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A NOTE ON $\mathcal{P}$-ADMISSIBLE SETS WITH URELEMENTS

JUDY GREEN

In [2] Barwise states that although the introduction of urelements into Zermelo-Fraenkel set theory is redundant, their introduction into the weaker Kripke-Platek theory for admissible sets is not. In this note* we will show that their introduction into the intermediate theory of power set admissible sets is once again redundant since all $\mathcal{P}$-admissible sets with urelements are of the same form as $\mathcal{P}$-admissible sets, i.e., $\mathsf{V}_M(\kappa) = \mathsf{H}_M(\kappa)$ where κ is a strong limit cardinal and $\kappa = \beth_\kappa$.

We assume familiarity with the formulation of the theory **KPU** (Kripke-Platek with urelements) and the language in which it is formulated (see [2]). We also assume familiarity with the hierarchy of set theoretic predicates due to Lévy [5], and the primitive recursive set functions of Jensen and Karp [4]. We expand the notation of [2] as follows:

Definition: A structure $\mathfrak{A}_{\mathfrak{M}} = (\mathfrak{M}; A, E, P, \ldots)$ for the language $\mathsf{L}(\epsilon, \mathcal{P}, \ldots)$ consists of

(1) a structure $\mathfrak{M} = \langle M, \ldots \rangle$ for the language L,
(2) a nonempty set A disjoint from M,
(3) a relation $E \subseteq (M \cup A) \times A$ to interpret ϵ,
(4) a function P from A into A to interpret $\mathcal{P}$, and
(5) other functions, relations, and constants on $M \cup A$ which interpret the other symbols in $\mathsf{L}(\epsilon, \mathcal{P}, \ldots)$.

In the language $\mathsf{L}(\epsilon, \mathcal{P}, \ldots)$ variables are distinguished to allow quantification over M (urelements), A (sets), and $A \cup M$. The variables used are, respectively: $p, q, r, \ldots$; $a, b, c, d, \ldots$; and $x, y, z, \ldots$.

Definition: The theory $\mathcal{P}$-**KPU** consists of the universal closures of the axioms of

extensionality: $\quad \forall x(x \epsilon a \leftrightarrow x \epsilon b) \rightarrow a = b$,

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foundation:	$\exists a\phi(a) \rightarrow \exists a(\phi(a) \wedge \forall b \in a \sim \phi(b))$ *for all formulas* $\phi(a)$ *in which* b *is not free*,
pair:	$\exists a(x \in a \wedge y \in a)$,
union:	$\exists b \forall y \in a \forall x \in y(x \in b)$,
Δ_0 in $\mathcal{P}$-collection:	$\forall x \in a \exists y \phi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b \phi(x, y)$ *for all* Δ_0 *in* $\mathcal{P}$ *formulas* $\phi(x, y)$ *in which* b *is not free, and*
power set:	$\forall a \exists b(b = \mathcal{P}(a))$ $b = \mathcal{P}(a) \leftrightarrow \forall c(c \in b \leftrightarrow \forall d(d \in c \rightarrow d \in a))$.

We let $\mathsf{P}_M(a)$ denote the power set of $a \cup M$ and define the universe of sets, V_M, using P_M instead of the usual power set operation. I.e.,

$$\mathsf{V}_M(0) = 0$$
$$\mathsf{V}_M(\alpha + 1) = \mathsf{P}_M(\mathsf{V}_M(\alpha))$$
$$\mathsf{V}_M(\lambda) = \bigcup_{\alpha<\lambda} \mathsf{V}_M(\alpha) \text{ if } \lambda \text{ is a limit ordinal.}$$

We call a structure $\mathfrak{A}_{\mathfrak{M}}$ for $\mathsf{L}(\epsilon, \mathcal{P}, \ldots)$ $\mathcal{P}$-admissible if $\mathfrak{A}_{\mathfrak{M}}$ is a model of $\mathcal{P}$-**KPU**, E is the restriction to $A \cup M$ of the membership relation ϵ_M of $\mathsf{V}_M \cup M$, A is a transitive$_M$ subset of V_M, i.e., $x \in_M y \in_M A$ implies $x \in_M A$, and P is the restriction to A of P_M. As in the case of $\mathcal{P}$-admissible sets without urelements, this definition is equivalent to the following: E is the restriction to $A \cup M$ of ϵ_M, P is the restriction to A of P_M, A is a transitive$_M$ subset of V_M which is Prim $\mathcal{P}$ closed (i.e., is closed under the primitive recursive in $\mathcal{P}$ set functions) and which satisfies the Δ_0 in $\mathcal{P}$ collection scheme.

