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CONCERNING THE POSTULATE-SYSTEMS OF SUBTRACTIVE ABELIAN GROUPS

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It is rather well-known, *cf.*, e.g., [5], p. 256, that in the field of Groups Theory, instead of operation $a + b$, the inverse operation $a - b$, which is defined as follows: $a - b = c$.≡. $a = b + c$ for all elements a, b, and c of the given algebra, can be accepted as a sole primitive notion for Abelian Groups. Throughout this paper the Abelian Groups based on this inverse operation will be called the Subtractive Abelian Groups. In [5] A. Tarski has established two postulate systems for such algebras. And, recently, in [1], R. Güting constructed another axiomatization for these systems. In [2] and [3] S. Leśniewski accepted a ternary functor $\boldsymbol{\varphi}(a\,b\,c)$ defined as follows: $\boldsymbol{\varphi}(a\,b\,c)$.≡. $a + b = c$ for all elements a, b, and c of the given algebra, as a sole primitive functor and has used it to construct two single axioms for Groups Theory and Abelian Groups Theory, respectively. Analogously, a ternary functor defined as follows: $\boldsymbol{\rho}(a\,b\,c)$.≡. $a - b = c$ for all elements a, b, and c of the given algebra, can be accepted as a sole primitive notion for Subtractive Abelian Groups. And, in [5], p. 256, Tarski announced without a proof that formulas *T1* and *T2*, presented in section **1** below, can be accepted as an axiom-system for such algebras.

A main purpose of this paper is to present a proof that formula *S1*, see section **1**, which I constructed in the style of Leśniewski's axioms mentioned above, can be accepted as a single axiom of the Subtractive Abelian Groups. As a by-product of the deductions given below it will also be shown that all postulate-systems mentioned above are inferentially equivalent.[1]

1 Although it is not necessary, in order to present the deductions which follow in the next sections in a uniform way, we accept here that the carrier sets which occur in two formalizations $\mathfrak{G}_1$ and $\mathfrak{G}_2$ of Subtractive Abelian Groups represent the same unempty arbitrary set.

1. This paper is written in the style of Leśniewski's works [2] and [3].

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1.1 *Any algebraic structure*

$$\mathfrak{G}_1 = \langle \mathbf{A}, - \rangle$$

where - is a binary operation defined on the carrier set $\mathbf{A}$, *is a subtractive Abelian Group, if it satisfies one of the following sets of postulates:*

a) *the first postulate-system of Tarski, cf.* [5], *i.e.:*

A1 $[AB]:A, B \epsilon \mathbf{A} .\supset. A - B \epsilon \mathbf{A}$

A2 $[ABC]:A, B, C \epsilon \mathbf{A} .\supset. A - (B - (C - (A - B))) = C$

b) *the second postulate-system of Tarski, cf.* [5], *i.e., the formula A1 and*

B1 $[AB]:A, B \epsilon \mathbf{A} .\supset. A - (A - B) = B$

B2 $[ABC]:A, B, C \epsilon \mathbf{A} .\supset. A - (B - C) = C - (B - A)$

c) *the postulate-system of Güting, cf.* [1], *i.e., the formulas A1, B1, and*

C1 $[ABC]:A, B, C \epsilon \mathbf{A} .\supset. (A - B) - C = (A - C) - B$

1.2 *Any algebraic structure*

$$\mathfrak{G}_2 = \langle \mathbf{A}, \boldsymbol{\rho} \rangle$$

where $\boldsymbol{\rho}$ *is a ternary functor defined on the carrier set* $\mathbf{A}$, *is a Subtractive Abelian Group, if it satisfies one of the following sets of postulates:*

d) *Tarski's postulate-system for* $\boldsymbol{\rho}$, *cf.* [5], *i.e.:*

T1 $[AB]:A, B \epsilon \mathbf{A} .\supset. [\exists C] . C \epsilon \mathbf{A} . \boldsymbol{\rho}(ABC)$

T2 $[ABCDEFG]:A, B, C, D, E, F, G \epsilon \mathbf{A} . \boldsymbol{\rho}(ABD) . \boldsymbol{\rho}(CDE) .$
$\boldsymbol{\rho}(BEF) . \boldsymbol{\rho}(AFG) .\supset. C = G$

e) *my axiom for* $\boldsymbol{\rho}$, *i.e.,*

S1 $[ABC] :: \boldsymbol{\rho}(BAC) .\equiv:\cdot [\exists DERS] . \boldsymbol{\rho}(DBE) . \boldsymbol{\rho}(RSC) :\cdot [HI] :: \boldsymbol{\rho}(AHI) .\equiv:\cdot$
$[\exists KLMN] . \boldsymbol{\rho}(IKL) . \boldsymbol{\rho}(MNH) \therefore [OP] : \boldsymbol{\rho}(OCH) . \boldsymbol{\rho}(BPI) .\supset. O = P$

1.3 It will be proved below in sections **2**, **3**, and **4** that

$$\{A1;\ B1;\ C1\} \rightarrow \{A1;\ B1;\ B2\} \rightarrow \{A1;\ A2\} \rightarrow \{T1;\ T2\} \rightarrow \{S1\} \rightarrow \{A1;\ B1;\ C1\}$$

Remark: Since in [5] Tarski announced his system $\{T1;\ T2\}$ without a proof, the deductions presented in section **3** below are entirely mine. The deductions given in section **4** differ considerably from those which Leśniewski presented in [3] in order to establish his single axiom for Abelian Groups.

1.4 Expressions of the form "$a \epsilon A$" and similar ones, which guarantee that an algebraic formula belonging to, say, algebra $\mathfrak{A}$ is closed with respect to the carrier set of $\mathfrak{A}$, do not occur in axiom *S1*. The elimination of such formulas from *S1* is possible due to an application of a method which is discussed, e.g., in [2], pp. 319-320. In the case of the systems under consideration in this paper, the following particular form of this method is used:

Clearly, formulas of the form "$a \epsilon A$" are not algebraic formulas, but they belong to this or that system of logic or set-theory on which the algebraic systems under consideration are based. Hence, we are able to eliminate the membership constant ϵ from the formulas under discussion by introducing to the accepted system of logic or set-theory the following definition:

Df1 $[aA]: a \epsilon A .\equiv. \mathfrak{I}\langle A\rangle \{a\}$

Obviously, the definiendum of *Df1* has a form of many-link expressions in the sense of Leśniewski, but, since the carrier set $\mathbf{A}$ in the systems considered in this paper is fixed, functor $\mathfrak{I}\langle\mathbf{A}\rangle$ can be used, as a constant. Then, we introduce to the system $\{T1; T2\}$, *cf.* section **2.3** below, the following definition:

DI $[ABC]: A \epsilon \mathbf{A} . B \epsilon \mathbf{A} . C \epsilon \mathbf{A} . A - B = C .\equiv. \boldsymbol{\rho}(ABC)$

and, moreover, we prove that the following formula

$$[A] \therefore \mathfrak{I}\langle\mathbf{A}\rangle \{A\} .\equiv: [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$$

is a thesis of system $\{T1; T2\}$. Hence, in the fields of systems $\{T1; T2\}$ and $\{S1\}$ we are able to introduce the definition

D1 $[A] \therefore \mathfrak{I}\langle\mathbf{A}\rangle \{A\} .\equiv: [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$

which together with definition *Df1* implies at once:

$$[A] \therefore A \epsilon \mathbf{A} .\equiv: [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$$

Furthermore, since in the field of $\{S1\}$ the following formula

$$[ABC]: A, B, C \epsilon \mathbf{A} . A - B = C .\equiv. \boldsymbol{\rho}(ABC)$$

is provable, *cf.* section **4**, we will know that the use of definitions *DI* and *D1* is correct.

2 *The proof that:* $\{A1; B1; C1\} \rightarrow \{A1; B1; B2\} \rightarrow \{A1; A2\} \rightarrow \{T1; T2\}$. Remark: In subsections **2.1** and **2.2** the use of axiom *A1* is assumed tacitly.

