Notre Dame Journal of Formal Logic
Volume XVI, Number 4, October 1975
NDJFAM

THE LOGIC OF THE SYNTHETIC A PRIORI

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Modal logic, which recognizes two kinds of truth, the analytic and the contingent, and the corresponding two kinds of falsity, is well suited to the logical needs of those philosophies which recognize precisely those four modal values; for example, the conceptual pragmatism of C. I. Lewis, logical empiricism, and, among earlier philosophies, that of Hume, who distinguished 'relations of ideas' and 'matters of fact', and that of Leibniz, who contrasted truths based respectively on the law of contradiction and the principle of sufficient reason.

But there are philosophies which recognize also, and insist upon the importance of, the *synthetically* necessary and the corresponding kind of falsity; for example, various forms of realism, phenomenology, and neo-Kantianism, and of course the philosophy of Kant himself.

The purpose of the present paper* is to propose a six-valued calculus of propositions suited to the logical needs of those latter philosophies. Our procedure will be to adopt a standard system of modal logic and to add to it appropriately. From among the several closely related systems of modal logic we choose C. I. Lewis' S2, which he, the modern founder of modal logic, regarded as *the* System of Strict Implication, and which is strong enough for our purposes. He set it forth in considerable detail in [1], Chapter VI and Appendixes II and III, a presentation which will frequently be referred to in what follows in this paper.

We must, however, change the *readings* which Lewis ordinarily gave to his principal modal symbols. He usually read '$\sim\Diamond\sim p$' as 'p is necessary'; but, since we recognize two kinds of necessity, let us read it rather as 'p is analytically necessary'. He usually read '$\sim\Diamond p$' as 'p is impossible'; but since we recognize two kinds of impossibility, the one associated with analytic necessity, the other with synthetic necessity, let us read it as 'p is strictly impossible'—extending the use of his word 'strict'. Similarly,

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*I am indebted to William T. Parry, who read an earlier version of this paper at my request and made valuable suggestions.

Received May 23, 1972

'$\Diamond p$', which he read as 'p is possible', we shall read as 'p is strictly possible'; and '$\Diamond \sim p$', which he read as 'p is possibly false', we shall read as 'it is strictly possible that p is false' (or, preferably, as 'not-p is strictly possible').

We may call these four readings (as thus revised) the major forms or modes of strict expression, or, more briefly, the *strict modes.* We shall also call Lewis' distinctive symbols *strict symbols*, and his logic *strict logic.* Though our readings differ from the ones which Lewis customarily used, they are consistent with his intentions (since he identified the necessary with the analytic) and could have been accepted by him. From his point of view, though redundant they are correct. From our point of view they are needfully explicit and not redundant. Our interpretative revision makes no change, of course, in his symbolic structure. We turn now to our additions to Lewis' system.

1 *Postulates, etc.*

(a) *The Categorial Modes.* Four additional modes of expression present themselves at the outset. They may be called the *categorial modes:*

(1) p is synthetically necessary;
(2) p is categorially impossible;
(3) p is categorially possible;
(4) not-p is categorially possible.

Of these, (1) is familiar, owing to the history of philosophy; the other three presumably are not, but may be explained as follows: (2), to say that p is categorially impossible means that not-p is synthetically necessary; (3), to say that p is categorially possible means that p is not categorially impossible (or, alternatively and equivalently, it means that not-p is not synthetically necessary); the meaning of (4) is derived immediately from that of (3) by the substitution of not-p for p.

It will be noted that the four categorial modes are related to each other in the same manner as the strict modes. For 'p is *strictly* impossible' is equivalent to 'not-p is *analytically* necessary'; and 'p is strictly possible' is equivalent to 'p is not strictly impossible' (or alternatively 'not-p is not analytically necessary'); the meaning of 'not-p is strictly possible' following immediately from 'p is strictly possible'.

(b) *Primitive Symbol.* Any one of the four categorial modes could serve as primitive. The temptation is strong to choose (3), 'p is categorially possible', for the sake of obvious analogy with Lewis' presentation, in which the modal primitive is 'p is strictly possible'. But let us rather choose (1), 'p is synthetically necessary', which has the advantage of being more familiar and philosophically more important than (3) and the others, and which has already been used to explain them; and let us symbolize it by

$$\mathbf{S}p$$

which is accordingly our primitive symbol. (This choice of notation is suggested by the fact that **S** is the capitalized initial of the first word in 'synthetically necessary'.[1]

(c) *Formation Rule.* Our rule to be added to Lewis' (implicit) formation rules is:

If α is well-formed, $\mathbf{S}\alpha$ is well-formed.

(d) *Readings in Lieu of Definitions.*

$\mathbf{S}{\sim}p$	p is categorially impossible
${\sim}\mathbf{S}{\sim}p$	p is categorially possible
${\sim}\mathbf{S}p$	not-p is categorially possible

In Lewis' system, since it does not recognize synthetic necessity, the statement that p is contingently true may be symbolized as $p . \Diamond {\sim} p$; and similarly 'p is contingently false' as ${\sim} p . \Diamond p$. But in the present system these analyses are insufficient. We have instead:

$p . \Diamond {\sim} p . {\sim}\mathbf{S}p$	p is contingently true
${\sim}p . \Diamond p . {\sim}\mathbf{S}{\sim}p$	p is contingently false

We are now provided with symbols corresponding to the six *values* of the system: ${\sim}\Diamond{\sim}p$; $\mathbf{S}p$; ${\sim}\Diamond p$; $\mathbf{S}{\sim}p$; $p . \Diamond {\sim} p . {\sim}\mathbf{S}p$; ${\sim}p . \Diamond p . {\sim}\mathbf{S}{\sim}p$.

