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SUBTRACTIVE ABELIAN GROUPS

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1 *Introduction* In an earlier paper of the author [1], the operation of subtraction was used to develop a simple set of axioms for Boolean algebras. It became apparent that the requirement of a distinguished element 1 and of the axioms

K1 $(a - b) - c = (a - c) - b$
K2 $1 - (1 - a) = a$
K3 $a - a = 1 - 1$
K4 $a - (a - b) = a - (1 - b)$

defines a Boolean algebra if one sets $a' = 1 - a$, $a \wedge b = a - b'$, and $a \vee b = (a' - b')$.

Thomas M. Hearne and Carl G. Wagner [2], also using subtraction, have defined a generalized Boolean algebra by means of only three axioms including K1. As the axioms K1-K3 are valid in every additive group with the usual definition of $a - b$ and with an arbitrary choice of the distinguished element 1, the question arises which further axiom has to be added to these three axioms to define an Abelian group. It suffices, in fact, to replace K4 by $a - (a - b) = b$. This obviates axiom K2 and, together with K1, even implies axiom K3. Let, therefore, a subtractive group be a system $\langle S, -\rangle$ which satisfies the axioms

S1 $(a - b) - c = (a - c) - b$
S2 $a - (a - b) = b$.

It is easy to show that these two axioms are satisfied in any Abelian group. It is the purpose of this paper to show that S1 and S2 may also be used to define an Abelian group.

Let us first demonstrate the independence and consistency of these axioms. Consider the set $S = \{a, b\}$ and let $a - a = a - b = b - b = a$, $b - a = b$. Then S1 holds, but not S2, for $a - (a - b) = a - a = a \neq b$. If instead we put $x - y = y$ for all $x, y \in S$, axiom S2 is satisfied, but not S1, since $(a - a) - b = b$ while $(a - b) - a = a$. If, finally, we set $a - a = b - b = a$ and $a - b = b - a = b$ both axioms are satisfied. This shows their consistency.

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2 *A more general structure* Instead of axiom S1 we will initially only assume the existence of an element 0 with the property that $a - 0 = a$ for all $a \in S$. S2 implies

Lemma 1 *$a - b = c$ if and only if $a - c = b$.*

Proof: If $a - b = c$ then $a - c = a - (a - b) = b$.

Corollary *$a - a = 0$ for all a.*

We are now in a position to define addition by $a + b = a - (0 - b)$. Then 0 is a neutral element for this addition. For we have for all $a = a + 0 = a - (0 - 0) = a - 0 = a$ and $0 + a = 0 - (0 - a) = a$. Further to every element a there belongs an additive inverse, namely $0 - a$, since $a + (0 - a) = a - (0 - (0 - a)) = a - a = 0$ and $(0 - a) + a = (0 - a) - (0 - a) = 0$.

3 *Subtractive and additive groups* We now assume both axioms S1 and S2. We shall first derive some lemmas which prepare the proof of the existence of an element 0 with the property that $a - 0 = a$ for all a. This allows us to assume the results of **2**. We then will prove the commutative and associative laws for addition in S. The following results are valid for all $a, b, c \in S$.

Lemma 2 *$a - b = (c - b) - (c - a) = (a - c) - (b - c)$.*

Proof: In view of S2 and S1

$$a - b = (c - (c - a)) - b = (c - b) - (c - a).$$

This implies $b - c = (a - c) - (a - b)$ and Lemma 1 now yields the second equation.

Lemma 3 *$a - a = b - b$.*

Proof: Using Lemma **2**, S1 and S2 we obtain

$$a - a = (a - b) - (a - b) = (a - (a - b)) - b = b - b.$$

There exists, therefore, a unique element 0 such that $a - a = 0$ for all a. Hence, by Lemma 1, $a - 0 = a$ for all a.

