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ARITHMETIC OPERATIONS ON ORDINALS

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1 *Introduction** We characterize addition and multiplication of ordinal numbers. We assume familiarity with the basic properties of ordinal arithmetic (Sierpiński [3], Chapter 14). Although our discussion is informal, it could be formalized within Gödel-Bernays set theory, e.g., within the axiom system consisting of groups A, B, C, and D of Gödel [1].

Greek letters, sometimes with subscripts, will denote ordinals; "On" will denote the class of all ordinals. As usual, "+" and "·" stand for ordinal addition and multiplication, respectively. Braces will designate proper classes as well as sets.

2 *Addition* Let + be any binary operation on On that is such that for all ordinals α, β, and γ,

1) $\alpha + 0 = \alpha$;
2) if $\beta \leqslant \gamma$, then $\alpha + \beta \leqslant \alpha + \gamma$;
3) if $\beta \leqslant \gamma$, then there is a unique δ such that $\beta + \delta = \gamma$.

In Proposition 2.1 and its corollary, we assume that + is a binary operation on On that satisfies 1), 2), and 3).

Proposition 2.1 *Let α, β, and γ be ordinals. If $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

Proof: $\alpha = \alpha + 0 \leqslant \alpha + \gamma$, by 1) and 2). Thus, if $\alpha + \beta = \alpha + \gamma$, then by 3), $\beta = \gamma$. By 2), $\alpha + \beta \leqslant \alpha + \gamma$; therefore, we must have $\alpha + \beta < \alpha + \gamma$.

Corollary *For all ordinals α, β, and γ, $\beta < \gamma$ if and only if $\alpha + \beta < \alpha + \gamma$.*

Define $+_1$, $+_2$, and $+_3$ on On as follows:

For $\alpha, \beta \in \mathrm{On}$,

$$\alpha +_1 \beta = \beta;$$
$$\alpha +_2 0 = \alpha,$$

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and for $\beta > 0$,

$$\alpha +_2 \beta = \begin{cases} \beta, \text{ if } \alpha \neq \beta, \\ 0, \text{ if } \alpha = \beta; \end{cases}$$

$$\alpha +_3 \beta = \alpha.$$

Then $+_1$ satisfies 2) and 3), but not 1); $+_2$ satisfies 1) and 3), but not 2); $+_3$ satisfies 1) and 2), but not 3), as does the Hessenberg natural sum (Hessenberg [2]). It is well-known that $+$ satisfies 1), 2), and 3); we now show that $+$ is the only binary operation on On which does so.

Theorem 2.1 *Let* $\boldsymbol{+}$ *be any binary operation on* On *that satisfies* 1), 2), *and* 3). *Then for all ordinals* α *and* β,

$$\alpha \mathbin{\boldsymbol{+}} \beta = \alpha + \beta.$$

Thus $\boldsymbol{+} = +$.

Proof: We utilize the Principle of Transfinite Induction. Let

$$A = \{\beta: \text{ for all } \alpha, \alpha \mathbin{\boldsymbol{+}} \beta = \alpha + \beta\}.$$

Then, by 1), $0 \in A$. Suppose $\beta \in A$; let α be an arbitrary ordinal. Surely $\alpha < \alpha + \beta^+$; let δ be the unique ordinal that satisfies $\alpha \mathbin{\boldsymbol{+}} \delta = \alpha + \beta^+$. Then

$$\alpha \mathbin{\boldsymbol{+}} \beta = \alpha + \beta < \alpha + \beta^+ = \alpha \mathbin{\boldsymbol{+}} \delta.$$

By the Corollary of Proposition 2. 1, $\beta < \delta$. Thus $\beta^+ \leqslant \delta$ and

$$\alpha + \beta = \alpha \mathbin{\boldsymbol{+}} \beta < \alpha \mathbin{\boldsymbol{+}} \beta^+ \leqslant \alpha \mathbin{\boldsymbol{+}} \delta = \alpha + \beta^+ = (\alpha + \beta)^+.$$

It follows that $\alpha \mathbin{\boldsymbol{+}} \beta^+ = \alpha + \beta^+$. Suppose $\gamma \subseteq A$, where γ is a limit ordinal. Fix α. Then

(1) $\alpha + \gamma$ is the smallest ordinal, δ, such that $\alpha \mathbin{\boldsymbol{+}} \beta < \delta$ for every $\beta < \gamma$.

Since $\alpha \mathbin{\boldsymbol{+}} \beta < \alpha + \gamma$ for every $\beta < \gamma$, it follows that $\alpha + \gamma \leqslant \alpha \mathbin{\boldsymbol{+}} \gamma$. Let δ be the unique ordinal that satisfies $\alpha \mathbin{\boldsymbol{+}} \delta = \alpha + \gamma$. Then $\gamma \leqslant \delta$, by (1). Therefore,

$$\alpha \mathbin{\boldsymbol{+}} \gamma \leqslant \alpha \mathbin{\boldsymbol{+}} \delta = \alpha + \gamma.$$

Hence $\alpha \mathbin{\boldsymbol{+}} \gamma = \alpha + \gamma$.