They are to be distinguished from, though they overlap, the symbols for the four strict and the four categorial modes: ${\sim}\Diamond{\sim}p$; ${\sim}\Diamond p$; $\Diamond p$; $\Diamond{\sim}p$; $\mathbf{S}p$; $\mathbf{S}{\sim}p$; ${\sim}\mathbf{S}{\sim}p$; ${\sim}\mathbf{S}p$.

An unfamiliar form of implication, which we may call *categorial implication*, belongs to the system:

$\mathbf{S}(p \supset q)$ p categorially implies q

(e) *Postulates.*

1.1 $\mathbf{S}p . \prec\!\!3 . p$
1.2 $\mathbf{S}p . \prec\!\!3 . \Diamond{\sim}p$
1.3 $\Diamond(pq) . {\sim}\mathbf{S}{\sim}(pq) : \prec\!\!3 . {\sim}\mathbf{S}{\sim}p$
1.4 $\mathbf{S}p . \mathbf{S}q : \prec\!\!3 . \mathbf{S}(pq)$
1.5 $\mathbf{S}p . \mathbf{S}(p \supset q) : \prec\!\!3 . \mathbf{S}q$

(f) *Discussion of the Postulates.* It will be observed that in each postulate the symbol in the *principal* position is strict. That is as it should be, for all laws of logic are analytic.

The postulates carry a double burden: (1) that of formulating distinctive categorial properties, and (2) that of expressing interconnections between the strict and the categorial.

Postulate 1.1, which is justified by the meaning of 'necessity', is what we may call the *categorial analogue of* Lewis' 18.42: ${\sim}\Diamond{\sim}p \prec\!\!3 p$. This

1. In the earlier version of this paper categorial possibility was adopted as the primitive. Parry convinced me that that choice was a 'tactical mistake'.

phrase will be used frequently in what follows. To say that a formula β is the categorial analogue of a formula α means that α contains at least one strict symbol in *subordinate* position and that β is identical with α except that each such symbol in α is replaced in β by the corresponding categorial symbol; i.e., the symbol for analytic necessity is replaced by that for synthetic necessity, that for strict impossibility by that for categorial impossibility, etc. Note that the symbol in the *principal* position in α remains unchanged in β. Thus 18.42 and 1.1 are the same except that the symbol for analytic necessity, being in a subordinate position in the former, is replaced in the latter by the symbol for synthetic necessity, while the symbol for strict implication, which is in the principal position in the former, is unchanged in the latter.

Postulate 1.2 is justified by the established meaning of the terms 'analytic' and 'synthetic', according to which they are exclusive of each other. Hence what is synthetically necessary is not analytically necessary, and what is analytically necessary is not synthetically necessary. This exclusiveness is formulated by 1.2. For, by double negation, the postulate is strictly equivalent to $\mathbf{S}p\,.\,{-\!\!\!3}\,.\sim(\sim\Diamond\sim p)$ (that p is synthetically necessary strictly implies that it is not analytically necessary); and, by transposition, it is strictly equivalent to $\sim\Diamond\sim p\,.\,{-\!\!\!3}\,.\sim(\mathbf{S}p)$ (that p is analytically necessary strictly implies that p is not synthetically necessary).

This postulate has certain consequences which at first acquaintance may seem unacceptably paradoxical; for example, three theorems from the next section:

2.73 $\sim\mathbf{S}(p \supset p)$
2.74 $\sim\mathbf{S}(p \vee \sim p)$
2.75 $\sim\mathbf{S}\sim(p\,.\sim p)$

But the sense of strangeness and questionableness should vanish when one recognizes that these theorems do not deny, or challenge, or throw doubt on the laws of identity, excluded middle, and contradiction. They merely point out that since those laws are analytic they are therefore not synthetic.

There is another and perhaps more fundamental version of the alleged paradox.[2] Postulate 1.2 has the consequence (Theorem 2.62): $\sim\Diamond p\,.\,{-\!\!\!3}\,.\sim\mathbf{S}\sim p$ (if p is strictly impossible it is categorially possible). This may be regarded as unacceptable on the ground that whatever is strictly impossible must be impossible in *every* sense; for example, it must be physically impossible.

To this we reply as follows. Let us take the example of physical impossibility. If p is strictly impossible then indeed it is physically impossible. But equally, if it is categorially impossible it is physically impossible. To be physically possible it must be both strictly and categorially possible (and must also satisfy the *empirical* criteria for physical possibility laid down by the physicist). The fact that in violating

2. This objection was suggested to me by Parry.

one of the first two requirements p automatically satisfies the other—as is entailed by the exclusiveness of the analytically necessary and the synthetically necessary—does not save p from being physically impossible and should not be regarded as objectionable. When categorial ideas have been admitted to the system, the generalization that whatever is strictly impossible must be impossible in every sense does not hold. It is replaced by the principle that whatever is *a priori* impossible (i.e., either strictly or categorially impossible) is impossible in every *a posteriori* sense.