2.1 Let us assume axioms *A1*, *B2*, and *C1*. Then:

C2 $[ABC]: A, B, C \epsilon \mathbf{A} .\supset. B - C = (A - C) - (A - B)$

PR $[ABC]: \mathrm{Hp}(3) .\supset.$
$B - C = (A - (A - B)) - C = (A - C) - (A - C)$
$[A1; 1; 2; 3; B1; C1, B/A - B]$

B2 $[ABC]: A, B, C \epsilon \mathbf{A} .\supset. A - (B - C) = C - (B - A)$

PR $[ABC]: \mathrm{Hp}(3) .\supset.$
$A - (B - C) = ((B - (B - C)) - (B - A) = C - (B - A)$
$[A1; 1; 2; 3; C2, A/B, B/A, C/B - C; B1, A/B, B/C]$

Thus, $\{A1; B1; C1\} \rightarrow \{A1; B1; B2\}$.

2.2 Let us assume axioms *A1*, *B1*, and *B2*. Then:

A2 $[ABC]: A, B, C \epsilon \mathbf{A} .\supset. A - (B - (C - (A - B))) = C$

PR $[ABC]: \mathrm{Hp}(3) .\supset.$

$A - (B - (C - (A - B))) = A - (B - (B - (A - C)))$ [$A1$; 1; 2; 3; $B2, A/C, B/A, C/B$]
$= A - (A - C) = C$ [$B1, A/B, B/A - C; B1, B/C$]

Thus, $\{A1; B1; B2\} \rightarrow \{A1; A2\}$.

2.3 Let us assume axioms $A1$ and $A2$. Then:

DI $[ABC]: A \epsilon \mathbf{A} . B \epsilon \mathbf{A} . C \epsilon \mathbf{A} . A - B = C .\equiv. \boldsymbol{\rho}(ABC)$
$A3$ $[ABX]: A, B, X \epsilon \mathbf{A} .\supset. A - B = C .\equiv. \boldsymbol{\rho}(ABC)$ [$DI, C/X$; *cf.* $D2$ in section **4**]
$A4$ $[A] \therefore \mathfrak{I}\langle \mathbf{A} \rangle \{A\} .\supset: [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$
PR $[A] \therefore \mathrm{Hp}(1) .\supset:$
2. $A \epsilon \mathbf{A}$. [$Df1, a/A, A/\mathbf{A}$; 1, *cf.* section **1.4**]
3. $A - A \epsilon \mathbf{A}$. [$A1, B/A$; 2]
4. $A - A = A - A$. [3]
$[\exists DE]$.
5. $\boldsymbol{\rho}(ADE)$: [$DI, B/A, C/A - A$; 2; 3; 4]
$[\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$ [5]
$A5$ $[A] \therefore [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA) :\supset. \mathfrak{I}\langle \mathbf{A} \rangle \{A\}$
PR $[A] \therefore \mathrm{Hp}(1) :\supset.$
2. $A \epsilon \mathbf{A}$. [1; $DI, B/D, C/E; D1, A/D, B/A, C/E; D1, A/D, B/E, C/A$]
$\mathfrak{I}\langle \mathbf{A} \rangle \{A\}$ [$Df1, a/A, A/\mathbf{A}$; 1]
$A6$ $[A] \therefore \mathfrak{I}\langle \mathbf{A} \rangle \{A\} .\equiv: [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$ [$A4$; $A5$; *cf.* section **1.4**]
$T1$ $[AB]: A, B \epsilon \mathbf{A} .\supset. [\exists C] . C \epsilon \mathbf{A} . \boldsymbol{\rho}(ABC)$
PR $[AB]: \mathrm{Hp}(2) .\supset.$
3. $A - B \epsilon \mathbf{A}$. [$A1$; 1; 2]
4. $A - B = A - B$. [3]
$[\exists C] . C \epsilon \mathbf{A} . \boldsymbol{\rho}(ABC)$ [DI; 1; 2; 3; 4]
$T2$ $[ABCDEFG]: A, B, C, D, E, F, G \epsilon \mathbf{A} . \boldsymbol{\rho}(ABD) . \boldsymbol{\rho}(CDE)$.
$\boldsymbol{\rho}(BEF) . \boldsymbol{\rho}(AFG) .\supset. C = G$
PR $[ABCDEFG]: \mathrm{Hp}(11) .\supset.$
12. $A - B = D$. [$DI, C/D$; 8]
13. $C - D = E$. [$DI, A/C, B/D, C/E$; 9]
14. $B - E = F$. [$DI; A/B, B/E, C/F$; 10]
15. $A - F = G$. [$DI, B/F, C/G$; 11]
$C = A - (B - (C - (A - B))) = A - (B - (C - D)) = A - (B - E)$
$= A - F = G$ [$A1; A2$; 1; 2; 3; 4; 5; 6; 7; 12; 13; 14; 15]

Thus, $\{A1; A2\} \rightarrow \{T1; T2\}$. Furthermore, $\{A1; A2\} \rightarrow \{A3; A6\}$.

3 Let us assume axioms $T1$ and $T2$. Then:

$D1$ $[A] \therefore [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA) :\equiv. \mathfrak{I}\langle \mathbf{A} \rangle \{A\}$
$T3$ $[A] \therefore A \epsilon \mathbf{A} .\equiv: [\exists DE]: \boldsymbol{\rho}(ADE) .\mathrm{v}. \boldsymbol{\rho}(DAE) .\mathrm{v}. \boldsymbol{\rho}(DEA)$ [$D1; Df1, a/A, A/\mathbf{A}$]
$T4$ $[ABC]: \boldsymbol{\rho}(ABC) .\supset. A \epsilon \mathbf{A} . B \epsilon \mathbf{A} . C \epsilon \mathbf{A}$
[$T3, D/B, E/C; T3, A/B, D/A, E/C; T3, A/C, D/A, E/B$]

$T5$ $[ABCDEFG]:\boldsymbol{\rho}(ABD).\boldsymbol{\rho}(CDE).\boldsymbol{\rho}(BEF).\boldsymbol{\rho}(AFG).\supset. C=G$
$[T2;\ T4,\ C/D;\ T4,\ A/C,\ B/D,\ C/E;\ T4,\ A/B,\ B/E,\ C/F;$
$T4,\ B/F,\ C/G;\ 1;\ 2;\ 3;\ 4]$

$T6$ $[ABCDEF]:\boldsymbol{\rho}(ABD).\boldsymbol{\rho}(CDE).\boldsymbol{\rho}(BEF).\supset.\boldsymbol{\rho}(AFC)$
PR $[ABCDEF]:\mathrm{Hp}(3).\supset.$
$[\exists G].$
4. $\boldsymbol{\rho}(AFG).$ $[T4,\ C/D;\ 1;\ T4,\ A/B,\ B/E,\ C/F;\ 3;\ T1,\ B/F]$
5. $C=G.$ $[T5;\ 1;\ 2;\ 3;\ 4]$
$\boldsymbol{\rho}(AFC)$ $[4;\ 5]$

$T7$ $[AB]:A,\ B\,\epsilon\,\mathbf{A}.\supset.[\exists C].C\,\epsilon\,\mathbf{A}.\boldsymbol{\rho}(ACB)$
PR $[AB]::\mathrm{Hp}(2).\supset\therefore$
$[\exists C]\therefore$
3. $\boldsymbol{\rho}(ABC):$ $[T1;\ 1;\ 2]$
$[\exists D]:$
4. $\boldsymbol{\rho}(BCD).$ $[T4,\ C/D;\ 3;\ T1,\ B/C;\ 1]$
$[\exists E].$
5. $\boldsymbol{\rho}(BDE)$ $[T4,\ A/B,\ B/C,\ C/D;\ 4;\ T1,\ A/B,\ B/D]$
6. $\boldsymbol{\rho}(AEB)\therefore$ $[T6,\ D/C,\ C/B,\ E/D,\ F/E;\ 3;\ 4;\ 5]$
$[\exists C].C\,\epsilon\,\mathbf{A}.\boldsymbol{\rho}(ACB)$ $[T4,\ A/B,\ B/D,\ C/E;\ 5;\ 6]$

$T8$ $[ABCKHIR]:\boldsymbol{\rho}(ABC).\boldsymbol{\rho}(KCH).\boldsymbol{\rho}(HKI).\boldsymbol{\rho}(BIR).\supset. A=R$
PR $[ABCKHIR]:\mathrm{Hp}(4).\supset.$
$[\exists D].$
5. $\boldsymbol{\rho}(BHD).$ $[T4;\ 1;\ T4,\ A/K,\ B/C,\ C/H;\ 2;\ T1,\ A/B,\ B/D]$
6. $\boldsymbol{\rho}(ADK).$ $[T6,\ D/C,\ C/K,\ E/H,\ F/D;\ 1;\ 2;\ 5]$
$A=R$ $[T5,\ A/B,\ B/H,\ C/A,\ E/K,\ F/I,\ G/R;\ 5;\ 6;\ 3;\ 4]$

