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# DEFINITIONAL BOOLEAN CALCULI

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It is often said that the truths of the propositional calculus follow (informally, to be sure) just from the meanings of the connectives and negation. Yet, formal developments of propositional calculi do not perspicuously reveal such a relation; an axiomatization may be said to "contain" the properties of the operators therein embedded but they do not do so in any clear way. What we should like is a formal development of propositional calculi with intuitive rules of inference and with axioms which are, or are like, truth tables.

In this paper I shall develop a class of formal systems, to be called "definitional boolean systems", which can, so to speak, be understood to reveal that the theorems of a propositional calculus do follow from (what amount to) definitions of the appropriate operators. Using "0" and "1" for the truth values "false" and "true" I shall, for example, represent the sense of negation by:

$$N0 = 1$$

and

$$N1 = 0.$$

We may associate these formulas with the proposition: if the truth value of a proposition, $X$, is false (true) then the truth value of the negation of $X$ is true (false).

Following through for certain other operators, we represent the meanings of:

| | | | | |
|---|---|---|---|---|
| $N0.\ N0 = 1$ | $A00.\ A00 = 0$ | $K00.\ K00 = 0$ | $C00.\ C00 = 1$ | $D00.\ D00 = 1$ |
| $N1.\ N1 = 0$ | $A01.\ A01 = 1$ | $K01.\ K01 = 0$ | $C01.\ C01 = 1$ | $D01.\ D01 = 1$ |
| | $A10.\ A10 = 1$ | $K10.\ K10 = 0$ | $C10.\ C10 = 0$ | $D10.\ D10 = 1$ |
| | $A11.\ A11 = 1$ | $K11.\ K11 = 1$ | $C11.\ C11 = 1$ | $D11.\ D11 = 0$ |

For convenience, I have named these formulas for the expressions to the left of the identity sign. I shall call these sets the **N** set, **A** set, **K** set, **C** set, and **D** set.

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The reader is no doubt wondering how, if the axioms are not to contain variables, formulas with variables may be deduced from them. We shall need just two rules of inference, an identity substitution rule allowing the substitution of an expression for one proved or assumed identical to it, and a generalization rule which allows the introduction of expressions containing variables in the following manner: from a formula assumed or proved true when "0" is substituted for an expression, $e$, in that formula, and a formula which is different only in that "1" occurs for $e$, we may assert that formula (with the "$e$"). In other words, this rule may be understood to claim that if a formula is true when one of its expression is false and also true when that same expression is true then the formula is true.

We proceed to develop a set of definitional boolean calculi.

*Terms, expressions and formulas:* A sign, $s$, is a term if and only if $s$ is one of the following: 0, 1, $x_1$, $x_2$, . . . , $x_n$ ($n$ is any natural numeral). "0" and "1" shall be called constants; the $x$'s are variable.

A sign, $s$, is a function letter if and only if $s$ is one of the following: $N, A, K, C$, or $D$.

An expression is any finite linear string of elements which are either terms or function letters.

A sign, $s$, is an identity sign if and only if $s$ is "=".

A formula is any finite linear string of elements which are either terms, function letters, or the identity sign.

*Well formed expressions* (wfe's):

1. Terms are wfe's.
2. If $u$ and $v$ are wfe's then $Nu$, $Auv$, $Kuv$, $Cuv$, and $Duv$ are wfe's.
3. No other expression is a wfe.

*Well formed formulas* (wff's): A formula is well formed if and only if it has the form "$u = v$", where $u$ and $v$ are wfe's. Note that the formulas of the **N** set, **A** set, **K** set, **C** set and **D** set are well formed.

*Instances:* An instance of a formula (or of an expression) is that formula (or expression) with constants occurring everywhere for variables, the same constant being substituted for each occurrance of the same variable. If there is no variable in a formula (or expression) we shall say the formula is its own instance. It is clear that an instance of a wfe or a wff is well formed.

*Atomic wfe's and atomic wff's:* An atomic wfe is a wfe containing just one function letter, e.g., $D0x_2$. An atomic wff is a wff with the form "atomic wfe = constant", e.g., $Kx_3x_7 = 1$. A defining instance is an instance of an atomic wff, e.g., $C11 = 1$. The value of a defining instance is its right most constant, e.g., the value of "$N0 = 1$" is 1.