Postulate 1.3 is related to Lewis' Consistency Postulate (19.01), $\Diamond(pq) \prec \Diamond p$, which is the distinctive postulate of S2, but the relation is somewhat complicated. One might expect that the categorial analogue, $\sim\mathbf{S}\sim(pq) \prec \sim\mathbf{S}\sim p$, would hold, but that turns out to be not the case. For, assuming that formula and writing $\sim p/q$, we could then assert the antecedent (by 2.75) and hence by detachment could assert $\sim\mathbf{S}\sim p$ (for all values of p). Thus we would arrive at the result that all propositions are categorially possible and hence that no proposition is categorially impossible, from which it follows also (by $\sim p/p$) that no proposition is synthetically necessary. So the assertion of the categorial analogue of the Consistency Postulate would destroy the whole point of the present system and reduce it to a mere redundant version of Lewis' logic. Therefore that formula must be rejected.

Rather, however, than *merely* rejecting it there is advantage in asserting it *subject to appropriate limitation.* And that is precisely what is done by Postulate 1.3. The appropriate limitation, or restriction, or proviso is arrived at by comparing the substitutional device, $\sim p/q$, with the alternative substitutions which produce the same untoward result. What all these substitutions have in common is the fact that they specify values of q which are (strictly) inconsistent with p. Therefore the appropriate limitation is the requirement that p and q shall not be inconsistent with each other, i.e., that $\Diamond(pq)$.

This proviso may be attached to the formula in either of two ways:

(1) $\Diamond(pq) . \prec : \sim\mathbf{S}\sim(pq) . \prec . \sim\mathbf{S}\sim p$
(2) $\Diamond(pq) . \sim\mathbf{S}\sim(pq) : \prec . \sim\mathbf{S}\sim p$

The second, the weaker, is adopted as our Postulate 1.3. The reason for the choice is the fact that if (1), the stronger, were adopted, we should then wish, for the sake of uniformity, to formulate its consequences in the same stronger form with the proviso standing by itself as the antecedent of the entire expression; but S2 would be insufficient for the deduction of several of the theorems in that form and S5 would be required. In Postulate 1.3 and its consequences, the presence of the proviso lessens, of course, but does not destroy the usefulness of the formulae to which it is attached. These formulae *do* hold—except in the extreme cases, the nuisance cases, in which the relevant propositions are strictly inconsistent with each other.

Postulate 1.4. Section 5 of Chapter VI of [1] consists of theorems which are consequences of the Consistency Postulate. Their categorial analogues fall into two groups. Those in one group are all refutable as they stand in essentially the same manner as the categorial analogue of the Consistency Postulate itself; they all hold only with proviso; and in that limited form they are all deducible from 1.3. Those in the second group hold without limitation. Postulate 1.4 is one of that group. The others in that group are deducible from it.

Postulate 1.5 is the categorial analogue of the principle: $\sim\Diamond\sim p\,.\,p \prec q : \prec .\sim\Diamond\sim q$. It is one of the two forms of what may be called categorial *modus ponens*, the other being 2.31.

One might have expected to find among the postulates a syllogistic principle, the analogue of Lewis' 11.6. But it, like the analogue of the Consistency Postulate, turns out to hold only with proviso. In that limited form (5.4) it is deducible from 1.5 with the assistance of 1.3.

(g) *Use of* S2 *in proofs*. Principles of S2—definitions, postulates, transformation rules, and theorems—will be used in our proofs of theorems. As in Lewis' presentation, so here, his transformation rules, [1], pp. 125-6, are used tacitly. They, as well as his definitions, postulates, and theorems, are to be construed as applying to any well formed formulae in our system. The following principles will be referred to by their serial numbers in [1]:

11.01	$p \vee q \;.{=}.\; \sim(\sim p \sim q)$
11.02	$p \prec q \;.{=}.\; \sim\Diamond(p \sim q)$
11.03	$p = q \;.{=}{:}\; p \prec q \,.\, q \prec p$
11.2	$pq \,.\prec.\, p$
12.77	$p \prec q : qr \,.\prec.\, s \therefore \prec : pr \,.\prec.\, s$
13.2	$p \,.\prec.\, p \vee q$
13.5	$p \vee \sim p$
13.7	$q \,.\prec.\, p \vee \sim p$
14.01	$p \supset q \;.{=}.\; \sim(p \sim q)$
14.1	$p \prec q \,.\prec.\, p \supset q$
14.2	$p \supset q \;.{=}.\; \sim p \vee q$
14.21	$pq \;.{=}.\; \sim(\sim p \vee \sim q)$
14.29	$p \,.\, p \supset q : \prec .\, q$
16.33	$p \prec q \;.{=}{:}\; p \,.\prec.\, pq$
16.35	$p \;.{=}{:}\; p \vee q \,.\, p$
16.38	$p \;.{=}{:}\; p \,.\, q \vee \sim q$
16.73	$p \,.\vee.\, qr :{=}: p \vee q \,.\, p \vee r$
16.8	$p \,.\supset.\, qr :{=}: p \supset q \,.\, p \supset r$
18.4	$p \prec \Diamond p$
18.42	$\sim\Diamond\sim p \prec p$
18.8	$\sim\Diamond(p \sim p)$
18.81	$\sim\Diamond\sim(p \vee \sim p)$
19.61	$p \prec q \,.\, p \prec r : \prec : p \,.\prec.\, qr$
19.65	$p \prec r \,.\, q \prec r : \prec : p \vee q \,.\prec.\, r$
19.68	$p \prec r \,.\, q \prec s : \prec : pq \,.\prec.\, rs$