$T9$ $[ABCHKV]:\boldsymbol{\rho}(CBK).\boldsymbol{\rho}(ABC).\boldsymbol{\rho}(KCH).\boldsymbol{\rho}(CAV).\supset. H=V$
PR $[ABCHKV]\therefore\mathrm{Hp}(4).\supset:$
$[\exists I]:$
5. $\boldsymbol{\rho}(HKI).$ $[T4,\ A/K,\ B/C,\ C/H;\ 3;\ T1,\ A/H,\ B/K]$
$[\exists R].$
6. $\boldsymbol{\rho}(BIR).$ $[T4;\ 2;\ T4,\ A/H.\ B/K,\ C/I;\ 5;\ T1,\ A/B,\ B/I]$
7. $A=R:$ $[T8;\ 2;\ 3;\ 5;\ 6]$
8. $\boldsymbol{\rho}(BIA):$ $[6;\ 7]$
$H=V$ $[T5,\ A/C,\ D/K,\ C/H,\ E/I,\ F/A,\ G/V;\ 1;\ 5;\ 8;\ 4]$

$T10$ $[AB]:A,\ B\,\epsilon\,\mathbf{A}.\supset.[\exists C].C\,\epsilon\,\mathbf{A}.\boldsymbol{\rho}(CAB)$
PR $[AB]\therefore\!\cdot\,\mathrm{Hp}(2).\supset::$
$[\exists C]::$
3. $\boldsymbol{\rho}(ACB)\therefore$ $[T7;\ 1;\ 2]$
$[\exists D]\therefore$
4. $\boldsymbol{\rho}(CDA):$ $[T4,\ B/C,\ C/B;\ 3;\ T7,\ A/C,\ B/A]$
$[\exists E]:$
5. $\boldsymbol{\rho}(ADE).$ $[T4,\ A/C,\ B/D,\ C/A;\ 4;\ T1,\ B/D]$
$[\exists F].$
6. $\boldsymbol{\rho}(EAF).$ $[T4,\ B/D,\ C/E;\ 5;\ T1,\ A/E,\ B/A]$
7. $F=B.$
$[T9,\ C/A,\ B/D,\ K/E,\ A/C,\ H/F,\ V/B;\ 5;\ 4;\ 6;\ 3]$

8. $\boldsymbol{\rho}(EAB) ::$ [6; 7]

$[\exists C] . C \epsilon \mathbf{A} . \boldsymbol{\rho}(CAB)$ $[T4, A/E, B/A, C/B; 8]$

T11 $[AB] : \boldsymbol{\rho}(AAB) . \supset . \boldsymbol{\rho}(ABA)$

PR $[AB] :: \mathrm{Hp}(1) . \supset \therefore$

$[\exists C] \therefore$

2. $\boldsymbol{\rho}(ACA) .$ $[T4, B/A, C/B; T7, B/A]$

$[\exists E] .$

3. $\boldsymbol{\rho}(EBC) .$ $[T4, B/A, C/B; 1; T4, B/C, C/A; 2; T10, A/B, B/C]$

4. $E = B .$ $[T5, B/A, D/B, C/E, E/C, F/A, G/B; 1; 3; 2; 1]$

5. $\boldsymbol{\rho}(BBC) :$ [3; 4]

$[\exists D] :$

6. $\boldsymbol{\rho}(ABD) .$ $[T4, B/A, C/B; 1; T1]$

$[\exists F] .$

7. $\boldsymbol{\rho}(FDB) .$ $[T4, C/D; 6; T10, A/D]$

8. $F = A .$ $[T5, C/F, E/B, F/C, G/A; 6; 7; 5; 2]$

9. $\boldsymbol{\rho}(ADB) .$ [7; 8]

10. $A = D \therefore$ $[T5, B/A, D/B, C/A, E/D, F/B, G/D; 1; 6; 9; 6]$

$\boldsymbol{\rho}(ABA)$ [6; 10]

T12 $[ABC] : \boldsymbol{\rho}(AAB) . \boldsymbol{\rho}(AAC) . \supset . B = C$

PR $[ABC] : \mathrm{Hp}(2) . \supset .$

3. $\boldsymbol{\rho}(ABA) .$ $[T11; 1]$

$B = C$ $[T9, C/A, K/A, H/B, V/C; 3; 3; 1; 2]$

T13 $[AB] : \boldsymbol{\rho}(AAB) . \supset . \boldsymbol{\rho}(BBB)$

PR $[AB] : \mathrm{Hp}(1) . \supset .$

2. $\boldsymbol{\rho}(ABA) .$ $[T11; 1]$

$[\exists C] .$

3. $\boldsymbol{\rho}(CBB) .$ $[T4, B/A, C/B; 1; T10, A/B]$

4. $C = B .$ $[T5, B/A, D/B, E/B, F/A, G/B; 1; 3; 2; 1]$

$\boldsymbol{\rho}(BBB)$ [3; 4]

T14 $[AB] : \boldsymbol{\rho}(ABA) . \supset . \boldsymbol{\rho}(AAB)$

PR $[AB] \therefore \mathrm{Hp}(1) . \supset :$

$[\exists C] :$

2. $\boldsymbol{\rho}(AAC) .$ $[T4, C/A; 1; T1, B/A]$

3. $\boldsymbol{\rho}(CCC) .$ $[T13, B/C; 2]$

$[\exists D] .$

4. $\boldsymbol{\rho}(DCB) .$ $[T4, B/A; 2; T4, C/A; 1; T10, A/C]$

5. $D = C .$ $[T5, B/A, D/C, C/D, E/B, F/A, G/C; 2; 4; 1; 2]$

6. $\boldsymbol{\rho}(CCB) .$ [4; 5]

7. $B = C :$ $[T12, A/C; 6; 3]$

$\boldsymbol{\rho}(AAB)$ [2; 7]

T15 $[ABCD] : \boldsymbol{\rho}(AAC) . \boldsymbol{\rho}(BBD) . \supset . C = D$

PR $[ABCD] :: \mathrm{Hp}(2) . \supset \therefore$

3. $\boldsymbol{\rho}(ACA) .$ $[T11, B/C; 1]$

4. $\boldsymbol{\rho}(CCC) \therefore$ $[T13, B/C; 1]$

$[\exists E] \therefore$

5. $\boldsymbol{\rho}(BCE) .$ $[T4, A/B, C/D; 2; T4, B/A; 1; T1, A/B, B/C]$

$[\exists F] .$

6. $\boldsymbol{\rho}(FEC)$. [$T4, A/B, B/C, C/E$; 5; $T10, A/E, B/C$]

7. $F = E$.
[$T5, A/B, B/C, D/E, C/F, E/C, F/C, G/E$; 5; 6; 4; 5]

8. $\boldsymbol{\rho}(EEC)$: [6; 7]
[$\exists G$]:

9. $\boldsymbol{\rho}(AEG)$. [$T4, B/A$; 1; $T4, A/E, B/E, C/G$; 8; $T1, B/E$]
[$\exists H$].

10. $\boldsymbol{\rho}(HGE)$. [$T4, B/E, C/G$; 9; $T10, A/G, B/E$]

11. $H = A$.
[$T5, B/E, D/G, C/H, F/C, G/A$; 9; 10; 8; 3]

12. $\boldsymbol{\rho}(AGE)$: [10; 11]

13. $B = E$ ∴ [$T5, B/A, D/C, C/B, F/G, G/E$; 1; 5; 9; 12]

14. $\boldsymbol{\rho}(BCB)$. [5; 13]

15. $\boldsymbol{\rho}(BBC)$. [$T14, A/B, B/C$; 14]

$C = D$ [$T12, A/B, B/C, C/D$; 15; 2]

T16 [ABC] : $\boldsymbol{\rho}(ABC)$.⊃. $\boldsymbol{\rho}(ACB)$

PR [ABC] ∴ Hp(1) .⊃:
[$\exists D$]:

2. $\boldsymbol{\rho}(AAD)$. [$T4$; 1; $T1, B/A$]

3. $\boldsymbol{\rho}(ADA)$. [$T11, B/D$; 2]
[$\exists F$].