Lemma 4 *$a + b = b + a$.*

Proof:
$$\begin{aligned} a + b &= a - (0 - b) && [cf., \text{ section } 2]\\ &= (0 - (0 - a)) - (0 - b) && [S2]\\ &= (0 - (0 - b)) - (0 - a) && [S1]\\ &= b - (0 - a) && [S2]\\ &= b + a. && [cf., \text{ section } 2] \end{aligned}$$

Lemma 5 *$(a + b) + c = a + (b + c)$.*

Proof:
$$\begin{aligned} (a + b) + c &= (b + a) + c && [\text{Lemma } 4]\\ &= (b - (0 - a)) - (0 - c) && [cf., \text{ section } 2]\\ &= (b - (0 - c)) - (0 - a) && [S1] \end{aligned}$$

$$= (b + c) + a \qquad [cf., \text{ section } 2]$$
$$= a + (b + c). \qquad [\text{Lemma } 4]$$

This completes the proof of the following

Theorem 1 *In every subtractive group* $\langle S, -\rangle$ *there exists a unique element* 0 *such that* $a - 0 = a$ *for all* $a \in S$. *If addition in* S *is defined by* $a + b = a - (0 - b)$, *then* $\langle S, +\rangle$ *is an Abelian group.*

We call the group $\langle S, +\rangle$ defined by Theorem 1 the additive group associated to the subtractive group $\langle S, -\rangle$. Vice versa, for every Abelian group $\langle S, +\rangle$ we may define $a - b$ as the unique solution of the equation $x + b = a$ and this way obtain a subtractive group $\langle S, -\rangle$ called the subtractive group associated to $\langle S, +\rangle$. The following two theorems show that in this way we obtain a one-to-one-correspondence between additive and subtractive groups.

Theorem 2 *Let* $\langle S, \sim\rangle$ *be the subtractive group associated to the additive group* $\langle S, +\rangle$ *which in turn is associated to a given subtractive group* $\langle S, -\rangle$. *Then* $a \sim b = a - b$ *for all* $a, b \in S$.

Proof: $a \sim b$ is the unique solution in S of the equation $x + b = a$. It suffices to show that $a - b$ also satisfies this equation. Now

$$\begin{aligned}(a - b) + b &= (a - b) - (0 - b) = (a - (0 - b)) - b && [S1]\\ &= (b - (0 - a)) - b && [\text{Lemma } 4]\\ &= (b - b) - (0 - a) && [S1]\\ &= 0 - (0 - a) = a && [\text{Lemma } 3;\ S2]\end{aligned}$$

Theorem 3 *Let* $\langle S, +\rangle$ *be an additive group and let* $\langle S, -\rangle$ *be the subtractive group associated to it. Let further* $\langle S, \oplus\rangle$ *be the additive group associated to* $\langle S, -\rangle$. *Then* $a \oplus b = a + b$ *for all* $a, b \in S$.

Proof: $a \oplus b = a - (0 - b)$ is the unique solution of the equation $x + (0 - b) = a$ where $0 - b$ is the unique solution of $y + b = 0$. We must show that $a + b$ also solves the equation $x + (0 - b) = a$. This, however, follows from $(a + b) + (0 - b) = (a + b) + y = a + (b + y) = a + (y + b) = a + 0 = a$.

We remark in closing that the only requirement for $\langle T, -\rangle$ to be a subgroup of a subtractive group $\langle S, -\rangle$ is that it be closed with regard to subtraction. Because of $a + b = a - (0 - b)$ it then follows that the additive group associated to $\langle T, -\rangle$ is also closed with respect to addition.

Note The difference operation has already been used by A. Tarski [3] to define an Abelian group. He shows, in fact, that a system $\langle G, -\rangle$ which satisfies the axiom

G1 $a - (b - (c - (a - b))) = c$

or the two axioms

G2 $a - (a - b) = b$
G3 $a - (b - c) = c - (b - a)$

is an Abelian group with addition defined by $a + b = a - ((a - a) - b)$. It is not difficult to show directly that S1 and S2 imply G1 and also G2 and G3 and vice versa.

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