Notre Dame Journal of Formal Logic
Volume XVI, Number 4, October 1975
NDJFAM

# ALGEBRAIC LOGIC WITH GENERALIZED QUANTIFIERS

CHARLES C. PINTER

1 *Introduction* The notion of languages with generalized quantifiers was introduced by A. Mostowski in [5]. Recently, this subject has attracted a great deal of attention and is currently undergoing a rapid development. In particular, the study of logic with the quantifier "there exist uncountably many" has become an important part of current investigations in model theory and set theory.

The object of this note is to describe the algebraic logic for calculi with generalized quantifiers. It is shown that the algebraic version of generalized quantifiers is a perfectly natural generalization of the usual notion of quantifiers in cylindric and polyadic algebras, and occurs naturally in Boolean algebras. We investigate the algebra of the structures which arise when generalized quantifiers are added to cylindric algebras, and characterize those cylindric algebras which admit generalized quantifiers. Finally, we give a few applications to the logic $\mathsf{L}(Q_1)$. Our notation and terminology is that of Henkin, Monk, and Tarski [3], except that we will say "quantifier" instead of "cylindrification".

2 *Algebraic Formulation of Generalized Quantifiers* Looking at the various extensions of quantification which have recently been studied (for example [1], [4], [5], [6]), a clear notion of generalized quantifiers is seen to emerge. Algebraically, this notion may be formulated in the following terms:

2.1 Definition Let $A$ be a Boolean algebra. By a *generalized quantifier* on $A$ we mean a function $\mathsf{q}: A \rightarrow A$ having the following properties:

Q1 $\mathsf{q}(x + y) = \mathsf{q}x + \mathsf{q}y$
Q2 $\mathsf{q}(x \cdot \mathsf{q}y) = \mathsf{q}x \cdot \mathsf{q}y$
Q3 $\mathsf{q}0 = 0$
Q4 $\mathsf{q}1 = 1$.

One immediately observes that quantifiers in the usual sense satisfy Q1-Q4. However, they also satisfy the inequality $x \leq \mathsf{q}x$, which does not hold for any other generalized quantifiers.

*Received August 8, 1974*

**2.2** Theorem *If* q *is a generalized quantifier on* $A$, *then the following hold for all* $x, y \in A$:

(i) *if* $x \leq y$, *then* $\mathrm{q}x \leq \mathrm{q}y$;
(ii) $\mathrm{qq}x = \mathrm{q}x$;
(iii) $\mathrm{q}x - \mathrm{q}y \leq \mathrm{q}(x - y)$;
(iv) $\mathrm{q}(-\mathrm{q}x) = -\mathrm{q}x$.

*Proof:* (i) follows immediately from Q1, and (ii) follows from Q2 and Q4, by letting $x = 1$. Now, $\mathrm{q}x \leq \mathrm{q}(x + y) = \mathrm{q}[(x - y) + y] = \mathrm{q}(x - y) + \mathrm{q}(y)$; thus, by Boolean algebra, $\mathrm{q}x - \mathrm{q}y \leq \mathrm{q}(x - y)$. Finally, we use Q4, (ii) and (iii) to prove (iv): $-\mathrm{q}x = \mathrm{q}1 - \mathrm{qq}x \leq \mathrm{q}(1 - \mathrm{q}x) = \mathrm{q}(-\mathrm{q}x)$. On the other hand, $\mathrm{q}(-\mathrm{q}x) \cdot \mathrm{q}x = \mathrm{q}(-\mathrm{q}x \cdot \mathrm{q}x) = \mathrm{q}0 = 0$, so $\mathrm{q}(-\mathrm{q}x) \leq -\mathrm{q}x$. Q.E.D.

In [2], Halmos proved that the range of every quantifier on $A$ is a "relatively complete" subalgebra of $A$, and that, conversely, every relatively complete subalgebra of $A$ is the range of a unique quantifier on $A$. In fact, each quantifier is *effectively* determined by the relatively complete subalgebra which it determines. We will now show that a similar result holds for generalized quantifiers. The next two theorems should be compared with Halmos ([2], Theorems 4 and 5).

Let $R$ designate the range of q, and let

$$I = \{x \in A : \mathrm{q}x = 0\}.$$

The following facts are immediately verified:

**2.3** Theorem

(i) *$R$ is a Boolean subalgebra of $A$.*
(ii) *$I$ is a Boolean ideal of $A$, and $R \cap I = \{0\}$.*
(iii) *For every $x \in A$, $\mathrm{q}x$ is the least $y \in R$ such that $x - y \in I$.*

We also have the converse of **2.3**, namely,

**2.4** Theorem *Let $R$ be a Boolean subalgebra of $A$ and $I$ a Boolean ideal of $A$ such that $R \cap I = \{0\}$. Suppose that for every $x \in A$, there is a least $y \in R$ such that $x - y \in I$. Then there is a unique generalized quantifier* q *on $A$ such that $R =$ range* q, *$I = \{x : \mathrm{q}x = 0\}$, and* q *is given by* **2.3** (iii).