4. $\boldsymbol{\rho}(BBF)$. [$T4$; 1; $T1, A/B$]

5. $D = F$. [$T15, C/D, D/F$; 2; 4]

6. $\boldsymbol{\rho}(BBD)$. [4; 5]
[$\exists G$].

7. $\boldsymbol{\rho}(GCB)$ [$T4$; 1; $T10, A/C$]

8. $G = A$: [$T5, D/C, C/G, E/B, F/D, G/A$; 1; 7; 6; 3]

$\boldsymbol{\rho}(ACB)$ [7; 8]

T17 [$ABCHIOP$] : $\boldsymbol{\rho}(BAC)$. $\boldsymbol{\rho}(AHI)$. $\boldsymbol{\rho}(OCH)$. $\boldsymbol{\rho}(BPI)$.⊃. $O = P$

PR [$ABCHIOP$] : Hp(4) .⊃.

5. $\boldsymbol{\rho}(BIP)$. [$T16, A/B, B/P, C/I$; 4]

$O = P$ [$T5, A/B, B/A, D/C, C/O, E/H, F/I, G/P$; 1; 3; 2; 5]

T18 [$ABCHIKLMN$] ∴ $\boldsymbol{\rho}(BAC)$. $\boldsymbol{\rho}(IKL)$. $\boldsymbol{\rho}(MNH)$: [OP] : $\boldsymbol{\rho}(OCH)$. $\boldsymbol{\rho}(BPI)$.
⊃. $O = P$:⊃. $\boldsymbol{\rho}(AHI)$

PR [$ABCHIKLMN$] :: Hp(4) :⊃∴

5. $\boldsymbol{\rho}(BCA)$ ∴ [$T16, A/B, B/A$; 1]
[$\exists T$] ∴

6. $\boldsymbol{\rho}(TCH)$.
[$T4, A/B, B/A$; 1; $T4, A/M, B/N, C/H$; 3; $T10, A/C, B/H$]

7. $\boldsymbol{\rho}(THC)$. [$T16, A/T, B/C, C/H$; 6]
[$\exists V$].

8. $\boldsymbol{\rho}(BVI)$.
[$T4, A/B, B/A$; 1; $T4, A/I, B/K, C/L$; 2; $T7, A/B, B/I$]

9. $T = V$. [4, $O/T, P/V$; 6; 8]

10. $\boldsymbol{\rho}(BTI)$: [8; 9]
[$\exists Z$]:

11. $\rho(ZIC)$.
$[T4, A/B, B/A; 1; T4, A/I, B/K, C/L; 2; T10, A/I, B/C]$
12. $\rho(BHZ)$. $[T6, A/B, B/T, D/I, C/Z, E/C, F/H; 10; 11; 6]$
13. $\rho(BZH)$. $[T16, A/B, B/H, C/Z; 12]$
$[\exists R]$.
14. $\rho(RIH)$.
$[T4, A/I, B/K, C/L; 2; T4, A/M, B/N, C/H; 3; T10, A/I, B/H]$
15. $R = A$ ∴
$[T5, A/B, B/T, D/I, C/R, E/H, F/C, G/A; 10; 14; 7; 5]$
16. $\rho(AIH)$. $[14; 15]$
$\rho(AHI)$ $[T16, B/I, C/H; 16]$

T19 $[ABC] : \rho(BAC) .\supset. [\exists DERS] . \rho(DBE) . \rho(RSC)$
PR $[ABC] :$ Hp(1) $.\supset.$
$[\exists E]$.
2. $\rho(BBE)$. $[T4, A/B, B/A; 1; T1, A/B]$
$[\exists DERS] . \rho(DBE) . \rho(RSC)$ $[2; 1]$

T20 $[AHI] : \rho(AHI) .\supset. [\exists KLMN] . \rho(IKL) . \rho(MNH)$
PR $[AHI]$ ∴ Hp(1) $.\supset:$
$[\exists L] :$
2. $\rho(IIL)$. $[T4, B/H, C/I; 1; T1, A/I, B/I]$
$[\exists N]$.
3. $\rho(HNH) :$ $[T4, B/H, C/I; 1; T7, A/H, B/H]$
$[\exists KLMN] : \rho(IKL) . \rho(MNH)$ $[2; 3]$

T21 $[ABC]$ :: $\rho(BAC)$.⊃∵ $[\exists DERS] . \rho(DBE) . \rho(RSC)$ ∵ $[HI]$ :: $\rho(AHI)$.≡∴ $[\exists KLMN] . \rho(IKL) . \rho(MNH)$ ∴ $[OP] : \rho(OCH) . \rho(BPI) .\supset. O = P$
$[T19; T17; T20; T18]$

T22 $[ABC]$ ∴ $A, B, C \epsilon \mathbf{A} : [HIOP] : \rho(AHI) . \rho(OCH) . \rho(BPI) .\supset. O = P :\supset.$ $\rho(BAC)$
PR $[ABC]$ ∴ Hp(4) $:\supset:$
$[\exists K] :$
5. $\rho(AKC)$. $[T7, B/C; 1; 3]$
6. $\rho(ACK)$. $[T16, B/K; 5]$
$[\exists D]$.
7. $\rho(BDC)$. $[T7, A/B, B/C; 2; 3]$
8. $A = D :$ $[4, H/K, I/C, O/A, P/D; 5; 6; 7]$
$\rho(BAC)$ $[7; 8]$

T23 $[BCOPRSV] : \rho(RCS) . \rho(SRV) . \rho(OCV) . \rho(BPB) .\supset. O = P$
PR $[BCOPRSV] :$ Hp(4) $.\supset.$
5. $\rho(BBP)$. $[T14, A/B, B/P; 4]$
6. $\rho(OVC)$. $[T16, A/O, B/C, C/V; 3]$
7. $\rho(SSO)$. $[T6, A/S, B/R, D/V, C/O, E/C, F/S; 2; 6; 1]$
$O = P$ $[T15, A/S, C/O, D/P; 7; 5]$

T24 $[BCRSV]$ ∴ $\rho(RCS) . \rho(SRV) .\supset: [OP] : \rho(OCV) . \rho(BPB) .\supset. O = P$ $[T23]$

T25 $[ABC]$ ∵ $B, C \epsilon \mathbf{A}$ :: $[HIKLMN]$ ∴ $\rho(IKL) . \rho(MNH)$ ∴ $[OP] : \rho(OCH)$. $\rho(BPI) .\supset. O = P :\supset. \rho(AHI)$ ∴⊃. $A \epsilon \mathbf{A}$
PR $[ABC]$ :: Hp(3) ::⊃∵
$[\exists R]$ ∵

4. $\boldsymbol{\rho}(CCR)::$ [*T1, A/C, B/C*; 2]

$[\exists S]::$

5. $\boldsymbol{\rho}(RCS)\therefore$ [*T4, A/C, B/C, C/R*; 4; *T1, A/R, B/C*]

$[\exists V]\therefore$

6. $\boldsymbol{\rho}(SRV):$ [*T4, A/R, B/C, C/S*; 5; *T1, A/S, B/R*]

7. $[OP]:\boldsymbol{\rho}(OCV).\boldsymbol{\rho}(BPB).\supset.O=P:$ [*T24*; 5; 6]

$[\exists W].$

8. $\boldsymbol{\rho}(BBW).$ [*T1, A/B*; 1]

9. $\boldsymbol{\rho}(AVB)\because$

[3, *H/V, I/B, K/B, L/W, M/S, N/R*; 8; 6; 7]

$A\epsilon\mathbf{A}$ [*T4, B/V, C/B*; 9]

T26 $[ABCDERS]\because\boldsymbol{\rho}(DBE).\boldsymbol{\rho}(RSC)::[HI]::\boldsymbol{\rho}(AHI).\equiv\therefore[\exists KLMN].\boldsymbol{\rho}(IKL).$ $\boldsymbol{\rho}(MNH)\therefore[OP]:\boldsymbol{\rho}(OCH).\boldsymbol{\rho}(BPI).\supset.O=P::\supset.\boldsymbol{\rho}(BAC)$

PR $[ABCDERS]\because\mathrm{Hp}(3)::\supset::$

4. $B\epsilon\mathbf{A}.$ [*T4, A/D, C/E*; 1]