19.71 $\Diamond p .=: \Diamond (pq) .\vee. \Diamond (p \sim q)$
19.87 $p \strictif q . \strictif : p .=. pq$

For the shortening of certain proofs it is desirable to have available also the following seven theorems, which Lewis had no occasion to record but which are readily deducible in S2. The serial numbers assigned to them suggest points at which they may be inserted in his account, the suffix 'a' indicating that they are added:

12.31a $\sim (pq) .=. \sim p \vee \sim q$
12.32a $\sim (p \vee q) .=. \sim p \sim q$
12.771a $p \strictif q : rq . \strictif . s \therefore \strictif : rp . \strictif . s$
16.721a $pq . r \vee s : \strictif : pr .\vee. qs$
16.722a $pq . \sim (qr) :=. pq \sim r$
16.82a $p \supset r . q \supset r :=: p \vee q .\supset. r$
18.521a $\sim \Diamond \sim p . \Diamond q : \strictif . \Diamond (pq)$

The following familiar principles are referred to by abbreviation or nickname:

Assoc-C	12.5	$(pq)r .=. p(qr) .=. q(pr) .=. (qp)r$, etc., etc.
Assoc-D	13.41	$p \vee (q \vee r) .=. (p \vee q) \vee r .=. q \vee (p \vee r) .=. (q \vee p) \vee r$, etc., etc.
DeM	12.31a	$\sim (pq) .=. \sim p \vee \sim q$
DeM	12.32a	$\sim (p \vee q) .=. \sim p \sim q$
Dist	16.72	$p . q \vee r :=: pq .\vee. pr$
DN	12.3	$p = \sim (\sim p)$
Id-E	12.11	$p = p$
Id-I	12.1	$p \strictif p$
LF	12.6	$pq . \strictif . r :=: q \sim r . \strictif . \sim p :=: p \sim r . \strictif . \sim q$
Perm-C	12.15	$pq .=. qp$
Perm-D	13.11	$p \vee q .=. q \vee p$
Syl	11.6	$p \strictif q . q \strictif r : \strictif . p \strictif r$
Taut-C	12.7	$p .=. pp$
Taut-D	13.31	$p .=. p \vee p$
Transp	12.44	$p \strictif q .=. \sim q \strictif \sim p$
Transp	12.2	$\sim p \strictif q .=. \sim q \strictif p$
Transp	12.45	$p \strictif \sim q .=. q \strictif \sim p$

Lewis' compendious statement of forms of the Ladd-Franklin principle (LF) is to be understood as including also the equivalences:

$$pq . \strictif . r :=: \sim rq . \strictif . \sim p :=: \sim rp . \strictif . \sim q$$

(which follow immediately by Perm-C from LF as he states it). We adopt Lewis' practice of regarding Assoc-C and Assoc-D as sufficient warrant for omitting brackets in multiple conjunctions and multiple disjunctions. Reference to the associative laws for that purpose will be tacit.

We turn now to a selection of theorems of what we may call *categorial logic.*

2 *Consequences* 1.1 *and* 1.2

2.1 $\mathbf{S}{\sim}p\,.\prec.\,{\sim}p$ [1.1 ${\sim}p/p$]
2.11 ${\sim}p\,.\prec.\,{\sim}\mathbf{S}p$ [1.1; Transp]
2.12 $p\,.\prec.\,{\sim}\mathbf{S}{\sim}p$ [2.1; Transp]
2.13 $\mathbf{S}p\,.\prec.\,{\sim}\mathbf{S}{\sim}p$ [1.1; 2.12; Syl]
2.14 $\mathbf{S}{\sim}p\,.\prec.\,{\sim}\mathbf{S}p$ [2.13 ${\sim}p/p$; DN]
2.15 $\mathbf{S}p\,.\prec.\,\Diamond p$ [1.1; 18.4; Syl]
2.16 $\mathbf{S}{\sim}p\,.\prec.\,\Diamond{\sim}p$ [2.15 ${\sim}p/p$]
2.17 ${\sim}\Diamond p\,.\prec.\,{\sim}\mathbf{S}p$ [2.15; Transp]
2.18 ${\sim}\Diamond{\sim}p\,.\prec.\,{\sim}\mathbf{S}{\sim}p$ [2.17 ${\sim}p/p$]
2.2 ${\sim}\mathbf{S}{\sim}p\,.\mathrm{v}.\,{\sim}\mathbf{S}p$ [2.14; 14.1; 14.2]
2.21 $\Diamond p\,.\mathrm{v}.\,{\sim}\mathbf{S}p$ [2.17; 14.1; 14.2; DN]
2.22 ${\sim}\mathbf{S}{\sim}p\,.\mathrm{v}.\,\Diamond{\sim}p$ [2.16; 14.1; 14.2]
2.3 $\mathbf{S}(p\supset q)\,.\prec.\,p\supset q$ [1.1 $p\supset q/p$]
2.31 $p\,.\,\mathbf{S}(p\supset q):\prec.\,q$ [14.29; 2.3; 12.771a]
2.32 ${\sim}q\,.\,\mathbf{S}(p\supset q):\prec.\,{\sim}p$ [2.31; LF]
2.33 ${\sim}\Diamond{\sim}p\,.\,\mathbf{S}q:\prec.\,\Diamond(pq)$ [2.15q/p; 18.521a; 12.771a]
2.4 $p\,.\prec\therefore{\sim}\Diamond{\sim}p:\mathrm{v}:\mathbf{S}p:\mathrm{v}:p\,.\,\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p$