*Proof:* For each $x \in A$, let $\mathrm{q}x$ be the least $y \in R$ such that $x - y \in I$. Q3 is immediate, and Q4 holds because $R \cap I = \{0\}$. Now, property **2.2** (i) is immediate, and from it we deduce that $\mathrm{q}x + \mathrm{q}y \leq \mathrm{q}(x + y)$. For the other half of equation Q1, we note that $x - (\mathrm{q}x + \mathrm{q}y) \leq x - \mathrm{q}x \in I$, $y - (\mathrm{q}x + \mathrm{q}y) \leq y - \mathrm{q}y \in I$, and therefore $(x + y) - (\mathrm{q}x + \mathrm{q}y) = [x - (\mathrm{q}x + \mathrm{q}y)] + [y - (\mathrm{q}x + \mathrm{q}y)] \in I$; thus, $\mathrm{q}(x + y) \leq \mathrm{q}x + \mathrm{q}y$, so we have Q1. Now, by the way q is defined, $\mathrm{q}x - \mathrm{qq}x \in I$; but $\mathrm{q}x - \mathrm{qq}x \in R$, so $\mathrm{q}x - \mathrm{qq}x = 0$, that is, $\mathrm{q}x \leq \mathrm{qq}x$. On the other hand, $\mathrm{qq}x$ is the *least* $y \in R$ such that $\mathrm{q}x - y \in I$, hence $\mathrm{qq}x \leq \mathrm{q}x$. We have established that $\mathrm{qq}x = \mathrm{q}x$, hence q is the identity function on $R$; it follows that $\mathrm{q}(-\mathrm{q}x) = -\mathrm{q}x$. This last formula, together with Q1, gives Q2. Q.E.D.

The relationship between generalized quantifiers and quantifiers in the usual sense is, once again, made plain. Indeed, a generalized quantifier q is a quantifier in the usual sense if and only if $I = \{0\}$.

The following characterization of generalized quantifiers will be needed in the next section. (Let us call a function $f$ *idempotent* if $ff = f$.)

**2.5** Theorem *Let $A$ be a Boolean algebra, and $f$: $A \to A$ a function. Then $f$ is a generalized quantifier iff $f$ is additive, idempotent, and its range is a Boolean subalgebra of $A$.*

*Proof:* The necessity of the condition is obvious. Now suppose $f$ is additive and idempotent, and range $f$ is a Boolean subalgebra of $A$. Q1 is satisfied by hypothesis. Now, $-fx \in$ *range* $f$, so

**2.6** $$f(-fx) = -fx.$$

From Q1 and **2.6**, one easily deduces Q2. We get Q3 from Q2 and **2.6** as follows: $f0 = f(fx \cdot -fx) = fx \cdot -fx = 0$. Finally, we deduce Q4 from Q3 and **2.6**. Q.E.D.

As an application of Theorem **2.5**, we note that every idempotent endomorphism is a generalized quantifier. For example, in polyadic algebras, every substitution $\mathsf{S}(\kappa/\lambda)$ and every constant is a generalized quantifier. It is therefore useful to observe (apparently this has not been noticed before) that by Theorem **2.4**, there is an effective way of constructing substitutions and constants if one is given their kernel and range. (It may be worth adding that: *if* q *is any generalized quantifier and* $J = \{x : \mathsf{q}x = 1\}$, *then* q *is an endomorphism iff $J$ is a dual ideal.*)

**3** *Cylindric Algebras and Generalized Quantifiers* One of the most important extensions of first-order logic is the logic $\mathsf{L}(\mathsf{Q}_1)$ which is formed by adding, to the first order logic with identity $\mathsf{L}$, the additional quantifier $\mathsf{Q}_1$ with the interpretation "there are uncountably many". In this section we introduce an expansion of cylindric algebras, which has the same relation to $\mathsf{L}(\mathsf{Q}_1)$ as cylindric algebras have to $\mathsf{L}$.

