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NOTE ON DUALITY IN PROPOSITIONAL CALCULUS

CHANDLER WORKS and WOLFGANG YOURGRAU

Current literature refers only vaguely to the principle of duality. In this note we formalize the principle and develop some simple metatheorems which enable one to dualize propositions without explicit recourse to de Morgan's rules or to contraposition.

Let P be a compound proposition whose truth value is a function of the truth values of the undecomposed mutually *independent* propositions $p_1, p_2, \ldots, p_i, \ldots, p_m$. Further, let the truth tables of these components be arranged in standard order. We then form a new propositional function, the *dual* of P, written as P^d, by negating the truth values of P and rearranging them in reversed order. Thus, P^d also depends on the same independent propositions as P.

After Boole,* we put $f(P) = 1$ when P is true and $f(P) = 0$ when P is false, so that $f(P)$ is a numerical function. It follows that $f(\sim P) = 1 - f(P)$. We represent the truth column for P by $f(P) = (a_1, a_2, \ldots, a_k, \ldots, a_n)$, where $a_k = 0$ or $a_k = 1$ and $n = 2^m$. Similarly, to another compound proposition, say Q, corresponds the numerical function $f(Q) = (b_1, b_2, \ldots, b_k, \ldots, b_n)$. Hence, $P \equiv Q$ if and only if $f(P) = f(Q)$, i.e. if and only if $a_k = b_k$ $(k = 1, 2, \ldots, n)$. We can now restate the above definition of the dual of P so that $f(P^d) = (1 - a_n, 1 - a_{n-1}, \ldots, 1 - a_{n-k+1}, \ldots, 1 - a_1)$. The k^{th} entry for $f(P^d)$ is $1 - a_{n-k+1}$.

Since the unary operation of dualizing is formed from two involutory operations, it is also involutory. That is,

$$(1) \qquad (P^d)^d \equiv P.$$

This can be shown rigorously by examining the k^{th} entry for $(P^d)^d$, which is

$$1 - (1 - a_{n-(n-k+1)+1}) = a_k.$$

**The Mathematical Analysis of Logic*, (New York: Barnes & Noble), pp. 20-21, 1965. Original publication: 1847.

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Because the six statements

$$P \equiv Q$$
$$f(P) = f(Q)$$
$$a_k = b_k$$
$$1 - a_{n-k+1} = 1 - b_{n-k+1}$$
$$f(P^{\mathbf{d}}) = f(Q^{\mathbf{d}})$$
$$P^{\mathbf{d}} \equiv Q^{\mathbf{d}}$$

are logically equivalent to each other,

(2) $$P \equiv Q \text{ if and only if } P^{\mathbf{d}} \equiv Q^{\mathbf{d}}.$$

Tautologies are dual with respect to self-contradictions. Let $\mathbf{t}$ be a tautology, i.e.

$$f(\mathbf{t}) = (1, 1, \ldots, 1)$$
$$f(\mathbf{t}^{\mathbf{d}}) = (0, 0, \ldots, 0) = f(\sim\mathbf{t}).$$

Therefore,

(3) $$\mathbf{t}^{\mathbf{d}} \equiv \sim\mathbf{t}.$$

The equivalence between dualizing and negating holds for tautologies and for self-contradictions, but obviously not for a proposition with a non-symmetric set of truth values.

Independent components may be treated as self-dual propositions: Since the p_i's are independent, $f(p_i)$, arranged in standard order, consists of strings of 2^{i-1} 1's alternating with strings of 2^{i-1} 0's and so

$$f(p_k) = f(p_k^{\mathbf{d}}).$$

Thus

(4) $$p_k^{\mathbf{d}} \equiv p_k.$$

The notation '$\sim P^{\mathbf{d}}$' may be used unambigously, for

(5) $$(\sim P)^{\mathbf{d}} \equiv \sim(P^{\mathbf{d}}).$$

The essential reasoning is that

$$f((\sim P)^{\mathbf{d}}) = (a_n, a_{n-1}, \ldots, a_{n-k+1}, \ldots, a_1) = f(\sim(P^{\mathbf{d}})).$$