PR [13.7] $p\,.\prec\therefore{\sim}\Diamond{\sim}p\,.\mathrm{v}.\,\mathbf{S}p:\mathrm{v}.\,{\sim}({\sim}\Diamond{\sim}p\,.\mathrm{v}.\,\mathbf{S}p)$ (1)
[DeM; DN] (1) $.=::\,p\,.\prec\therefore{\sim}\Diamond{\sim}p\,.\mathrm{v}.\,\mathbf{S}p:\mathrm{v}:\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p$ (2)
[16.33; Dist] (2) $.=::\,p\,.\prec\therefore p\,.\,{\sim}\Diamond{\sim}p:\mathrm{v}:p\,.\,\mathbf{S}p:\mathrm{v}:p\,.\,\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p$ (3)
[18.42; 19.87] $p\,.\,{\sim}\Diamond{\sim}p:=.\,{\sim}\Diamond{\sim}p$ (4)
[1.1; 19.87] $p\,.\,\mathbf{S}p:=.\,\mathbf{S}p$ (5)
[(4) . (5)] (3) .=. Q.E.D.

2.41 ${\sim}\Diamond{\sim}p:\mathrm{v}:\mathbf{S}p:\mathrm{v}:p\,.\,\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p\therefore\prec.\,p$ [18.42; 1.1; 11.2; 19.65]
2.42 $p\,.=\therefore{\sim}\Diamond{\sim}p:\mathrm{v}:\mathbf{S}p:\mathrm{v}:p\,.\,\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p$ [2.4; 2.41; 11.03]
2.43 ${\sim}p\,.=\therefore{\sim}\Diamond p:\mathrm{v}:\mathbf{S}{\sim}p:\mathrm{v}:{\sim}p\,.\,\Diamond p\,.\,{\sim}\mathbf{S}{\sim}p$ [2.42${\sim}p/p$; DN]
2.44 ${\sim}\Diamond{\sim}p:\mathrm{v}:\mathbf{S}p:\mathrm{v}:p\,.\,\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p:\mathrm{v}:{\sim}\Diamond p:\mathrm{v}:\mathbf{S}{\sim}p:\mathrm{v}:{\sim}p\,.\,\Diamond p\,.\,{\sim}\mathbf{S}{\sim}p$
[Id-I $p\vee{\sim}p/p$; 2.42; 2.43; 13.5]

2.44 shows that the six values of the system are *exhaustive*. For their exclusiveness see 2.8 and the comment introducing it.

2.6 $\mathbf{S}{\sim}p\,.\prec.\,\Diamond p$ [1.2${\sim}p/p$; DN]
2.61 ${\sim}\Diamond{\sim}p\,.\prec.\,{\sim}\mathbf{S}p$ [1.2; Transp]
2.62 ${\sim}\Diamond p\,.\prec.\,{\sim}\mathbf{S}{\sim}p$ [2.6; Transp]
2.7 $\Diamond p\,.\mathrm{v}.\,{\sim}\mathbf{S}{\sim}p$ [2.62; 14.1; 14.2; DN]
2.71 $p\prec q\,.\prec.\,{\sim}\mathbf{S}(p\supset q)$ [2.62 $p{\sim}q/p$; 11.02; 14.01]
2.72 $\mathbf{S}(p\supset q)\,.\prec.\,{\sim}(p\prec q)$ [2.71; Transp]
2.73 ${\sim}\mathbf{S}(p\supset p)$ [2.71 p/q; Id-I]
2.74 ${\sim}\mathbf{S}(p\vee{\sim}p)$ [2.73; 14.2; Perm-D]
2.75 ${\sim}\mathbf{S}{\sim}(p{\sim}p)$ [2.62 $p{\sim}p/p$; 18.8]

The detailed exhibition of the *exclusiveness* of the six values requires six lengthy theorems, of which only one need be given, as a sample:

2.8 $\mathbf{S}p\,.\prec:{\sim}({\sim}\Diamond{\sim}p)\,.\,{\sim}({\sim}\Diamond p)\,.\,{\sim}({\sim}\mathbf{S}p)\,.\,{\sim}(p\,.\,\Diamond{\sim}p\,.\,{\sim}\mathbf{S}p)\,.\,{\sim}({\sim}p\,.\,\Diamond p\,.\,{\sim}\mathbf{S}{\sim}p)$

PR [1.2; DN] $\mathbf{S}p \,.\, \prec\!\!\!\!-\!\!\!3 \,.\, \sim(\sim\Diamond\sim p)$ (1)

[2.15; DN] $\mathbf{S}p \,.\, \prec\!\!\!\!-\!\!\!3 \,.\, \sim(\sim\Diamond p)$ (2)

[Id-I $\mathbf{S}p/p$; DN] $\mathbf{S}p \,.\, \prec\!\!\!\!-\!\!\!3 \,.\, \sim(\sim\mathbf{S}p)$ (3)

[13.2 $\mathbf{S}p/p$, $\sim p \vee \sim\Diamond\sim p/q$; Assoc-D; 11.01; DN]
$\mathbf{S}p \,.\, \prec\!\!\!\!-\!\!\!3 \,.\, \sim(p \,.\, \Diamond\sim p \,.\, \sim\mathbf{S}p)$ (4)

[1.1; 13.2 $\sim\Diamond p \vee \mathbf{S}\sim p/q$; Syl; 11.01; DN] $\mathbf{S}p \,.\, \prec\!\!\!\!-\!\!\!3 \,.\, \sim(\sim p \,.\, \Diamond p \,.\, \sim\mathbf{S}\sim p)$ (5)

[19.61] (1)-(5) . $\prec\!\!\!\!-\!\!\!3$. Q.E.D.