**3.1** Definition By a cylindric algebra of dimension $\alpha$ with generalized quantifiers, we mean an algebraic structure

$$\mathfrak{A} = \langle A, +, \cdot, 0, 1, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda}, \mathsf{q}_\kappa\rangle_{\kappa,\lambda<\alpha}$$

where $\langle A, +, \cdot, -, 0, 1, \mathsf{c}_\kappa, \mathsf{d}_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ is a cylindric algebra, and the operations $\mathsf{q}_\kappa$ satisfy the following conditions for all $\kappa, \lambda < \alpha$ and all $x, y \in A$:

G1 $\mathsf{q}_\kappa(x + y) = \mathsf{q}_\kappa x + \mathsf{q}_\kappa y$
G2 $\mathsf{q}_\kappa \mathsf{d}_{\kappa\lambda} = 0$
G3 $\mathsf{q}_\kappa \mathsf{c}_\kappa = \mathsf{c}_\kappa$
G4 $\mathsf{c}_\kappa \mathsf{q}_\kappa = \mathsf{q}_\kappa$
G5 $\mathsf{q}_\kappa \mathsf{c}_\lambda x \leqslant \mathsf{c}_\lambda \mathsf{q}_\kappa x + \mathsf{q}_\lambda \mathsf{c}_\kappa x$.

By G3 and G4, it is immediate that $\mathsf{q}_\kappa \mathsf{q}_\kappa = \mathsf{q}_\kappa$ and range $\mathsf{q}_\kappa$ = range $\mathsf{c}_\kappa$. Thus, by Theorem **2.5**, the operations $\mathsf{q}_\kappa$ are generalized quantifiers.

**3.2** Lemma *Let $\mathfrak{A}$ be a cylindric algebra with generalized quantifiers. The following hold for all $x \in A$ and $\kappa, \lambda < \alpha$:*

(i) $c_\lambda q_\kappa x \leq q_\kappa c_\lambda x$;
(ii) $q_\kappa(x - d_{\kappa\lambda}) = q_\kappa x$, *if* $\kappa \neq \lambda$;
(iii) $q_\kappa s_\lambda^\kappa = s_\lambda^\kappa$;
(iv) $s_\lambda^\kappa q_\kappa = q_\kappa$;
(v) $q_\lambda s_\lambda^\kappa = q_\kappa s_\kappa^\lambda$.

*Proof:* Because $q_\kappa$ is additive and $x \cdot d_{\kappa\lambda} \leq d_{\kappa\lambda}$, we have, for $\kappa \neq \lambda$,

**3.3** $$q_\kappa(x \cdot d_{\kappa\lambda}) \leq q_\kappa d_{\kappa\lambda} = 0.$$

Now, $q_\kappa x = q_\kappa[(x - d_{\kappa\lambda}) + (x \cdot d_{\kappa\lambda})] = q_\kappa(x - d_{\kappa\lambda}) + q_\kappa(x \cdot d_{\kappa\lambda}) = q_\kappa(x - d_{\kappa\lambda})$, **3.3** having been used in the last equation. We therefore have (ii). By G3 and G4, range $q_\kappa$ = range $c_\kappa$ = range $s_\lambda^\kappa$ (for $\kappa \neq \lambda$), hence (iii) and (iv) are immediate. Using G5 and **3.3**, we have:

$$q_\lambda s_\lambda^\kappa x = q_\lambda c_\kappa(x \cdot d_{\kappa\lambda}) \leq q_\kappa c_\lambda(x \cdot d_{\kappa\lambda}) + c_\kappa q_\lambda(x \cdot d_{\kappa\lambda}) = q_\kappa c_\lambda(x \cdot d_{\kappa\lambda}) = q_\kappa s_\kappa^\lambda x;$$

thus, we have (v). Finally, $x \leq c_\lambda x$, and therefore $c_\lambda q_\kappa x \leq c_\lambda q_\kappa c_\lambda x = c_\lambda q_\kappa s_\kappa^\lambda c_\lambda x = c_\lambda q_\lambda s_\lambda^\kappa c_\lambda x = q_\lambda s_\lambda^\kappa c_\lambda x = q_\kappa s_\kappa^\lambda c_\lambda x = q_\kappa c_\lambda x$. Q.E.D.

By **3.2** (i), condition G5 may be written in the more "symmetrical" form

G5′ $$q_\kappa c_\lambda x + c_\kappa q_\lambda x = q_\lambda c_\kappa x + c_\lambda q_\kappa x.$$

A complete set of axioms for $L(Q_1)$ is given in [4]. These axioms may be written in algebraic form as follows:

K1 $q_\kappa(d_{\kappa\lambda} + d_{\kappa\mu}) = 0$, *if* $\kappa \neq \lambda, \mu$
K2 $c_\kappa^\delta(-x + y) \leq -q_\kappa x + q_\kappa y$
K3 $q_\kappa x = q_\lambda s_\lambda^\kappa x$, *if* $\lambda \notin \Delta x$
K4 $q_\kappa c_\lambda x \leq c_\lambda q_\kappa x + q_\lambda c_\kappa x$.