*Minor expressions:* A minor expression of a wfe, $E$, is an atomic wfe contained in $E$, e.g., the minor expressions of "$CCN01Kx_1x_2$" are $N0$, 1, and $Kx_1x_2$. Note that if a minor expression contains no variables it is the left most wfe of "$N0 = 1$".

Note that elements of the **N, A, K, C** and **D** sets are defining instances. Note also that atomic wff's "inconsistent" with these, e.g., $N0 = 0$, are defining instances.

*A consistent set of defining instances:* A set of defining instances is consistent if and only if it is false that the set contains two defining instances with the same left most wfe's but different right most constants. $\{N0 = 0, N0 = 1\}$ is an inconsistent set.

*The value of an instance of a wfe:* "The value, with respect to defining instances $f_1, f_2, \ldots, f_k$, of a wfe, $F$" means the result of the following calculation (if it has a unique result): the first step of the calculation maps the wfe, $E$, onto a wfe produced by substituting the values of the minors of $E$ for these minors. The $n$'th step of the calculation maps the result of the $n$ - 1'th step, $E'$, onto a wfe produced by substituting for the minors of $E'$ the values of these minors. In calculating the value of an instance of a wfe it is necessary to use the defining instances of that wfe, i.e., those formulas which give the values of atomic wfe's containing constants. If the set of defining instances of a wfe does not contain the set of all minors of expressions which are results of steps of a calculation of the value of a wfe then "the value of that wfe" is undefined. Futhermore, since the result of such a calculation must be unique, if a set of defining instances contains two formulas, one giving 1 and the other 0 as the value of a minor, then the value is also undefined. If the defining instances belong to the union of the **N, A, K, C** and **D** sets, then we shall call values of instances of wfe's normal values.

*Identities:* We shall call an instance of a wff true with respect to a set, **S**, of defining instances if and only if the value of the wfe to the right of the identity sign with respect to **S** is the value with respect to **S** of the wfe to the left of the identity sign. A wff, $F$, is an identity with respect to **S** if and only if all its instances are true. We shall not call $F$ an identity if the value of a wfe in $F$ is not defined. If the set **S** is a subset of the union of the **N, A, K, C** and **D** sets then if $F$ is an identity with respect to **S**, $F$ is a normal identity.

*Rules of Inference:* We shall want a rule of inference which will allow substitution of equals for equals.

**R1.** $u_2 = u_1, f \vdash g$ *where* $u_2$ *and* $u_1$ *are wfe's and* $f$ *and* $g$ *are wff's such that*

(1) *if* $f$ *contains neither* $u_2$ *nor* $u_1$ *then* $g$ *is* $f$,

(2) *if* $f$ *contains* $u_2$ $(u_1)$ $g$ *is a formula produced by substituting* $u_2$ $(u_1)$ *in* $f$ *in one of its occurrences.*

Some people may prefer to regard **R1** as a schema for an infinite class of inference rules since, (1) consequences drawn by means of it do not depend upon the order of the wfe's in "$u_1 = u_2$" and (2) though it allows a substitution in only one occurrence there are potentially infinitely many places (the first occurrence from the left, the second, etc.) where the substitution may be made. It is clear that successive applications of **R1** will allow a substitution in every place or any desired number of places where that substitution is justified. **R1** is sufficient to generate the following derived rules:

**DR1.1** $u = v \vdash u = u$ *also*, $u = v \vdash v = v$
**DR1.2** $u = v \vdash v = u$
**DR1.3** $u = v,\ v = w \vdash u = w$

One might say, the properties of equivalence are packed into **R1**.

Suppose $u$ is contained in a wff, $f$, and that $v$ is a wfe. Let "$f_{u=v}$" denote the wff obtained by substituting $v$ for $u$ in $f$.

**R2.** $f_{u=0}, f_{u=1} \vdash f$.

This is the generalization rule.

Metatheorem 1. *If $f$ and $g$ are identities with respect to the set,* **S**, *of defining instances, and if $h$ is either an* **R1** *or an* **R2** *consequence of $f$ and $g$, containing no function letters not in* **S**, *then $h$ is an identity with respect to* **S**. ("Identityhood" is hereditary.)