5. $C\epsilon\mathbf{A}\therefore$ [*T4, A/R, B/S*; 2]

6. $[HIOP]:\boldsymbol{\rho}(AHI).\boldsymbol{\rho}(OCH).\boldsymbol{\rho}(BPI).\supset.O=P::$ [3]

7. $[HIKLMN]\therefore\boldsymbol{\rho}(IKL).\boldsymbol{\rho}(MNH)\therefore[OP]:\boldsymbol{\rho}(OCH).\boldsymbol{\rho}(BPI).\supset.O=P:\supset.$ $\boldsymbol{\rho}(AHI)\therefore$ [3]

8. $A\epsilon\mathbf{A}.$ [*T25*; 4; 5; 7]

$\boldsymbol{\rho}(BAC)$ [*T22*; 8; 4; 5; 6]

S1 $[ABC]::\boldsymbol{\rho}(BAC).\equiv\because[\exists DERS].\boldsymbol{\rho}(DBE).\boldsymbol{\rho}(RSC)\because[HI]::\boldsymbol{\rho}(AHI).\equiv\therefore$ $[\exists KLMN].\boldsymbol{\rho}(IKL).\boldsymbol{\rho}(MNH)\therefore[OP]:\boldsymbol{\rho}(OCH).\boldsymbol{\rho}(BPI).\supset.O=P$

[*T21*; *T26*]

Thus, $\{T1;\ T2\}\rightarrow\{S1\}$.

4 Let us assume axiom *S1*. Then:

S2 $[ABC]:\boldsymbol{\rho}(BAC).\supset.[\exists DE].\boldsymbol{\rho}(DBE)$ [*S1*]

S3 $[ABCHI]:\boldsymbol{\rho}(BAC).\boldsymbol{\rho}(AHI).\supset.[\exists KLMN].\boldsymbol{\rho}(IKL).\boldsymbol{\rho}(MNH)$ [*S1*]

S4 $[ABCHIOP]:\boldsymbol{\rho}(BAC).\boldsymbol{\rho}(AHI).\boldsymbol{\rho}(OCH).\boldsymbol{\rho}(BPI).\supset.O=P$ [*S1*]

S5 $[ABCHIKLMN]\therefore\boldsymbol{\rho}(BAC).\boldsymbol{\rho}(IKL).\boldsymbol{\rho}(MNH):[OP]:\boldsymbol{\rho}(OCH).\boldsymbol{\rho}(BPI).\supset.$ $O=P:\supset.\boldsymbol{\rho}(AHI)$ [*S1*]

S6 $[AHI]:\boldsymbol{\rho}(AHI).\supset.[\exists MN].\boldsymbol{\rho}(MNH)$

PR $[AHI]:\mathrm{Hp}(1).\supset.$

$[\exists DE].$

2. $\boldsymbol{\rho}(DAE).$ [*S2, B/A, A/H, C/I*; 1]

$[\exists MN].\boldsymbol{\rho}(MNH)$ [*S3, B/D, C/E*; 2; 1]

S7 $[AHI]:\boldsymbol{\rho}(AHI).\supset.[\exists KL].\boldsymbol{\rho}(IKL)$

PR $[AHI]:\mathrm{Hp}(1).\supset.$

$[\exists DE].$

2. $\boldsymbol{\rho}(DAE).$ [*S2, B/A, A/H, C/I*; 1]

$[\exists KL].\boldsymbol{\rho}(IKL)$ [*S3, B/D, C/E*; 2; 1]

S8 $[AHI]:\boldsymbol{\rho}(AHI).\supset.[\exists VW].\boldsymbol{\rho}(VIW)$

PR $[AHI]:\mathrm{Hp}(1).\supset.$

$[\exists KL].$

2. $\boldsymbol{\rho}(IKL).$ [*S7*; 1]

$[\exists VW].\boldsymbol{\rho}(VIW)$ [*S2, B/I, A/K, C/L*; 2]

S9 $[AHI]:\boldsymbol{\rho}(AHI) .\supset. [\exists QT].\boldsymbol{\rho}(HQT)$

PR $[AHI]:\mathrm{Hp}(1) .\supset.$

$[\exists MN].$

2. $\boldsymbol{\rho}(MNH).$ $[S6;\ 1]$

$[\exists QT].\boldsymbol{\rho}(HQT)$ $[S7,\ A/M,\ H/N,\ I/H;\ 2]$

S10 $[AHI]:\boldsymbol{\rho}(AHI) .\supset. [\exists RS].\boldsymbol{\rho}(RSA)$

PR $[AHI]:\mathrm{Hp}(1) .\supset.$

$[\exists DE].$

2. $\boldsymbol{\rho}(DAE).$ $[S2,\ B/A,\ A/H,\ C/I;\ 1]$

$[\exists RS].\boldsymbol{\rho}(RSA)$ $[S6,\ A/D,\ H/A,\ C/E;\ 2]$

S11 $[AHI]:\boldsymbol{\rho}(AHI) .\supset. [\exists DEKLMN].\boldsymbol{\rho}(ADE).\boldsymbol{\rho}(KAL).\boldsymbol{\rho}(MNA)$

$[1;\ S2,\ B/A,\ A/H,\ C/I;\ 1;\ S10;\ 1]$

S12 $[AHI]:\boldsymbol{\rho}(HAI) .\supset. [\exists DEKLMN].\boldsymbol{\rho}(ADE).\boldsymbol{\rho}(KAL).\boldsymbol{\rho}(MNA)$

$[S9,\ A/H,\ H/A;\ 1;\ 1;\ S6,\ A/H,\ H/A;\ 1]$

S13 $[AHI]:\boldsymbol{\rho}(HIA) .\supset. [\exists DEKLMN].\boldsymbol{\rho}(ADE).\boldsymbol{\rho}(KAL).\boldsymbol{\rho}(MNA)$

$[S7,\ A/H,\ H/I,\ I/A;\ 1;\ S8,\ A/H,\ H/I,\ I/A;\ 1;1]$

S14 $[AHI]\therefore\boldsymbol{\rho}(AHI).\vee.\boldsymbol{\rho}(HAI).\vee.\boldsymbol{\rho}(HIA) :\supset. [\exists DEKLMN].\boldsymbol{\rho}(ADE).$

$\boldsymbol{\rho}(KAL).\boldsymbol{\rho}(MNA)$ $[S11;\ S12;\ S13]$

D1 $[A]\therefore[\exists DE]:\boldsymbol{\rho}(ADE).\vee.\boldsymbol{\rho}(DAE).\vee.\boldsymbol{\rho}(DEA) :\equiv. \mathcal{J}\langle\mathbf{A}\rangle\ \{A\}$

S15 $[A]\therefore A\,\epsilon\,\mathbf{A} .\equiv: [\exists DE]:\boldsymbol{\rho}(ADE).\vee.\boldsymbol{\rho}(DAE).\vee.\boldsymbol{\rho}(DEA)$ $[D1;\ Df1]$

S16 $[ABC]:\boldsymbol{\rho}(ABC) .\supset. A\,\epsilon\,\mathbf{A}.B\,\epsilon\,\mathbf{A}.C\,\epsilon\,\mathbf{A}$

$[S15,\ D/B,\ E/C;\ S15,\ D/A,\ A/B,\ E/C;\ S15,\ D/A,\ E/B,\ A/C]$

S17 $[A]:A\,\epsilon\,\mathbf{A} .\supset. [\exists DEKLMN].\boldsymbol{\rho}(ADE).\boldsymbol{\rho}(KAL).\boldsymbol{\rho}(MNA)$

$[S15;\ S14,\ H/D,\ I/E]$

S18 $[A]:A\,\epsilon\,\mathbf{A} .\supset. [\exists C].\boldsymbol{\rho}(CAA)$

PR $[A]::\mathrm{Hp}(1) .\supset\therefore$

$[\exists DEMN]\therefore$

2. $\boldsymbol{\rho}(ADE).$

3. $\boldsymbol{\rho}(MNA):$ $[S17;\ 1]$

4. $[\exists C].\boldsymbol{\rho}(CAA).\vee.\boldsymbol{\rho}(NAA)\therefore$

$[S5,\ B/M,\ A/N,\ C/A,\ I/A,\ K/D,\ L/E,\ H/A;\ 3;\ 2;\ 3]$

$[\exists C].\boldsymbol{\rho}(CAA)$ $[4]$

S19 $[AB]:\boldsymbol{\rho}(BBB).\boldsymbol{\rho}(ABB) .\supset. A = B$

$[S4,\ A/B,\ C/B,\ H/B,\ I/B,\ O/A,\ P/B;\ 1;\ 1;\ 2;\ 1]$

S20 $[ABNVWQT]\therefore\boldsymbol{\rho}(NBA).\boldsymbol{\rho}(BVW).\boldsymbol{\rho}(QTB) .\supset: [\exists C].\boldsymbol{\rho}(CAB).\vee.\boldsymbol{\rho}(BBB)$

