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A NOTE ON THE STRUCTURE OF THE POWER SET

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In the theory of sets, one commonly finds existence theorems of the form

For every infinite set S of cardinality $\mathfrak{m}$ *there exists a subset T of* $\mathcal{P}(S)$ *such that* $\operatorname{card}(T) = 2^{\mathfrak{m}}$ *and* P,

where P is some property on the elements of T and '$\mathcal{P}(S)$' denotes the power set of S. Thus, for example, it is known that there exist, for every infinite set S of cardinality $\mathfrak{m}$, subsets F, G, H and L of $\mathcal{P}(S)$ each of cardinality $2^{\mathfrak{m}}$ and such that

(1) $\operatorname{card}(X \cap Y) < \mathfrak{m}$ for distinct $X, Y \in F$;
(2) $\operatorname{card}(X - Y) = \mathfrak{m}$ for distinct $X, Y \in G$;
(3) $\left(\bigcap_{i=1}^{s} A_i\right) \cap \left(\bigcap_{i=1}^{t} S - B_i\right) \neq \phi$ for distinct $A_1, \ldots, A_s, B_1, \ldots, B_t \in H$;
(4) $X \subseteq Y$ or $Y \subseteq X$ for each $X, Y \in L$.[1]

Following Sierpiński ([2], p. 29), let us call a property P *hereditary* if, whenever P holds for a set, it also holds for each of its subsets. Now the purpose of this note is to point out that any existence theorem of the above form can be strengthened to read ". . . there exist $2^{2^{\mathfrak{m}}}$ subsets of $\mathcal{P}(S)$ of cardinality $2^{\mathfrak{m}}$ such that P," provided that P is hereditary (as in (1)-(4)). This observation, which does not seem to appear in the literature, is based upon the following

Theorem. *For every infinite set of cardinality* $\mathfrak{p}$ *there exists* $2^{\mathfrak{p}}$ *subsets of cardinality* $\mathfrak{p}$.

Proof. Let S be any infinite set of cardinality $\mathfrak{p}$, and put

$$Z = \{X : X \subseteq S \text{ and } \operatorname{card}(X) = \mathfrak{p}\}$$

1. These and similar results can be found in [1], [3] and [4]. In the proof of (4), Wolk [4] employs the generalized continuum hypothesis.

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and $Z^* = \mathcal{P}(S)\text{-}Z$. Obviously $S = X \cup (S\text{-}X)$ for each $X \subseteq S$, from which it follows with the aid of the axiom of choice that card$(X) = \mathfrak{p}$ or card$(S\text{-}X) = \mathfrak{p}$. Thus, to see that card$(Z^*) \leqslant$ card(Z), simply map each $X \in Z^*$ onto $S\text{-}X$ in Z, thereby establishing a one-one correspondence between the elements of Z^* and those of a subset of Z. But now $\mathcal{P}(S) = Z \dot{\cup} Z^*$; and so again with the aid of the axiom of choice, either card$(Z) = 2^{\mathfrak{p}}$ or card$(Z^*) = 2^{\mathfrak{p}}$. In either case, $2^{\mathfrak{p}} \leqslant$ card(Z). But clearly card$(Z) \leqslant 2^{\mathfrak{p}}$; whence, card$(Z) = 2^{\mathfrak{p}}$. (In case $\mathfrak{p} = \aleph_0$ or $\mathfrak{p} = 2^{\aleph_0}$, this theorem is easily obtained without the axiom of choice. For a proof of the latter instance, see [1], p. 142.)

REFERENCES

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[4] Wolk, E. S., "A Theorem on Power Sets," *Amer. Math. Monthly*, vol. 72, (1965), pp. 397-398.

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