*Proof:* a. We prove metatheorem 1 for **R1**. We suppose that $f$ and $g$ are identities with respect to **S**. Suppose $f$ is "$u_1 = u_2$". If $g$ contains neither $u_1$ nor $u_2$ then $h$ is an identity with respect to **S** since $g$ is. Suppose $g$ contains $u_1$ ($u_2$). Then the value of an instance of $u_1$ ($u_2$) equals the value of an instance of $u_2$ ($u_1$). Then an instance of $h$ is true if and only if an instance of $g$ is true. Since $g$ is an identity with respect to **S** so is $h$.

b. We prove metatheorem 1 for **R2**. We suppose that $f$ and $g$ are identities with respect to **S**. If $f$ is "$h_{u=0}$" and $g$ is "$h_{u=1}$" where $u$ is a variable the union of the instances of $f$ and $g$ are exactly the set of instances of $h$. Since $f$ and $g$ are both identities with respect to **S** so is $h$. Suppose $f$ is "$h_{u=0}$" and $g$ is "$h_{u=1}$" where $u$ is not a variable. If $u$ is a constant then $h$ is either $f$ or $g$ and is an identity with respect to **S** if $f$ and $g$ are. If $u$ is not a constant then either its instances all have the same value or else some have the value "0" and some the value "1". In either case if $f$ and $g$ are identities with respect to **S** then so are their instances, and the instances of $h$ may be derived from the instances of $f$ and $g$ by use of **R1**. Then if $f$ and $g$ are identities with respect to **S** so are the instances of $h$. Then $h$ is an identity with respect to **S**.

Corollary 1.1. *"Normal identityhood" is hereditary for* **R1** *and* **R2**.

Corollary 1.2. *If $X$ is a system with the above defined wffs and rules of inference, and with axioms all of which are identities with respect to a consistent set,* **S**, *then some wff is not a theorem of $X$ ($X$ is consistent).*

*Proof:* Suppose $u = c$ is a defining instance for the axioms of $X$. The formula: $u = c'$, where $c'$ is 0(1) if $c$ is 1(0) is not an identity since **S** is consistent. Then $u = c'$ is not a theorem.

Corollary 1.3. *If $\{a_k\}$ is a consistent set of defining instances and $a_n$ is not in $\{a_k\}$ but is consistent with its members then $a_n$ is not a consequence of the members of $\{a_k\}$. (A consistent set of defining instance axioms are independent.)*

*Proof:* If $a_n$ is $u = c'$ where $a_n$ is $u = c$ then $a_n$ is not a consequence of $a'_n \cup \{a_k\}$ (Cor. 1.2.) Then $a_n$ is not a consequence of $\{a_k\}$.

*Definitional Boolean Calculi.* Suppose $F_1, F_2, \ldots, F_k$ are function letters. By "the definitional boolean calculus, **S** $F_1, F_2, \ldots, F_k$", we mean that system containing wfe's, wff's, rules **R1** and **R2**, and as axioms, the union of the $F_1, F_2, \ldots,$ and $F_k$ sets.

*The system* **NC**. We shall show the system **NC** contains a propositional calculus by showing that representations of Mendelson's axioms[1] are theorems in **NC** and that a representation of modus ponens is a derived rule of inference in **NC**.

We map each wff of Mendelson's system, $L$, onto a wff of **NC** in the following way: we put each wff of $L$ into Polish notation, substitute the variable $x_i$ of **NC** for the letter $A_i$ of $L$, and append "=1" to the wfe so obtained. The counterparts of Mendelson's three axiom schema in **NC** are:

$$A1\quad CuCvu = 1$$
$$A2\quad CCuCvwCCuvCuw = 1$$

and

$$A3\quad CCNvNuCCNvuv = 1$$

where $u$, $v$ and $w$ are metatheoretic variables for wfe's.