3 *Consequences of* 1.3. This section presents limited categorial analogues of theorems in Lewis' Section 5, which he entitled 'The Consistency Postulate and its Consequences'. Those of his theorems in that section which are irreversible implications have categorial analogues which require provisos. In the case of each of his theorems of equivalence in that section, the analogue of one of the two constituent implications similarly requires a proviso, but the analogue of the other does not and is treated in the next section. (This does not apply to the equivalences 19.57 and 19.58, which, containing no strict symbol in subordinate position, have no categorial analogues, and which as a matter of fact belong properly in an earlier section as is shown in Appendix III of [1], p. 505.)

For convenience of comparison the theorems in the present and the next section are given serial numbers related to the numbers of the corresponding theorems in Lewis' Section 5, namely, identical decimals. Thus the limited analogue of Lewis' 19.02 is our 3.02, and the (unlimited) analogue of his 19.61 is our 4.61. Most of the proofs in the present section are closely related to the ones given by Lewis for his corresponding theorems. It will be noted that the proviso varies from theorem to theorem depending on the substitutions required. From now on the use of Perm-C and of DN is so frequent and so obvious that reference to them will be tacit.

3.02	$\Diamond p \,.\, \sim\mathbf{S}\sim p : \prec\!\!\!\!-\!\!\!3 \,.\, \sim\mathbf{S}\sim(p \vee q)$	[1.3 $p \vee q/p$, p/q; 16.35]
3.13	$\Diamond(pq) \,.\, \sim\mathbf{S}\sim(pq) : \prec\!\!\!\!-\!\!\!3 \,.\, \sim\mathbf{S}\sim q$	[1.3 p/q, q/p]
3.14	$\Diamond(pq) \,.\, \sim\mathbf{S}\sim(pq) : \prec\!\!\!\!-\!\!\!3 : \sim\mathbf{S}\sim p \,.\, \sim\mathbf{S}\sim q$	[1.3; 3.13; 19.61]
3.16	$\Diamond(pq) \,.\, \mathbf{S}\sim p : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}\sim(pq)$	[1.3; LF]
3.17	$\Diamond(pq) \,.\, \mathbf{S}\sim q : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}\sim(pq)$	[3.13; LF]
3.2	$\Diamond\sim p \,.\, \sim\mathbf{S}p : \prec\!\!\!\!-\!\!\!3 \,.\, \sim\mathbf{S}(pq)$	[3.02 $\sim p/p$, $\sim q/q$; 14.21]
3.23	$\Diamond\sim p \,.\, \mathbf{S}(pq) : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}p$	[3.2; LF]
3.24	$\Diamond\sim q \,.\, \mathbf{S}(pq) : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}q$	[3.23 p/q, q/p]
3.25	$\Diamond\sim p\,\Diamond\sim q \,.\, \mathbf{S}(pq) : \prec\!\!\!\!-\!\!\!3 : \mathbf{S}p \,.\, \mathbf{S}q$	[3.23; 3.24; 19.68; Assoc-C; Taut-C]
3.26	$\Diamond p \,.\, \mathbf{S}\sim(p \vee q) : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}\sim p$	[3.02; LF]
3.27	$\Diamond q \,.\, \mathbf{S}\sim(p \vee q) : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}\sim q$	[3.26 p/q, q/p; Perm-D]
3.28	$\Diamond p\,\Diamond q \,.\, \mathbf{S}\sim(p \vee q) : \prec\!\!\!\!-\!\!\!3 : \mathbf{S}\sim p \,.\, \mathbf{S}\sim q$	[3.25 $\sim p/p$, $\sim q/q$; DeM]
3.32	$\Diamond p\,\Diamond q : \sim\mathbf{S}\sim p \,.\!\vee\!.\, \sim\mathbf{S}\sim q \therefore \prec\!\!\!\!-\!\!\!3 \,.\, \sim\mathbf{S}\sim(p \vee q)$	[3.28; LF; DeM]
3.33	$\Diamond(\sim p \sim q) \,.\, \mathbf{S}p : \prec\!\!\!\!-\!\!\!3 \,.\, \mathbf{S}(p \vee q)$	[3.16 $\sim p/p$, $\sim q/q$; 11.01]

When Lewis uses his dyadic symbol of consistency (*small circle*) we first translate it, by 18.1 and 18.3, before constructing the analogue.