As a corollary to (4) and (5),

(6) $$(\sim p)^{\mathbf{d}} \equiv \sim p.$$

The duality between conjunction and disjunction can easily be established. From truth tables we have

$$f(P \cdot Q) = f(P)\, f(Q);$$

the k^{th} entry of $f((P \cdot Q)^{\mathbf{d}})$ is $1 - a_{n-k+1}b_{n-k+1}$. Also from truth tables,

$$f(P \vee Q) = f(P) + f(Q) - f(P)\, f(Q).$$

Hence the k^{th} entry of $f(P^{\mathbf{d}} \vee Q^{\mathbf{d}})$ is

$$(1 - a_{n-k+1}) + (1 - b_{n-k+1}) - (1 - a_{n-k+1})(1 - b_{n-k+1}) = 1 - a_{n-k+1}b_{n-k+1}.$$

Consequently,

$$f((P \cdot Q)^{\mathbf{d}}) = f(P^{\mathbf{d}} \vee Q^{\mathbf{d}})$$

and

(7) $$(P \cdot Q)^{\mathbf{d}} \equiv P^{\mathbf{d}} \vee Q^{\mathbf{d}}.$$

A result analogous to theorem (7) issues almost immediately. Replacing P and Q by their duals, we obtain

$$(P^{\mathbf{d}} \cdot Q^{\mathbf{d}})^{\mathbf{d}} \equiv (P^{\mathbf{d}})^{\mathbf{d}} \vee (Q^{\mathbf{d}})^{\mathbf{d}}.$$

Employing (2) and then (1), we get, on communting,

(8) $$(P \vee Q)^{\mathbf{d}} \equiv P^{\mathbf{d}} \cdot Q^{\mathbf{d}}.$$

The dual of a conditional can be developed from the foregoing theorems by means of the familiar tautologies

$$P \supset Q \equiv {\sim}P \vee Q \equiv {\sim}(P \cdot {\sim}Q).$$

Because of (2), (8), and (5), the following three statements are logically true:

$$(P \supset Q)^{\mathbf{d}} \equiv ({\sim}P \vee Q)^{\mathbf{d}}$$
$$\equiv {\sim}P^{\mathbf{d}} \cdot Q^{\mathbf{d}}$$

(9) $$(P \supset Q)^{\mathbf{d}} \equiv {\sim}(Q^{\mathbf{d}} \supset P^{\mathbf{d}}).$$

The next four theorems are readily proved by applying methods similar to those used in (8) and (9).

(10) $$(P \equiv Q)^{\mathbf{d}} \equiv P^{\mathbf{d}} \wedge Q^{\mathbf{d}},$$

(11) $$(P \wedge Q)^{\mathbf{d}} \equiv (P^{\mathbf{d}} \equiv Q^{\mathbf{d}}),$$

where '$\wedge$' symbolizes exclusive disjunction.

(12) $$(P \downarrow Q)^{\mathbf{d}} \equiv P^{\mathbf{d}} / Q^{\mathbf{d}}.$$

(13) $$(P / Q)^{\mathbf{d}} \equiv P^{\mathbf{d}} \downarrow Q^{\mathbf{d}}.$$

To justify the dualization of quantifiers, we introduce a lemma: One can reformulate the original definition of duality so that

(14) $$F^{\mathbf{d}}(P_1, P_2, \ldots, P_r) \equiv {\sim}F({\sim}P_1{}^{\mathbf{d}}, {\sim}P_2{}^{\mathbf{d}}, \ldots, {\sim}P_r{}^{\mathbf{d}}) \text{ and}$$
$$p_i \equiv p_i{}^{\mathbf{d}}, \text{ where } {\sim}P^{\mathbf{d}} \equiv {\sim}(P^{\mathbf{d}}).$$