$[S5,\ B/N,\ A/B,\ C/A,\ I/B,\ K/V,\ L/W,\ M/Q,\ N/T,\ H/B;\ 1;\ 2;\ 3]$

S21 $[ABVWQTZ]\therefore\boldsymbol{\rho}(ZAA).\boldsymbol{\rho}(BVW).\boldsymbol{\rho}(QTB) .\supset: [\exists C].\boldsymbol{\rho}(CAB).\vee.\boldsymbol{\rho}(ABB)$

$[S5,\ B/Z,\ C/A,\ I/B,\ K/V,\ L/W,\ M/Q,\ N/T,\ H/B;\ 1;\ 2;\ 3]$

S22 $[AB]:A,\ B\,\epsilon\,\mathbf{A} .\supset. [\exists C].\boldsymbol{\rho}(CAB)$

PR $[AB]\,::\,\mathrm{Hp}(2) .\supset::$

$[\exists Z]::$

3. $\boldsymbol{\rho}(ZAA)::$ $[S18;\ 1]$

$[\exists MNKL]::$

4. $\boldsymbol{\rho}(MNA).$

5. $\boldsymbol{\rho}(AKL)\therefore$ $[S17;\ 1]$

$[\exists VWQT]\therefore$

6. $\boldsymbol{\rho}(BVW)\,.$ [S17, A/B; 2]
7. $\boldsymbol{\rho}(QTB):$ [S17, A/B; 2]
8. $[\exists C]\,.\,\boldsymbol{\rho}(CAB)\,.\!\vee\!.\,\boldsymbol{\rho}(NBA):$
[S5, B/M, A/N, C/A, I/A, M/Q, N/T, H/B; 4; 5; 7]
9. $[\exists C]\,.\,\boldsymbol{\rho}(CAB)\,.\!\vee\!.\,\boldsymbol{\rho}(BBB):$ [8; S20; 6; 7]
10. $[\exists C]\,.\,\boldsymbol{\rho}(CAB)\,.\!\vee\!.\,\boldsymbol{\rho}(ABB)\;\therefore\!\!\cdot$ [S21; 3; 6; 7]
$[\exists C]\,.\,\boldsymbol{\rho}(CAB)$ [9; 10; S19]

S23 $[AB]:\boldsymbol{\rho}(BBA)\,.\!\supset\!.\,\boldsymbol{\rho}(BAB)$
PR $[AB]:\mathrm{Hp}(1)\,.\!\supset\!.$
$[\exists C]\,.$
2. $\boldsymbol{\rho}(CAB).$ [S16, A/B, C/A; 1; S22]
3. $C=B\,.$ [S4, A/B, C/A, I/A, H/B, O/C, P/B; 1; 1; 2; 1]
$\boldsymbol{\rho}(BAB)$ [2; 3]

S24 $[AB]:\boldsymbol{\rho}(BBA)\,.\!\supset\!.\,\boldsymbol{\rho}(AAA)$
PR $[AB]:\mathrm{Hp}(1)\,.\!\supset\!.$
2. $\boldsymbol{\rho}(BAB)\,.$ [S23; 1]
$[\exists C]\,.$
3. $\boldsymbol{\rho}(CAA)\,.$ [S16, A/B, C/A; 1; S18]
4. $C=A\,.$ [S4, A/B, C/A, H/A, I/B, O/C, P/A; 1; 2; 3; 2]
$\boldsymbol{\rho}(AAA)$ [3; 4]

S25 $[A]:A\,\epsilon\,\mathbf{A}\,.\!\supset\!.\,[\exists C]\,.\,\boldsymbol{\rho}(ACA)$
PR $[A]\,\therefore\!\!\cdot\;\mathrm{Hp}(1)\,.\!\supset\!::$
$[\exists DE]::$
2. $\boldsymbol{\rho}(ADE)\,\therefore$ [S17; 1]
$[\exists MN]\,\therefore$
3. $\boldsymbol{\rho}(MND):$ [S16, B/D, C/E; 2; S17, A/D]
4. $[\exists C]\,.\,\boldsymbol{\rho}(ACA)\,.\!\vee\!.\,\boldsymbol{\rho}(DDA)::$
[S5, B/A, A/D, C/E, I/A, K/D, L/E, H/D; 2; 2; 3]
5. $[\exists C]\,.\,\boldsymbol{\rho}(ACA)\,.\!\vee\!.\,\boldsymbol{\rho}(AAA):$ [4; S24, B/D]
$[\exists C]\,.\,\boldsymbol{\rho}(ACA)$ [5]

S26 $[ABC]:\boldsymbol{\rho}(ADA)\,.\,\boldsymbol{\rho}(DBB)\,.\,\boldsymbol{\rho}(DDB)\,.\,\boldsymbol{\rho}(DAB)\,.\!\supset\!.\,\boldsymbol{\rho}(ADB)$
PR $[ABD]:\mathrm{Hp}(4)\,.\!\supset\!.$
5. $\boldsymbol{\rho}(BBB)\,.$ [S24, A/B, B/D; 3]
6. $D=B\,.$ [S19, A/D; 5; 2]
7. $\boldsymbol{\rho}(BAB)\,.$ [4; 6]
8. $\boldsymbol{\rho}(ABA)\,.$ [1; 6]
9. $A=B\,.$ [S4, A/B, C/B, H/A, I/B, O/A, P/B; 5; 7; 8; 5]
$\boldsymbol{\rho}(ADB)$ [1; 9]

S27 $[ABPMNVW]\,\therefore\,\boldsymbol{\rho}(APA)\,.\,\boldsymbol{\rho}(BVW)\,.\,\boldsymbol{\rho}(MNA)\,.\!\supset\!:\,[\exists C]\,.\,\boldsymbol{\rho}(ACB)\,.\!\vee\!.\,\boldsymbol{\rho}(PAB)$
[S5, B/A, A/P, C/A, I/B, K/V, L/W, H/A; 1; 2; 3]

S28 $[ABPQTVW]\,\therefore\,\boldsymbol{\rho}(APA)\,.\,\boldsymbol{\rho}(BVW)\,.\,\boldsymbol{\rho}(QTB)\,.\!\supset\!:\,[\exists C]\,.\,\boldsymbol{\rho}(ACB)\,.\!\vee\!.\,\boldsymbol{\rho}(PBB)$
[S5, B/A, A/P, C/A, I/B, K/V, L/W, M/Q, N/T, H/B; 1; 2; 3]

S29 $[ABPVWXY]\,\therefore\,\boldsymbol{\rho}(APA)\,.\,\boldsymbol{\rho}(BVW)\,.\,\boldsymbol{\rho}(XYP)\,.\!\supset\!:\,[\exists C]\,.\,\boldsymbol{\rho}(ACB)\,.\!\vee\!.\,\boldsymbol{\rho}(PPB)$
[S5, B/A, A/P, C/A, I/B, K/V, L/W, M/X, N/Y, H/P; 1; 2; 3]

S30 $[AB]:A,\,B\,\epsilon\,\mathbf{A}\,.\!\supset\!.\,[\exists C]\,.\,\boldsymbol{\rho}(ACB)$
PR $[AB]\,\because\!\because\,\mathrm{Hp}(2)\,.\!\supset\!\because\!\because$
$[\exists MN]\because\!\because$