Corollary 2.1 *If* $\boldsymbol{+}$ *is a binary operation on* On *that satisfies* 1), 2), *and* 3), *then* $\boldsymbol{+}$ *is associative.*

Corollary 2.2 *No commutative binary operation on* On *satisfies* 1), 2), *and* 3).

Let $\#$ be any binary operation on On that satisfies the following: for all ordinals α, β, and γ,

4) if $\beta < \gamma$, then $\alpha \# \beta < \alpha \# \gamma$;
5) $\beta \leqslant \gamma$ if and only if there is some δ such that $\beta \# \delta = \gamma$.

In Propositions 2.2 and 2.3, we assume that $\sharp$ is a binary operation on On that satisfies 4) and 5).

Proposition 2.2 *For all ordinals* β *and* γ, *if* $\beta < \gamma$, *then there is a unique* δ *such that* $\beta \sharp \delta = \gamma$.

Proposition 2.3 *For every ordinal* α, $\alpha \sharp 0 = \alpha$.

Proof: $\alpha \leqslant \alpha \sharp 0$, by 5). Suppose $\alpha < \alpha \sharp 0$. Let δ be the unique ordinal that satisfies $\alpha \sharp \delta = \alpha$. If $\delta \neq 0$, then $0 < \delta$ and, by 4),

$$\alpha \sharp 0 < \alpha \sharp \delta = \alpha \leqslant \alpha \sharp 0.$$

This contradiction establishes that $\alpha \sharp 0 = \alpha$.

Observe that $+_1$ satisfies 4) but not 5). Define $+_4$ on On by

$$\alpha +_4 \beta = \max\{\alpha, \beta\}, \text{ for all } \alpha, \beta \in \mathrm{On}.$$

Then $+_4$ satisfies 5) but not 4). Clearly, + satisfies both 4) and 5).

Theorem 2.2 *Let* $\sharp$ *be any binary operation on* On *that satisfies* 4) *and* 5). *Then for all ordinals* α *and* β,

$$\alpha \sharp \beta = \alpha + \beta.$$

Thus $\sharp = +$.

Proof: $\sharp$ satisfies 1), 2), and 3); the result follows from Theorem 2.1.

Corollary 2.3 *Let* $\natural$ *be any binary operation on* On *that satisfies the following:*

2) *if* $\beta \leqslant \gamma$, *then* $\alpha \natural \beta \leqslant \alpha \natural \gamma$;
5′) $\beta \leqslant \gamma$ *implies there is a unique* δ *such that* $\beta \natural \delta = \gamma$, *and* $\beta > \gamma$ *implies there is no* δ *such that* $\beta \natural \delta = \gamma$.

Then for all ordinals α *and* β, $\alpha \natural \beta = \alpha + \beta$.

Proof: It suffices to show that $\natural$ satisfies 4) and 5). Clearly 5′) implies that $\natural$ satisfies 5). Let α be an arbitrary ordinal and let $\beta < \gamma$. Then 5′) implies that $\alpha \natural \beta \neq \alpha \natural \gamma$. This together with 2) indicates that $\alpha \natural \beta < \alpha \natural \gamma$.

Observe that $+_1$ satisfies 2) but not 5′). Moreover, define $+_5$ on On as follows:

$$0 +_5 \beta = \beta, \text{ for all } \beta;$$

$$1 +_5 \beta = \begin{cases} \beta^+, & \text{if } \beta < \omega, \\ \beta, & \text{if } \omega \leqslant \beta; \end{cases}$$

for $\alpha \geqslant 2$, let

$$\alpha +_5 \beta = \begin{cases} 0, & \text{if } \alpha > \beta, \\ \beta, & \text{if } \alpha \leqslant \beta. \end{cases}$$

Then $+_5$ also satisfies 2) but not 5′). Furthermore, 5″) $\beta \leq \gamma$ if and only if there is a unique δ such that $\beta +_5 \delta = \gamma$.

Define $+_6$ on On as follows:

$$\alpha +_6 \beta = \begin{cases} 1, \text{ if } \alpha = \beta = 0; \\ 0, \text{ if } \alpha = 0 \text{ and } \beta = 1; \\ \alpha + \beta, \text{ otherwise.} \end{cases}$$

Then $+_6$ satisfies 5′) but not 2).

3 *Multiplication* Let $\times$ be a binary operation on On that is such that for all ordinals α, β, and γ,

1) if $\gamma < \alpha \times \beta$, then there are ordinals α_1 and β_1 that satisfy $\alpha_1 < \alpha$, $\beta_1 < \beta$, and $\gamma = \alpha \times \beta_1 + \alpha_1$;
2) if $\beta < \gamma$, then $\alpha \times \beta + \alpha \leq \alpha \times \gamma$.

It is well-known that $\cdot$ satisfies 1) and 2). Define $\times_1$ and $\times_2$ as follows: For all ordinals α and β,

$$\alpha \times_1 \beta \equiv 0;$$
$$\alpha \times_2 \beta = \alpha \cdot \beta^+.$$

Then $\times_1$ satisfies 1) but not 2), and $\times_2$ satisfies 2) but not 1).