3.44 $\Diamond(pqr) . \sim\mathbf{S}\sim(pqr) : \prec : \sim\mathbf{S}\sim(pq) . \sim\mathbf{S}\sim(pr)$
[3.14 pq/p, pr/q; Assoc-C; Taut-C]

3.46 $\Diamond(p \sim q \sim r) : \mathbf{S}(p \supset q) .\text{v}. \mathbf{S}(p \supset r) \therefore \prec . \mathbf{S}(p .\supset. q \vee r)$
[3.44$\sim q/q$, $\sim r/r$; LF; 11.01; DeM; 14.01]

3.48 $\Diamond(p \sim q \sim r) . \mathbf{S}(p \supset q) : \prec . \mathbf{S}(p .\supset. q \vee r)$
[3.46; 13.2 $\mathbf{S}(p \supset q)/p$, $\mathbf{S}(p \supset r)/q$; 12.771a]

3.5 $\Diamond(pq \sim r) : \mathbf{S}(p \supset r) .\text{v}. \mathbf{S}(q \supset r) \therefore \prec . \mathbf{S}(pq .\supset. r)$
[3.44$\sim r/p$, p/r; LF; DeM; 14.01]

3.51 $\Diamond(pq \sim r) . \mathbf{S}(p \supset r) : \prec . \mathbf{S}(pq .\supset. r)$
[3.5; 13.2 $\mathbf{S}(p \supset r)/p$, $\mathbf{S}(q \supset r)/q$; 12.771a]

3.6 $\Diamond(p \sim qr) . \mathbf{S}(p \supset q) : \prec . \mathbf{S}(pr .\supset. qr)$
[3.16 $p \sim q/p$, r/q; 16.722a r/q, q/r; 14.01]

3.62 $\Diamond(p \sim q) \Diamond(p \sim r) . \mathbf{S}(p .\supset. qr) : \prec : \mathbf{S}(p \supset q) . \mathbf{S}(p \supset r)$
[3.28 $p \sim q/p$, $p \sim r/q$; Dist; DeM; 14.01]

3.64 $\Diamond(p \sim q \sim r) . \mathbf{S}(p \supset q) : \prec . \mathbf{S}(p \vee r .\supset. q \vee r)$
[3.6$\sim q/p$, $\sim p/q$, $\sim r/r$; DeM; Transp]

3.66 $\Diamond(p \sim r) \Diamond(q \sim r) . \mathbf{S}(p \vee q .\supset. r) : \prec : \mathbf{S}(p \supset r) . \mathbf{S}(q \supset r)$
[3.62$\sim r/p$, $\sim p/q$, $\sim q/r$; Transp; 11.01]

3.68 $\Diamond(pq \sim r) \Diamond(pq \sim s) : \mathbf{S}(p \supset r) . \mathbf{S}(q \supset s) \therefore \prec . \mathbf{S}(pq .\supset. rs)$

PR [19.68] 3.16 $p \sim r/p$. 3.17 $q \sim s/q : \prec :: \Diamond(pq \sim r) \Diamond(pq \sim s) :$
$\mathbf{S}\sim(p \sim r) . \mathbf{S}\sim(q \sim s) \therefore \prec : \mathbf{S}\sim(pq \sim r) . \mathbf{S}\sim(pq \sim s)$ (1)
[1.4 $\sim(pq \sim r)/p$, $\sim(pq \sim s)/q$; DeM] $\mathbf{S}\sim(pq \sim r) . \mathbf{S}\sim(pq \sim s) : \prec :$
$\mathbf{S}\sim(pq \sim r .\text{v}. pq \sim s)$ (2)
[Dist; DeM] (2) $.{=}\therefore \mathbf{S}\sim(pq \sim r) . \mathbf{S}\sim(pq \sim s) : \prec . \mathbf{S}\sim[pq . \sim(rs)]$ (3)
[Syl; 14.01] (1) . (3) : $\prec$. Q.E.D.

3.681 $\Diamond(p \sim r \sim s) \Diamond(q \sim r \sim s) : \mathbf{S}(p \supset r) . \mathbf{S}(q \supset s) \therefore \prec . \mathbf{S}(p \vee q .\supset. r \vee s)$
[3.68 $\sim r/p$, $\sim s/q$, $\sim p/r$, $\sim q/s$; Transp; 11.01]

3.691 $\Diamond(pq) \Diamond(pr) : \sim\mathbf{S}\sim(pq) .\text{v}. \sim\mathbf{S}\sim(pr) \therefore \prec . \sim\mathbf{S}\sim(p . q \vee r)$
[3.62 $\sim q/q$, $\sim r/r$; LF; DeM; 14.01; 11.01]

3.7 $\Diamond(pq) \Diamond(p \sim q) : \sim\mathbf{S}\sim(pq) .\text{v}. \sim\mathbf{S}\sim(p \sim q) \therefore \prec . \sim\mathbf{S}\sim p$

PR [1.3; 1.3$\sim q/q$; 19.65] $\Diamond(pq) . \sim\mathbf{S}\sim(pq) :\text{v}: \Diamond(p \sim q)$.
$\sim\mathbf{S}\sim(p \sim q) \therefore \prec . \sim\mathbf{S}\sim p$ (1)
[16.721a] $\Diamond(pq) \Diamond(p \sim q) : \sim\mathbf{S}\sim(pq) .\text{v}. \sim\mathbf{S}\sim(p \sim q) \therefore \prec \therefore$
$\Diamond(pq) \sim\mathbf{S}\sim(pq) :\text{v}: \Diamond(p \sim q) . \sim\mathbf{S}\sim(p \sim q)$ (2)
[Syl] (1) . (2) : $\prec$. Q.E.D.