3. $\boldsymbol{\rho}(MNA)$ :: [$S17$; 1]
$[\exists QTVW]$ ::
4. $\boldsymbol{\rho}(BVW)$. [$S17$, A/B; 2]
5. $\boldsymbol{\rho}(QTB)$ :: [$S17$, A/B; 2]
$[\exists P]$ ::
6. $\boldsymbol{\rho}(APA)$ ∴ [$S25$; 1]
$[\exists XY]$ ∴
7. $\boldsymbol{\rho}(XYP)$: [$S16$, B/P, C/A; 6; $S17$, A/P]
8. $[\exists C] . \boldsymbol{\rho}(ACB) . \vee . \boldsymbol{\rho}(PAB)$: [$S27$; 6; 4; 3]
9. $[\exists C] . \boldsymbol{\rho}(ACB) . \vee . \boldsymbol{\rho}(PBB)$: [$S28$; 6; 4; 5]
10. $[\exists C] . \boldsymbol{\rho}(ACB) . \vee . \boldsymbol{\rho}(PPB)$: [$S29$; 6; 4; 7]
11. $[\exists C] . \boldsymbol{\rho}(ACB) . \vee . \boldsymbol{\rho}(APB)$ :: [9; 10; 8; $S26$, D/P; 6]
$[\exists C] . \boldsymbol{\rho}(ACB)$ [11]

S31 $[ABCHIO] : \boldsymbol{\rho}(BAC) . \boldsymbol{\rho}(AHI) . \boldsymbol{\rho}(OCH) . \supset . \boldsymbol{\rho}(BOI)$
PR $[ABCHIO] : \mathrm{Hp}(3) . \supset .$
$[\exists P]$.
4. $\boldsymbol{\rho}(BPI)$. [$S16$, A/B, B/A; 1; $S16$, B/H, C/I; 2; $S30$, A/B, B/I]
5. $O = P$. [$S4$; 1; 2; 3; 4]
$\boldsymbol{\rho}(BOI)$ [4; 5]

S32 $[A] : A \epsilon \mathbf{A} . \supset . [\exists C] . \boldsymbol{\rho}(AAC)$
PR $[A]$ :: $\mathrm{Hp}(1) . \supset$ ::
$[\exists B]$ ::
2. $\boldsymbol{\rho}(ABA)$ ∴ [$S25$; 1]
$[\exists C]$ ∴
3. $\boldsymbol{\rho}(BCB)$: [$S16$, C/A; 2; $S25$, A/B]
$[\exists D]$:
4. $\boldsymbol{\rho}(DAC)$. [$S16$, A/B, B/C, C/B; 3; $S22$; 1]
5. $\boldsymbol{\rho}(ADB)$. [$S31$, B/A, A/B, C/A, H/C, I/B, O/D; 2; 3; 4]
$[\exists E]$.
6. $\boldsymbol{\rho}(AEC)$. [$S16$, A/B, B/C, C/B; 3; $S30$, B/C; 1]
7. $A = E$: [$S4$, B/A, A/D, C/B, H/A, I/C, O/A, P/E; 5; 4; 2; 6]
8. $\boldsymbol{\rho}(AAC)$ :: [6; 7]
$[\exists C] . \boldsymbol{\rho}(AAC)$ [8]

S33 $[AB] : \boldsymbol{\rho}(ABA) . \supset . \boldsymbol{\rho}(AAB)$
PR $[AB]$ ∴ $\mathrm{Hp}(1) . \supset$:
$[\exists C]$:
2. $\boldsymbol{\rho}(AAC)$. [$S16$, C/A; 1; $S32$]
$[\exists D]$.
3. $\boldsymbol{\rho}(DCB)$. [$S16$, B/A; 2; $S16$, C/A; 1; $S22$, A/C]
4. $D = B$. [$S4$, B/A, H/B, I/A, O/D, P/B; 2; 1; 3; 1]
5. $\boldsymbol{\rho}(BCB)$: [3; 4]
$\boldsymbol{\rho}(AAB)$ [$S31$, B/A, A/B, C/A, H/C, I/B, O/A; 1; 5; 2]

S34 $[ABCD] : \boldsymbol{\rho}(ABC) . \boldsymbol{\rho}(ACD) . \supset . B = D$

PR $[ABCD] :: \mathrm{Hp}(2)\ .\supset\therefore$

$[\exists E] \therefore$

3. $\boldsymbol{\rho}(AAE)\ .$ $[S16;\ 1;\ S32]$

4. $\boldsymbol{\rho}(EEE)\ .$ $[S24,\ A/E,\ B/A;\ 3]$

$[\exists F]\ .$

5. $\boldsymbol{\rho}(FEC)\ .$ $[S16,\ B/A,\ C/E;\ 3;\ S16;\ 1;\ S22,\ A/E,\ B/C]$

6. $F = C\ .$ $[S4,\ B/A,\ C/E,\ H/C,\ I/D,\ O/F,\ P/C;\ 3;\ 2;\ 5;\ 2]$

7. $\boldsymbol{\rho}(CEC)\ .$ $[5;\ 6]$

$[\exists G]\ .$

8. $\boldsymbol{\rho}(GDE)\ .$
$[S16,\ B/C,\ C/D;\ 2;\ S16,\ B/A,\ C/E;\ 3;\ S22,\ A/D,\ B/E]$

9. $G = B\ .$
$[S4,\ B/A,\ A/C,\ C/D,\ H/E,\ I/C,\ O/G,\ P/B;\ 2;\ 7;\ 8;\ 1]$

10. $\boldsymbol{\rho}(BDE)\ .$ $[8;\ 9]$

$[\exists H]\ .$

11. $\boldsymbol{\rho}(HEB)\ .$ $[S16,\ B/A,\ C/E;\ 3;\ S16;\ 1;\ S22,\ A/E]$

12. $H = B\ .$
$[S4,\ B/A,\ C/E,\ H/B,\ I/C,\ O/H,\ P/B;\ 3;\ 1;\ 11;\ 1]$

13. $\boldsymbol{\rho}(BEB)\ .$ $[11;\ 12]$

14. $\boldsymbol{\rho}(BBE)\ :$ $[S33,\ A/B,\ B/E;\ 13]$

$[\exists K] :$

15. $\boldsymbol{\rho}(DKE)\ .$
$[S16,\ B/C,\ C/D;\ 2;\ S16,\ B/A,\ C/E;\ 3;\ S30,\ A/D,\ B/E]$

$[\exists L]\ .$

16. $\boldsymbol{\rho}(LEK)\ .$
$[S16,\ A/D,\ B/K,\ C/E;\ 15;\ S22,\ A/E,\ B/K]$

17. $L = D :$
$[S4,\ A/D,\ C/E,\ H/K,\ I/E,\ O/L,\ P/D;\ 10;\ 15;\ 16;\ 10]$

18. $\boldsymbol{\rho}(DEK) :$ $[16;\ 17]$

19. $\boldsymbol{\rho}(DDE)\ .$ $[S31,\ B/D,\ A/E,\ C/K,\ H/E,\ I/E,\ O/D;\ 18;\ 4;\ 15]$

20. $\boldsymbol{\rho}(DED) \therefore$ $[S23,\ A/E,\ B/D;\ 19]$

$B = D$ $[S4,\ A/D,\ C/E,\ H/D,\ I/E,\ O/D,\ P/B;\ 10;\ 19;\ 20;\ 14]$

S35 $[ABC] : \boldsymbol{\rho}(ABC)\ .\supset.\ \boldsymbol{\rho}(ACB)$

PR $[ABC] : \mathrm{Hp}(1)\ .\supset.$

$[\exists D]\ .$

2. $\boldsymbol{\rho}(ADB)\ .$ $[S16;\ 1;\ S30]$

3. $D = C\ .$ $[S34,\ B/D,\ C/B,\ D/C;\ 2;\ 1]$

$\boldsymbol{\rho}(ACB)$ $[3;\ 4]$

S36 $[AB] : A,\ B \epsilon \mathbf{A}\ .\supset.\ [\exists C]\ .\ C \epsilon \mathbf{A}\ .\ \boldsymbol{\rho}(ABC)$

PR $[AB] : \mathrm{Hp}(2)\ .\supset.$

$[\exists C]\ .$

3. $\boldsymbol{\rho}(ACB)\ .$ $[S30;\ 1;\ 2]$

4. $\boldsymbol{\rho}(ABC)\ .$ $[S35,\ B/C,\ C/B;\ 3]$

5. $C \epsilon \mathbf{A}\ .$ $[S16;\ 4]$

$[\exists C]\ .\ C \epsilon \mathbf{A}\ .\ \boldsymbol{\rho}(ABC)$ $[5;\ 4]$

S37 $[ABCD] : \boldsymbol{\rho}(ABC)\ .\ \boldsymbol{\rho}(ABD)\ .\supset.\ C = D$

PR $[ABCD] : \mathrm{Hp}(2)\ .\supset.$

3. $\boldsymbol{\rho}(ADB)$. [$S35$, C/D; 2]

$C = D$ [$S34$, B/D, C/B, D/C; 3; 1]

S38 $[ABCDEFG]:\boldsymbol{\rho}(ABD).\boldsymbol{\rho}(DCE).\boldsymbol{\rho}(ACF).\boldsymbol{\rho}(FBG).\supset.\ E = G$

PR $[ABCDEFG]:\mathrm{Hp}(4).\supset.$

5. $\boldsymbol{\rho}(ADB)$. [$S35$, C/D; 1]

6. $\boldsymbol{\rho}(AFC)$. [$S35$, B/C, C/F; 3]

$[\exists H]$.