In addition, G3 and G4 are implicit in the manner of treating bound variables.

It is easy to see that K1-K4, together with the implicitly given G3 and G4, are equivalent to G1-G5. The fact that G1 and G2 can be derived from K1-K4 is proved in [4], Lemma 1.9. In the other direction, K1 is an immediate consequence of G1 and G2; K2 may be written in the form $q_\kappa x - q_\kappa y \leq c_\kappa(x - y)$, and this follows from G1 by Theorem **2.2** (iii); and K3 follows from Lemma **3.2** (v).

**4** *Characterization of Cylindric Algebras Which Admit Generalized Quantifiers* Let $\mathfrak{A}$ be a cylindric algebra with generalized quantifiers, and for each $\kappa < \alpha$, let $I_\kappa = \{x : q_\kappa x = 0\}$. By the results of section **2**, $I_\kappa$ is a Boolean ideal of $A$ and the following conditions hold:

D1 (*range* $c_\kappa$) $\cap I_\kappa = \{0\}$
D2 *for every* $x \in A$, *there is a least* $y \in$ *range* $c_\kappa$ *such that* $x - y \in I_\kappa$
D3 $d_{\kappa\lambda} \in I_\kappa$
D4 $c_\lambda x \in I_\kappa \Rightarrow c_\kappa x \in I_\lambda$, *for every* $x \in I_\lambda$.

(The last two conditions hold by G2 and G5, respectively.)

Conversely, we also have the following:

*Let* $\mathfrak{A}' = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}\rangle_{\kappa,\lambda<\alpha}$ *be a cylindric algebra with a family* $\langle I_\kappa\rangle_{\kappa<\alpha}$ *of Boolean ideals satisfying* D1-D4. *If we define* $q_\kappa$ *by*

**4.1** $q_\kappa x =$ *the least* $y \in$ *range* $c_\kappa$ *such that* $x - y \in I_\kappa$,

*then* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda}, q_\kappa\rangle_{\kappa,\lambda<\alpha}$ *is a cylindric algebra with generalized quantifiers.*

By the results of section **2**, it remains only to verify G5. But this follows easily when we apply D4 to the formula $x - q_\lambda c_\kappa x - c_\lambda q_\kappa x$.

**5** *Some Applications to the Logic* $L(Q_1)$ If K2 is written in the form $q_\kappa x - q_\kappa y \leq c_\kappa(x - y)$, we can see that $x - y = 0$ implies $q_\kappa x - q_\kappa y = 0$; that is, $x \leq y$ implies $q_\kappa x \leq q_\kappa y$. From this relation, together with K4, one immediately deduces K3, as in our proof of Lemma **3.2** (v). It follows that in Keisler's axiomatization of $L(Q_1)$, the axiom

$$(Qx)\phi(x) \leftrightarrow (Qy)\phi(y)$$

is redundant.

In the proof of Lemma **2.5**, we showed that the formula $q_\kappa 1 = 1$ can be deduced from G1, G3, and G4. Thus, in $L(Q_1)$, the formula $(Qx)(x \equiv x)$ is a consequence of the axioms given by Keisler. It follows that there are no countable standard models for $L(Q_1)$. (In [4], it is mistakenly asserted that such models may exist.)

## REFERENCES

[1] Fuhrken, G., "Languages with the added quantifier, 'there exist at least $\aleph_\alpha$'," in *The Theory of Models*, edited by J. Addison, L. Henkin, and A. Tarski, North-Holland, Amsterdam (1965), pp. 121-131.

[2] Halmos, P., "Algebraic logic, I. Monadic Boolean algebras," *Compositio Mathematicae*, vol. 12 (1955), pp. 217-249.

[3] Henkin, L., D. Monk, and A. Tarski, *Cylindric Algebras*, *Part I*, North-Holland, Amsterdam (1971).

[4] Keisler, J., "Logic with the quantifier 'there exist uncountably many'," *Annals of Mathematical Logic*, vol. 1 (1970), pp. 1-93.

[5] Mostowski, A., "On a generalization of quantifiers," *Fundamenta Mathematicae*, vol. 44 (1957), pp. 12-36.

[6] Yasuhara, M., "An axiomatic system for first-order languages with an equicardinality quantifier," *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 633-640.

*Bucknell University*
*Lewisburg, Pennsylvania*