Theorem 3.1 *Let* $\times$ *be a binary operation on* On *that satisfies* 1) *and* 2). *Then for all ordinals* α *and* β, $\alpha \times \beta = \alpha \cdot \beta$. *Thus* $\times = \cdot$.

Proof: Let

$$A = \{\beta\colon \text{ for all } \alpha, \alpha \times \beta = \alpha \cdot \beta\}.$$

$0 \in A$ because otherwise, $0 < \alpha \times 0$ would require that there be an ordinal $\beta_1 < 0$, by 1). Suppose $\beta^+ \subseteq A$ but $\beta^+ \notin A$. Then for some α, $\alpha \times \beta^+ \neq \alpha \cdot \beta^+$. Then, by 2),

$$\alpha \cdot \beta^+ = \alpha \cdot \beta + \alpha = \alpha \times \beta + \alpha \leq \alpha \times \beta^+.$$

It follows that $\alpha \cdot \beta^+ < \alpha \times \beta^+$. Thus $\alpha > 0$; by 1), there are $\beta_1 \leq \beta$ and $\alpha_1 < \alpha$ for which

$$\alpha \cdot \beta^+ = \alpha \times \beta_1 + \alpha_1 = \alpha \cdot \beta_1 + \alpha_1 \cdot 1 < \alpha \cdot \beta_1 + \alpha \cdot 1 = \alpha \cdot \beta_1^+ \leq \alpha \cdot \beta^+.$$

This inequality is false; hence $\beta^+ \in A$. Let γ be a limit ordinal for which $\gamma \subseteq A$. If α is an arbitrary ordinal and if $\beta < \gamma$, then

$$\alpha \cdot \beta^+ = \alpha \cdot \beta + \alpha = \alpha \times \beta + \alpha \leq \alpha \times \gamma.$$

Since $\alpha \cdot \gamma$ is the smallest ordinal for which $\alpha \cdot \beta^+ < \alpha \cdot \gamma$ for every $\beta < \gamma$, it follows that $\alpha \cdot \gamma \leq \alpha \times \gamma$. If $\alpha \cdot \gamma < \alpha \times \gamma$, then there are $\alpha_1 < \alpha$ and $\gamma_1 < \gamma$ for which

$$\alpha \cdot \gamma = \alpha \times \gamma_1 + \alpha_1 = \alpha \cdot \gamma_1 + \alpha_1 \cdot 1 < \alpha \cdot \gamma_1 + \alpha \cdot 1 = \alpha \cdot \gamma_1^+ < \alpha \cdot \gamma.$$

This contradiction establishes that $\alpha \cdot \gamma = \alpha \times \gamma$.

Corollary 3.1 *Let* $\otimes$ *be a binary operation on* On *that satisfies*

3) *for every* $\alpha > 0$ *and for every* β *there is a unique* $\langle \zeta, \rho \rangle$ *with* $0 \leq \rho < \alpha$ *for which* $\beta = \alpha \otimes \zeta + \rho$;
4) *if* $\beta \leq \gamma$, *then* $\alpha \otimes \beta \leq \alpha \otimes \gamma$;
5) $0 \otimes \beta = 0$.

Then $\times = \cdot$.

Proof: It suffices to show that $\otimes$ satisfies 1) and 2).

1): Suppose $\gamma < \alpha \otimes \beta$. Clearly, 5) implies that $\alpha > 0$. By 3), $\gamma = \alpha \otimes \zeta + \rho$, where $\rho < \alpha$. Finally, 4) implies that $\zeta < \beta$.

2): Let $\beta < \gamma$. Then $0 \otimes \beta + 0 = 0 \otimes \beta \leq 0 \otimes \gamma$. If $\alpha > 0$, it follows that $\alpha \otimes \beta \leq \alpha \otimes \gamma$. Thus for some unique ρ_0, $\alpha \otimes \gamma = \alpha \otimes \beta + \rho_0$. By 3), $\rho_0 \geq \alpha$; hence $\alpha \otimes \beta + \alpha < \alpha \otimes \gamma$.

Note that $\times_1$ satisfies 4) and 5), but not 3). Define $\otimes_1$ and $\otimes_2$ on On as follows:

for all ordinals α and β:

$$\alpha \otimes_1 \beta = \begin{cases} 0, \text{ if } \alpha = \beta = 1, \\ 1, \text{ if } \alpha = 1 \text{ and } \beta = 0, \\ \alpha \cdot \beta, \text{ otherwise;} \end{cases}$$

$$\alpha \otimes_2 \beta = \begin{cases} 1, \text{ if } \alpha = 0, \\ \\ \alpha \cdot \beta, \text{ otherwise.} \end{cases}$$

Then $\otimes_1$ satisfies 3) and 5), but not 4); $\otimes_2$ satisfies 3) and 4), but not 5).

4 *Remark* In [4], we characterize the Hessenberg natural sum and generalizations of this operation.

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