We define the rank and transitive closure functions on $A \cup M$ as usual, i.e., $\rho_M(x) = \bigcup \{\rho_M(y) + 1 \mid y \in_M x\}$ and $\mathsf{TC}_M(x) = x \cup \bigcup \{\mathsf{TC}_M(y) \mid y \in_M x\}$, and note that both of these functions are primitive recursive. We also note that V_M is a primitive recursive in $\mathcal{P}$ function. As in the case without urelements, at the α'th stage of construction of the universe we have all sets of rank less than α, i.e., $\mathsf{V}_M(\alpha) = \{a \mid \rho_M(a) < \alpha\}$. Let $\mathsf{ord}(A)$ be the set of ordinals in A.

Lemma *If* $\mathfrak{A}_{\mathfrak{M}}$ *is* $\mathcal{P}$*-admissible then* $A = \mathsf{V}_M(\mathsf{ord}(A))$.

Proof: This follows directly from the fact that A is closed under the functions ρ_M, P_M, and V_M.

Lemma *If* $\mathfrak{A}_{\mathfrak{M}}$ *is* $\mathcal{P}$*-admissible and* $a \in A$, *then* $|a| \in A$.

Proof: Suppose $a \in A$ and f is an isomorphism from a onto $|a|$. The relation r defined on $a \times a$ by $\langle x, y\rangle \in r$ iff $f(x) \in f(y)$ is an element of A since A is closed under the functions $\times$ and P_M. If g is the function which defines the r predecessors of elements of a, i.e., if $x \in_M a$ $g(r, x) = \{z \mid \langle z, x\rangle \in r\} = \{(b)_0 \mid b \in r \wedge (b)_1 = x\}$, then g is primitive recursive and hence Σ_1 definable on A. Since r is an element of A, f can now be seen to have the Σ_1 definition:

$$f(x) = \alpha \leftrightarrow \exists c \exists b(c = g(r, x) \wedge \mathsf{fcn}(b) \wedge \mathsf{dm}(b) = c \wedge \mathsf{rg}(b) = \alpha \wedge \forall y \in c \exists d(d = g(r, y) \wedge b(y) = \mathsf{rg}(b \upharpoonright d))).$$

Hence by Σ replacement (see [1]) $f \in A$, i.e., since f is Σ on A, $\mathrm{dm}(f) \in A$ and $\mathrm{rg}(f) \subseteq A$ we have $f \in A$. But $|a| = \mathrm{rg}(f)$, so $|a| \in A$.

A similar proof shows:

Lemma: *If* $\mathfrak{A}_{\mathfrak{M}}$ *is* $\mathscr{P}$*-admissible then* $\mathrm{ord}(A)$ *is a cardinal.*

Theorem: *If* $\mathfrak{A}_{\mathfrak{M}}$ *is* $\mathscr{P}$*-admissible, then* $A = V_M(\kappa) = H_M(\kappa)$ *where* κ *is a strong limit cardinal such that* $\kappa = \beth_\kappa$.

Proof: Since $\mathrm{ord}(A) = \kappa$ is a cardinal and A is closed under the function P_M, κ is a strong limit cardinal. Since $A = V_M(\kappa)$ is closed under the cardinality function, $V_M(\kappa) \subseteq H_M(\kappa)$. Since $|\rho_M(a)| \leqslant |TC_M(a)|$ for all sets $a \in V_M$ (see [5]) we have $H_M(\kappa) \subseteq V_M(\kappa)$. Finally A's closure under the cardinality function and the function V_M gives $\kappa = \beth_\kappa$.

As a final remark we note that using exactly the same methods as in the case without urelements, i.e., consistency properties [3], we get the $\mathrm{Cf}\ \omega$ compactness theorem of Barwise and Karp for $\mathscr{P}$-admissible sets with urelements: *If* $\mathfrak{A}_{\mathfrak{M}}$ *is* $\mathscr{P}$*-admissible and* $A = V_M(\kappa)$ *with* $\mathrm{cf}(\kappa) = \omega$, *then* $\mathfrak{A}_{\mathfrak{M}}$ *is* Σ_1 *compact.*

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Rutgers, The State University of New Jersey
Camden, New Jersey