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THEORIES OF TYPES AND ORDERED PAIRS

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It is generally taken for granted that the Modern Theory of Types with its strictly linear type hierarchy is slightly weaker than the Simple Theory of Types with its complicated branching one. The reason for this supposition is the alleged banishment of relations between objects of unlike type from the Modern Theory, and with them such classical results as Cantor's Theorem. To be sure, such heterogeneous relations can be simulated (and the banished theorems along with them) but the simulation is not quite satisfying because it blurs fundamental distinctions of type.[1]

The fact is that Wiener's original representation of ordered pairs, although only given for a few pairs of relative types, did not present such difficulty.[2] It has been, perhaps, the nearly universal adoption of Kuratowski's simpler construction that has blinded us to this fact. In what follows I propose a trivial modification of the Kuratowski construction which restores severed connections by allowing the possibility of true heterogeneous ordered pairs and hence also heterogeneous relations and all the classical results requiring their existence for proof. The point, then, is that the Modern Theory of Types is fully equivalent to the Simple Theory of Types.

In what follows 'Λ^{n+1}' will denote the class of all objects of type n which

1. For a brief description and comparison of these two theories in which the allegation is made and the usual method of simulation presented, see W. V. Quine, *Set Theory and its Logic*, revised edition (Belnap Press, Cambridge, Mass., 1971), pp. 249-265. For a more complete description of the Simple Theory of Types, see Hilbert and Ackermann, *Principles of Mathematical Logic* (Chelsea, New York, 1950), pp. 152-161.

2. Norbert Wiener, "A simplification of the logic of relations," *Proceedings of the Cambridge Philosophical Society*, vol. 17 (1914), pp. 387-390, reprinted in Jean van Heijenoort, *From Frege to Gödel* (Harvard University Press, Cambridge, Mass., 1967), pp. 224-227.

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are not self-identical (the empty class of such objects), and braces will be used in the usual way to designate classes by enumerating their members.

Definition Scheme 1:

$$\mathsf{I}_0 x^n = \{\{x^n\}, \Lambda^{n+1}\}$$
$$\mathsf{I}_{k+1} x^n = \{\mathsf{I}_k x^n\}$$

This definition is doubly schematic: for each of the infinitely many types n, it represents an infinite number of definitions, one for each subscript.

Definition Scheme 2:

$$\langle x^m, y^n\rangle = \{\{\mathsf{I}_{\max(m,n)-m} x^m\}, \{\mathsf{I}_{\max(m,n)-m} x^m, \mathsf{I}_{\max(m,n)-n} y^n\}\}$$

Examination will show that this is just the Kuratowski pair of $\mathsf{I}_{\max(m,n)-m} x^m$ and $\mathsf{I}_{\max(m,n)-n} y^n$. The result of this change will turn out to be a notion of ordered pair which is similar to Wiener's in allowing the pairing of objects of unlike types without any blurring of distinctions of type. And since either $\max(m,n) - m = 0$ or $\max(m,n) - n = 0$, the type of a pair of objects will be four greater than the larger of the two types.

The proofs of the following metatheorems are simple, never requiring more than an easy induction in the metalanguage for their proofs. For this reason I will only give brief indications of how to construct them.

Metatheorem 1 *If the expression*

$$\Lambda^h \in \mathsf{I}_k x^n$$

is both well-formed and true, then $k = 0$ *and* $h = n + 1$.

(The proof is by induction on k using Definition Scheme 1.)

Metatheorem 2 *For every* h, k, m, n, *if*

$$\mathsf{I}_h x^m = \mathsf{I}_k y^m$$

is well-formed, then it logically implies

$$x^m = y^n.$$

(The proof is by induction on h using the previous metatheorem and the fact that by Definition Scheme 1, $\mathsf{I}_k x^m$ is either a singleton (just in case $k \neq 0$) or the unordered pair composed of singleton x^m and the appropriate empty set.)

Metatheorem 3 *For every* m, n, h, k, *if*

$$\langle x^m, y^n\rangle = \langle z^h, w^k\rangle,$$

is well-formed, then it logically implies

$$\mathsf{I}_{\max(m,n)-m} x^m = \mathsf{I}_{\max(h,k)-h} z^h \text{ and } \mathsf{I}_{\max(m,n)-n} y^n = \mathsf{I}_{\max(h,k)-k} w^k$$

(by Definition Scheme 2 and the corresponding fact for Kuratowski pairs); it also implies

$$x^m = z^h \text{ and } y^n = w^k$$

(from the above fact together with Metatheorem 2). And since identity is homogeneous, we can infer that $m = h$ and $n = k$ from the fact that

$$\langle x^m, y^n \rangle = \langle z^h, w^k \rangle$$

is well-formed.

The second part of Metatheorem 3 is the main result of this essay and tells us that the definition of ordered pair given in Definition Scheme 2 provides us with true heterogeneous ordered pairs (and with them heterogeneous relations and the theorems whose proofs require the existence of such relations) which preserve all distinctions of type. With heterogeneous relations, the Modern Theory of Types becomes fully equivalent to the Simple Theory of Types.

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