 569, 581, 583, 587, 593, 611, 614, 616, 626, 629, 631, 638, 641, 643, 650, 652, 656, 658, 662, 664, 716, 717, 718, 719, 721, 722, 723, 725, 726, 728, 731, 732, 733, 735, 736, 738, 741, 742, 744, 747, 751, 752, 753, 755, 756, 758, 761, 762, 763, 765, 766, 767, 769, 770, 772, 773, 775, 777, 781, 782, 784, 787, 788, 790, 791, 793, 795, 799, 802, 804, 811, 812, 814, 817, 818, 820, 821, 823, 825, 829, 832, 834, 841, 844, 846, 856, 857, 858, 859, 860, 862, 863, 864, 865, 867, 868, 869, 871, 872, 874, 877.

35. 2111212
1222121

51, 148, 150, 177, 230, 233, 239, 249, 292, 301, 304, 387, 447, 456, 459, 480, 483, 489, 491, 493, 499, 501, 511, 513, 533, 535, 552, 561, 576, 602, 641, 646, 656, 661, 662, 667, 702, 705, 708, 729, 731, 733, 736, 742, 766, 768, 772, 774, 775, 780, 790, 792, 793, 798, 802, 807, 812, 815, 818, 821, 830, 839, 842, 851, 854, 859, 861, 862, 863, 865, 867, 869, 872, 875, 877.

64. 2222222
1111111

1—877.

The process of isolating the analytic wff's is this: *making every variable unique*, a single 128 row two-valued tt is constructed for each punctuation (using '1' as analytic, '2' as non-analytic, the standard initial assignment of values to variables, and the standard definition of the implication operator). Wherever a '2' occurs in a final column, noting the couplet number of the corresponding initial assignment of 1's and 2's gives by means of CHART III, the non-analytic wff's.

To illustrate this process, consider the very first punctuation. Where every variable is made unique, a tt is constructed for *CpCzCqCsCtCuv*. If p is assigned 64 1's followed by 64 2's, r assigned 32 1's followed by 32 2's, etc., the result is that this wff (call it 1.877) is non-analytic under and only under the assignment 1111112. It follows, obviously, that 1.2, 1.65, 1.66, . . . 1.587, . . . etc., (see CHART III, Couplet Number 1.) are all non-analytic.

The total number of 13 letter analytic wff's turns out to be 28,495. By identifying variables, the number of single axiom candidates reduces to 741. Of course, in any case where one of these 741 cannot be eliminated later, wff's, originally cancelled through it by identifying variables, must be re-instated as single axiom candidates. Fortunately, as it turns out, there are only two of these (see below, numbers 51.224 and 64.230).

In order to eliminate as many of the 741 single axiom candidates as possible, matrices must be found under which the candidate is analytic and the known single axiom non-analytic. Since Łukasiewicz's axiom is complete, a matrix showing its independence from a wff eliminates that wff as a candidate for single axiom status. The necessary matrices are as follows: (an asterisk denotes selected, analytic, values; and to the right of each matrix are given the values under which Łukasiewicz's single axiom is non-analytic)

I

C	1	2	3
*1	1	2	3
2	1	1	3
3	1	1	1

$p = 2$
$r = 3$
$q = 3$
$s = 1$

II

C	1	2	3	4
*1	1	2	3	3
2	1	1	3	3
3	1	2	1	1
4	1	1	1	1

$p = 2$
$r = 4$
$q = 1$
$s = 3$

III

C	1	2	3	4
*1	1	2	3	3
2	1	1	3	3
3	1	2	1	1
4	1	2	2	1

$p = 3$
$r = 1$
$q = 2$
$s = 4$

IV

C	1	2	3	4	5
*1	1	2	4	4	4
*2	1	1	4	4	4
3	1	2	1	2	2
4	1	1	1	1	1
5	1	2	3	3	3

$p = 5$
$r = 4$
$q = 4$
$s = 5$

V

C	1	2	3	4	5
*1	1	1	1	4	4
*2	1	1	4	4	4
*3	1	1	1	4	4
4	1	1	1	1	1
5	1	1	1	1	1

$p = 3$
$r = 1$
$q = 1$
$s = 2$

VI

C	1	2	3
*1	1	2	3
2	1	1	2
3	1	1	1

$p = 2$
$r = 2$
$q = 3$
$s = 1$

When matrices I, II, and III (those used by Łukasiewicz in his original paper) are applied to our 741 candidates, the three matrices eliminate all except these:

(1) 47.453 *CCprCqCCCpspr*
(2) 47.514 *CCrpCqCCCpsrp*
(3) 51.438 *CCprCCCpqpCsr*
(4) 51.500 *CCrpCCCpqrCsp*
(5) 55.453 *CCprCCCqCpspr*
(6) 55.514 *CCrpCCCqCpsrp*
(7) 63.484 *CCCpqrCsCCrpp*
(8) 64.462 *CCCpqpCCprCsr*
(9) 64.480 *CCCpqrCCrpCsp*
(10) 74.575 *CCCqCpspCCprr*
(11) 74.580 *CCCqCpsrCCrpp*

Matrix IV eliminates 47.453, 51.438, 55.453, 64.462, and 74.575. Matrix V eliminates 47.514, 55.514, 63.483, and 74.580. Of the 741, the only two wff's remaining are 51.500 and 64.480, the latter being the known single axiom. When variables were identified, 51.224, *CCprCCCrppCqr*, was set aside by 51.500 and 64.230, *CCCprrCCrpCqp*, was set aside by 64.480. But these two candidates are re-instated only temporarily and readily, then, eliminated by Matrix VI.

Thus the only possible single axiom other than

(I) 64.480 *CCCpqrCCrpCsp* is (II) 51.500 *CCrpCCCpqrCsp*.

NOTES

1. Łukasiewicz, Jan, "The Shortest Axiom of the Implicational Calculus of Propositions," *Proceedings of the Royal Irish Academy*, vol. 52, sect. A ((I) was originally discovered by Łukasiewicz in 1936.)

2. Ibid., p. 33.

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