3.72 $\Diamond(pq) \Diamond(p \sim q) . \mathbf{S}\sim p : \prec : \mathbf{S}(p \supset \sim q) . \mathbf{S}(p \supset q)$ [3.7; LF; 14.21; 14.01]

3.73 $\Diamond(\sim pq) \Diamond(\sim p \sim q) . \mathbf{S}p : \prec : \mathbf{S}(q \supset p) . \mathbf{S}(\sim q \supset p)$ [3.72$\sim p/p$; Transp]

3.74 $\Diamond(p \sim q) . \mathbf{S}\sim p : \prec . \mathbf{S}(p \supset q)$ [3.16$\sim q/q$; 14.01]

3.75 $\Diamond(\sim pq) . \mathbf{S}p : \prec . \mathbf{S}(q \supset p)$ [3.74$\sim p/p$, $\sim q/q$; Transp]

For the analogues of the constituent implications in Lewis' important equivalence 19.8, see 3.28 and 4.8. For the analogues of 19.81, see 3.25 and 1.4. For the analogues of 19.82, see 3.32 and 4.82.

4 *Consequences of* 1.4.

4.61 $\mathbf{S}(p \supset q) . \mathbf{S}(p \supset r) : \prec . \mathbf{S}(p .\supset. qr)$ [1.4 $p \supset q/p$, $p \supset r/q$; 16.8]

4.65 $\mathbf{S}(p \supset r) . \mathbf{S}(q \supset r) : \prec . \mathbf{S}(p \vee q . \supset . r)$ [1.4 $p \supset r/p$, $q \supset r/q$; 16.82a]

4.69 $\sim \mathbf{S} \sim (p . q \vee r) . \prec : \sim \mathbf{S} \sim (pq) . \vee . \sim \mathbf{S} \sim (pr)$ [1.4 $\sim p \vee \sim q/p$, $\sim p \vee \sim r/q$; 16.73; Transp; DeM]

4.7 $\sim \mathbf{S} \sim p . \prec : \sim \mathbf{S} \sim (pq) . \vee . \sim \mathbf{S} \sim (p \sim q)$ [4.69 $\sim q/r$; 16.38]

4.72 $\mathbf{S}(p \supset \sim q) . \mathbf{S}(p \supset q) : \prec . \mathbf{S} \sim p$ [4.7; Transp; DeM; 14.01]

4.73 $\mathbf{S}(q \supset p) . \mathbf{S}(\sim q \supset p) : \prec . \mathbf{S} p$ [4.72 $\sim p/p$; Transp]

4.8 $\mathbf{S} \sim p . \mathbf{S} \sim q : \prec . \mathbf{S} \sim (p \vee q)$ [1.4 $\sim p/p$, $\sim q/q$; DeM]

4.82 $\sim \mathbf{S} \sim (p \vee q) . \prec : \sim \mathbf{S} \sim p . \vee . \sim \mathbf{S} \sim q$ [1.4 $\sim p/p$, $\sim q/q$; Transp; DeM]

5 *Consequences of* 1.5.

5.1 $\sim \mathbf{S} q . \mathbf{S}(p \supset q) : \prec . \sim \mathbf{S} p$ [1.5; LF]

5.11 $\mathbf{S} \sim q . \mathbf{S}(p \supset q) : \prec . \mathbf{S} \sim p$ [1.5 $\sim q/p$, $\sim p/q$; Transp]

5.12 $\sim \mathbf{S} \sim p . \mathbf{S}(p \supset q) : \prec . \sim \mathbf{S} \sim q$ [5.11; LF]

5.15 $\mathbf{S} \sim q . \mathbf{S} \sim (p \sim q) : \prec . \mathbf{S} \sim p$ [5.11; 14.01]

5.2 $\sim \mathbf{S} \sim p . \mathbf{S} q : \prec . \sim \mathbf{S} \sim (pq)$ [5.12 $\sim q/q$; LF; 14.01]

5.21 $\sim \Diamond \sim p . \mathbf{S} q : \prec . \sim \mathbf{S} \sim (pq)$ [2.18; 5.2; 12.77]

5.3 and 5.31 are lemmas to 5.4, which establishes the categorial analogue of the syllogistic principle 11.6, with the least demanding proviso.

5.3 $\Diamond (pq \sim r) : \mathbf{S}(p \supset q) . \mathbf{S}(q \supset r) \therefore \prec . \mathbf{S}(p \supset r)$
[5.15 $p \sim r/p$, $p \sim q/q$; 16.722a $\sim r/p$, p/q, $\sim q/r$; 3.17 $q \sim r/q$; 12.771a; 14.01; Assoc-C]

5.31 $\Diamond (p \sim q \sim r) : \mathbf{S}(p \supset q) . \mathbf{S}(q \supset r) \therefore \prec . \mathbf{S}(p \supset r)$
[5.15 $p \sim r/p$, $q \sim r/q$; 16.722a $\sim r/q$, q/r; 3.16 $p \sim q/p$, $\sim r/q$; 12.771a; 14.01; Assoc-C]

5.4 $\Diamond (p \sim r) : \mathbf{S}(p \supset q) . \mathbf{S}(q \supset r) \therefore \prec . \mathbf{S}(p \supset r)$
[5.3; 5.31; 19.65; Dist; 19.71]

REFERENCE

[1] Lewis, C. I., and C. H. Langford, *Symbolic Logic*, second edition, Dover Publications, New York (1959). The second edition is essentially the same as the first (1932) except for the addition of Appendix III.

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