Lemma 2. *Suppose u is a wfe. The following are theorem schema:*

| | | |
|---|---|---|
| a. | $C1u = u$ | $C10$, $C11$, **R2** |
| b. | $C0u = 1$ | $C00$, $C01$, **R2** |
| c. | $Cu1 = 1$ | $C01$, $C11$, **R2** |
| d. | $Cuu = 1$ | $C00$, $C11$, **R2** |
| e. | $1 = 1$ | $N0$, $N0$, **R1** |
| f. | $0 = 0$ | N1, N1, **R1** |
| g. | $u = u$ | $e$, $f$, **R2** |
| h. | $CN10 = 1$ | $C00$, $N1$, **R1** |
| i. | $CN00 = 0$ | C10, N0, **R1** |
| j. | $CNu0 = u$ | $h$, $i$, **R2** |

**DR 2.** *If u and v are wfe's then* $u = 1, Cuv = 1 \vdash_C v = 1$. (Modus Ponens).

*Proof:* 1. $u = 1$
2. $Cuv = 1$

1. In Mendelson's system, *Introduction to Mathematical Logic*, Van Nostrand, 1964.

3. $C1w = v$ *Lemma* 2a
4. $C1v = 1$ 1, 2 **R1**
5. $v = 1$ 3, 4 **R1**

Metatheorem 2.1. $\vdash_{\mathbf{NC}} A1$

*Proof:*
1. $C0Cv0 = 1$ *Lemma* 2b
2. $Nv1 = 1$ *Lemma* 2c
3. $C1Cv1 = 1$ $C11$, 2, **R1**
4. $CuCvu = 1$ 1, 3, **R2**

Metatheorem 2.2. $\vdash_{\mathbf{NC}} A2$

*Proof:*

| | | |
|---|---|---|
| 1. | $C0Cvw = 1$ | Lemma 2b |
| 2. | $C0v = 1$ | Lemma 2b |
| 3. | $C0w = 1$ | Lemma 2b |
| 4. | $CC0vC0w = 1$ | $C11$, 2, 3, **R1** (twice) |
| 5. | $CC0CvwCC0vC0w = 1$ | $C11$, 1, 4, **R1** (twice) |
| 6. | $C1v = v$ | *Lemma* 2a |
| 7. | $C1w = w$ | *Lemma* 2a |
| 8. | $Cvw = Cvw$ | *Lemma* 2g |
| 9. | $Cvw = CC1vC1w$ | 6, 7, 8, **R1** (twice) |
| 10. | $C1Cvw = Cvw$ | *Lemma* 2a |
| 11. | $CCvwCvw = 1$ | *Lemma* 2d |
| 12. | $CC1CvwCC1vC1w = 1$ | 9, 10, 11, **R1** (twice) |
| 13. | $CCuCvwCCuvCuw = 1$ | 5, 12, **R2** |

Metatheorem 2.3. $\vdash_{\mathbf{NC}} A3$

| | | |
|---|---|---|
| 1. | $CNv0 = v$ | *Lemma* 2j |
| 2. | $Cvv = 1$ | *Lemma* 2d |
| 3. | $CCNv0v = 1$ | 1, 2, **R1** |
| 4. | $CCNvN01 = 1$ | *Lemma* 2c |
| 5. | $CCNvN0CCNv0v = 1$ | 3, 4, **R1** |
| 6. | $CNvN1 = v$ | 1, N1, **R1** |
| 7. | $CvCCNv1v = 1$ | *Metatheorem* 2.1 |
| 8. | $CCNvN1CCNv1v = 1$ | 6. 7, **R1** |
| 9. | $CCNvNuCCNvuv = 1$ | 5, 8, **R2** |

It is, in a way, unnecessary work to have proved **DR2**, and metatheorems 2.1, 2.2 and 2.3. Though these proofs help to show what can be done in **NC**, and to suggest what can be done in other definitional systems, a much stronger result is possible from which it follows that **NC** contains a propositional calculus. It is clear that the tautologies of any propositional calculus using one or more of negation, conjunction, disjunction, implication and stroke map onto wfe's. It is easy to see that mappings of tautologies of a propositional calculus are normal identities, since the test for tautologousness exactly parallels the method for determining normal identityhood of a wff: wfe = 1. Then if we can show that a definitional boolean system, $X$, contains all those normal identities which contain function letters representing functions adequate for a propositional calculus then $X$ must contain a propositional calculus. Suppose system $X$ contains function letters adequate for a propositional calculus. We show first:

Metatheorem 3. $\vdash_X 0 = 0$ and $\vdash_X 1 = 1$.