7. $\boldsymbol{\rho}(HBC)$. [$S16$, C/D; 1; $S16$, A/D, B/E; 2; $S22$, A/B, B/C]

8. $\boldsymbol{\rho}(AHE)$. [$S31$, B/A, A/D, C/B, H/C, I/E, O/H; 5; 2; 7]

9. $\boldsymbol{\rho}(HCB)$. [$S35$, A/H; 7]

10. $\boldsymbol{\rho}(AHG)$. [$S31$, B/A, A/F, H/B, I/G, O/H; 6; 4; 9]

$E = G$ [$S37$, B/H, C/E, D/G; 8; 10]

Since theses *S36* and *S37* are the consequences of *S1*, we can introduce into the system the following definition:

D2 $[ABX]:A\epsilon\mathbf{A}.B\epsilon\mathbf{A}.X\epsilon\mathbf{A}.\supset:A - B = X.\equiv.\boldsymbol{\rho}(ABX)$ [$S36$; $S37$]

Then:

S39 $[ABC]:A\epsilon\mathbf{A}.B\epsilon\mathbf{A}.C\epsilon\mathbf{A}.A - B = C.\equiv.\boldsymbol{\rho}(ABC)$

[$D2$, X/C; $D2$, X/C; $S16$]

A1 $[AB]:A, B\epsilon\mathbf{A}.\supset.A - B\epsilon\mathbf{A}$

PR $[AB]:\mathrm{Hp}(2).\supset.$

$[\exists C]$.

3. $C\epsilon\mathbf{A}$.

4. $\boldsymbol{\rho}(ABC)$. [$S36$; 1; 2]

5. $A - B = C$. [$S39$; 4]

$A - B\epsilon\mathbf{A}$ [5; 3]

B1 $[AB]:A, B\epsilon\mathbf{A}.\supset.A - (A - B) = B$

PR $[AB]\therefore\mathrm{Hp}(2).\supset:$

$[\exists C]:$

3. $\boldsymbol{\rho}(ABC)$. [$S36$; 1; 2]

$[\exists D]$.

4. $\boldsymbol{\rho}(ACD)$. [$S16$; 3; $S36$, B/C]

5. $B = D$. [$S34$; 3; 4]

6. $A - C = D$. [$S39$; B/C, C/D]

7. $A - C = B$. [6; 5]

8. $A - B = C:$ [$S39$; 3]

$A - (A - B) = B$ [7; 8]

C1 $[ABC]:A, B, C\epsilon\mathbf{A}.\supset.(A - B) - C = (A - C) - B$

PR $[ABC]::\mathrm{Hp}(3).\supset::$

$[\exists D]::$

4. $\boldsymbol{\rho}(ABD)$. [$S36$; 1; 2]

5. $A - B = D\therefore$ [$S39$, C/D; 4]

$[\exists E]\therefore$

6. $\boldsymbol{\rho}(DCE)$. [$S16$, C/D; 4; $S36$, A/D, B/C; 3]

7. $D - C = E$. [$S39$, A/D, B/C, C/E; 6]

8. $(A - B) - C = E :$ [7; 5]
$[\exists F]$.
9. $\boldsymbol{\rho}(ACF)$. [$S36$, B/C; 1; 3]
10. $A - C = F$. [$S39$, B/C, C/F; 9]
$[\exists G]$.
11. $\boldsymbol{\rho}(FBG)$. [$S16$, B/C, C/F; 10; 2; $S36$, A/F]
12. $F - B = G$. [$S39$, A/F, C/G; 11]
13. $(A - C) - B = G$. [12; 10]
14. $E = G ::$ [$S38$; 4; 6; 9; 11]
$(A - B) - C = (A - C) - B$ [8; 13; 14]

Thus, $\{S1\} \rightarrow \{A1; B1; C1\}$. Furthermore, $\{S1\} \rightarrow \{S39\}$.

5 It follows from sections **2**, **3**, and **4** that

$$\{A1; B1; C1\} \rightarrow \{A1; B1; B2\} \rightarrow \{A1; A2\} \rightarrow \{T1; T2\} \rightarrow \{S1\} \rightarrow \{A1; B1; C1\}$$

Moreover, it was proved in section **2.3** that formulas *A3* and *A6* which correspond to definitions *D1* and *D2* respectively, *cf.* section **4**, are the consequences of axioms *A1* and *A2*. And, vice versa, in section **4** it has been shown that *S1* implies *S39*, i.e., the formula which corresponds to definition *DI* in section **2.3**. Hence, the proof that the systems under consideration are inferentially equivalent is complete.

6 Clearly, the axioms

I $[ABC] :: \boldsymbol{\varphi}(ABC) .\equiv :. [\exists DEFG] . \boldsymbol{\varphi}(ADE) . \boldsymbol{\varphi}(CFG) :. [HI] :: \boldsymbol{\varphi}(HBI) .\equiv :. [\exists KLMN] . \boldsymbol{\varphi}(KHL) . \boldsymbol{\varphi}(MNI) :. [OP] : \boldsymbol{\varphi}(OCI) . \boldsymbol{\varphi}(PAH) .\supset . O = P$

and

II $[ABC] :: \boldsymbol{\varphi}(BAC) .\equiv :. [\exists DEFG] . \boldsymbol{\varphi}(ADE) . \boldsymbol{\varphi}(FGC) :. [HI] :: \boldsymbol{\varphi}(HBI) .\equiv :. [\exists KLMN] . \boldsymbol{\varphi}(HKL) . \boldsymbol{\varphi}(MIN) :. [OP] : \boldsymbol{\varphi}(OCI) . \boldsymbol{\varphi}(PAH) .\supset . O = P$

of Leśniewski for theory of Groups and theory of Abelian Groups respectively, *cf.* [2], p. 321, and [3], p. 243, and axiom

S1 $[ABC] :: \boldsymbol{\rho}(BAC) .\equiv :. [\exists DERS] . \boldsymbol{\rho}(DBE) . \boldsymbol{\rho}(RSC) :. [HI] :: \boldsymbol{\rho}(AHI) .\equiv :. [\exists KLMN] . \boldsymbol{\rho}(IKL) . \boldsymbol{\rho}(MNH) :. [OP] : \boldsymbol{\rho}(OCH) . \boldsymbol{\rho}(BPI) .\supset . O = P$

have a very similar structure. But, while both axioms of Leśniewski are compact, *cf.* [4], p. 62, point (d), axiom *S1* does not possess this important property. Namely, an inspection of the deductions presented in section **4** shows at once that all theorems proven there are the consequences of the theses *S2, S3, S4,* and *S5* which are derived from axiom *S1* in virtue of propositional calculus and the quantification theory. Whence, the sixth component of *S1* obtainable by the same means, viz.

S40 $[ABCDERS] :: \boldsymbol{\rho}(DBE) . \boldsymbol{\rho}(RSC) :. [HI] :: \boldsymbol{\rho}(AHI) .\equiv :. [\exists KLMN] . \boldsymbol{\rho}(IKL) . \boldsymbol{\rho}(MNH) :. [OP] : \boldsymbol{\rho}(OCH) . \boldsymbol{\rho}(BPI) . O = P :. \supset . \boldsymbol{\rho}(BAC)$ [*S1*]

follows from the theses *S2-S5, cf.* section **5**. Therefore the components of

S1 are not mutually independent, and, consequently, axiom *S1* is not compact in the sense of Leśniewski. On the other hand, it is easy to prove the mutual independency of the formulas *S2-S5*. As yet, I was unable to find a compact formula of the same length as *S1* which could be accepted as a single axiom of Subtractive Abelian Groups.

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