Abbreviating

$$\vec{p} = p_1, p_2, \ldots, p_m,$$
$${\sim}\vec{p} = {\sim}p_1, {\sim}p_2, \ldots, {\sim}p_m,$$
$${\sim}\vec{p}_{\mathbf{d}} = {\sim}(p_1^{\mathbf{d}}), {\sim}(p_2^{\mathbf{d}}), \ldots, {\sim}(p_m^{\mathbf{d}}),$$

from (14) one has

$$F^{\mathbf{d}}(P(\vec{p})) \equiv {\sim}F({\sim}P^{\mathbf{d}}(\vec{p})).$$

By recursion

$$F^{\mathbf{d}}(P(\vec{p})) \equiv {\sim}F({\sim}{\sim}P({\sim}\vec{p}^{\mathbf{d}})).$$

Hence

(15) $$F^{\mathbf{d}}(P(\vec{p})) \equiv {\sim}F(P({\sim}\vec{p})).$$

This expression is equivalent to the initial definition, for negating the p_i amounts precisely to reversing the order of the main truth values. It is assumed that (15) applies even when p_i is open.

The converse derivation of (14) from (15) requires no more than a few steps. If $F \equiv P \equiv p_i$ then $p_i^{\mathbf{d}} \equiv {\sim}({\sim}p_i) \equiv p_i$. In addition

$$F^{\mathbf{d}}(P_1, P_2, \ldots, P_r) \equiv {\sim}F(P_1({\sim}\vec{p}), P_2({\sim}\vec{p}), \ldots, P_r({\sim}\vec{p})).$$

Now, (15) yields ${\sim}P^{\mathbf{d}}(\vec{p}) \equiv P({\sim}\vec{p})$, which gives (14).

Using this lemma, we can easily dualize quantifiers. Since $(\forall x)P(x)$ and $(\exists x)P(x)$ are functions of $P(x)$ and as

$$[(\exists x)P(x)] \equiv {\sim}[(\forall x){\sim}P(x)]$$

is a tautology, by the lemma

(16) $$[(\exists x)P(x)]^{\mathbf{d}} \equiv (\forall x)P^{\mathbf{d}}(x).$$

Similarly,

(17) $$[(\forall x)P(x)]^{\mathbf{d}} \equiv (\exists x)P^{\mathbf{d}}(x).$$

To summarize theorems (4) through (13) as well as (16) and (17), one has the result:

(18) *Any proposition P is dualized by performing all the following operations on the logical constants in P.*

Interchange

(a) conjunction and disjunction;
(b) equivalence and exclusive disjunction;
(c) alternative denial and joint denial;
(c) universal and existential quantifiers.

Replace every conditional by its negated converse.

The next theorem is fundamental to the utilization of duality.

(19) *If and only if* $P \equiv \mathbf{t}$ *then* ${\sim}P^{\mathbf{d}} \equiv \mathbf{t}$.

For, under either of these conditions,

$$f(P) = (1, 1, \ldots, 1)$$
$$f(P^{\mathbf{d}}) = (0, 0, \ldots, 0).$$

(19) may be derived more formally from (3), (2), and (5).

From (9) and (19) we establish:

(20) *If and only if* $P \supset Q \equiv \mathbf{t}$ *then* $Q^{\mathbf{d}} \supset P^{\mathbf{d}} \equiv \mathbf{t}$.

The importance of this theorem arises from the concept of entailment as the logical necessity of a conditional.

The final theorem follows from the tautology

$$(P \equiv Q) \equiv (P \supset Q) \cdot (Q \supset P).$$

(21) *If and only if* $(P \equiv Q) \equiv \mathbf{t}$ *then* $(P^{\mathbf{d}} \equiv Q^{\mathbf{d}}) \equiv \mathbf{t}$.

Theorems (18) and (21) show on inspection that all commutative, associative, and distributive laws as well as the de Morgan rules must occur in dual pairs.

University of Denver
Denver, Colorado