This has been shown for **NC**. Each of the **N, A, K, C** and **D** sets contain pairs of formulas one setting a wfe equal to 0 and one setting a wfe equal to 1. Using **R1** on itself twice we get $\vdash_X 0 = 0$ and $\vdash_X 1 = 1$.

Metatheorem 3.1. *Suppose $f$ is a normal identity containing no function letter $X$ does not contain. Then all instances of $f$ are theorems of $X$.*

*Proof:* Suppose $f'$ is an instance of $f$ and $f'$ is $u = v$. Consider the method for determining $f'$ a normal identity. It involves two calculations, one of which reduces $u$ to a constant, **c**, and the other of which reduces $v$ to that same constant. Using these calculations we can construct proofs proving $u = \mathbf{c}$ and $v = \mathbf{c}$, hence $u = v$, in the following manner:

1) $\mathbf{c} = \mathbf{c}$ is the first step of the proof.

2) We move from the $i-1$'th step to the $i$'th step of the calculation by substituting a constant for a wfe containing constants. We may make steps in the proof correspond, in reverse order, to steps in the calculation by reversing this process, substituting a wfe in the left wfe of a prior step for the constant asserted (by an axiom) equal to it. That is, if $s_1, s_2, \ldots, s_n$ are steps in the calculation and $p_1, p_2, \ldots, p_n$ are sequences of sequential steps in the proof:

| | | |
|---|---|---|
| $s_n$ is **c** | corresponds to | $p_1$) $\mathbf{c} = \mathbf{c}$ |
| $s_i$ is the result of substituting a constant for a wfe. | | $p_{n-i}$) Those steps resulting in $w = \mathbf{c}$ where $w$ is the result of the reverse substitution. |

Methatheorem 3.2. *Suppose $f$ is a normal identity containing no function letter $X$ does not contain. Then $f$ is a theorem of $X$.*

*Proof:* It is sufficient to show that $f$ follows from its instances. Let $f(u_1, u_2, \ldots, u_j)$ stand for that formula derived from $f$ by either substituting a constant $u_i$ for a variable letter $x_i$ everywhere in $f$ or by substituting $x_i$ for $x_i$ everywhere in $f$. Order the instances of $f$ in the following manner:

| | |
|---|---|
| $f(0, 0, \ldots, 0)$ | where the next member of the sequence is obtained by the |
| $f(0, 0, \ldots, 1)$ | binary addition of 1 to the sequence within the parentheses |
| $\vdots$ | of the prior member. |
| $f(1, 1, \ldots, 1)$ | |

From successive pairs we may prove:

$$\begin{array}{l} f(0, 0, \ldots, 0, x_n) \\ f(0, 0, \ldots, 1, x_n) \\ \quad\vdots \\ f(1, 1, \ldots, 1, x_n) \end{array} \qquad \text{(I)}$$

or reduce the $2^n$ instances to $2^{n-1}$ theorems of $X$.

Proceeding in the same way, from successive pairs of (I) we may prove the $2^{n-2}$ theorems:

$$\begin{array}{l} f(0, 0, \ldots, 0, x_{n-1}, x_n) \\ \quad\vdots \\ f(1, 1, \ldots, 1, x_{n-1}, x_n) \end{array}$$

and finally we may prove $f(x_1, x_2, \ldots, x_n)$ which is $f$.

This proves the metatheorem. Notice that a rule weaker than **R2** suffices, since **R2** allows introducing any wfe for constants though it is necessary only to allow the introduction of variables.

*Concluding Remarks.* Naturally, it is of interest to know why any of this is of any interest. Apart from the mathematical attractions of the system, the power of the two inference rules, **R1** and **R2**, are revealed. It is nice to know how far one can go just with substitution and a simple, intuitive rule like **R2**. But more importantly, it is interesting to learn that the theorems of a boolean calculus (which includes the propositional calculus) do really, in an important sense, follow just from the matrix definitions of boolean functions, expressed as propositions.

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