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NONSTANDARD PROBABILITY

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Introduction[1] If F is a finite set, there is a natural probability measure on F given by $\mu_F(A) = \|A\|/\|F\|$, where $A \subseteq F$ and $\|X\|$ denotes the number of elements in a finite set X. In their paper [1], A. R. Bernstein and F. Wattenberg have shown the existence of a *finite set F such that if A is a Lebesgue measurable subset of [0, 1], then $\mu_F(A) = \|{}^*A \cap F\|/\|F\| \simeq m(A)$, where m is Lebesgue measure. The measure μ_F is called a sample measure. Since Lebesgue measure on [0, 1] is the measure induced by the uniform distribution on [0, 1], it is natural to ask for what other probability distributions on the real numbers can a similar result be shown. If the notion of sample measure is generalized, then the result of [1] may be extended to arbitrary real measures induced by probability distributions. This generalization and extension form the main portion of this paper. As an application of this extended result two nonstandard theorems of the central limit type are stated.

Preliminaries Let *R be an enlargement of the real number system R. Each set or concept in R will receive the prefix '*' when denoting the corresponding set or concept in *R, e.g., a *finite subset of *R is a subset on which there is an internal bijection onto an initial segment of *N, the enlargement of the natural numbers. (See [3] or [4] for more details.) Let $\langle z_n \rangle_{n \in N}$ be a sequence (necessarily external) in *R. Then $\lim_n z_n = z$ will mean $z \in R$ and, for every standard $\epsilon > 0$, there exists an $m \in N$ such that $n \geq m$ implies $|z_n - z| < \epsilon$. Equivalently, $\lim_n z_n = z$ means the z_n are eventually near-standard and z is the limit of the standard parts of these z_n.

A double sequence $\{\{X_{nk}\}\}$ will be a collection of random variables with $n \in N$, $1 \leq k \leq k_n$, and $k_n \to \infty$ as $n \to \infty$. A double sequence $\{\{X_{nk}\}\}$ is said to

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be infinitesimal if and only if, for each standard $\epsilon > 0$, $\lim_n \max_k P\{|X_{nk}| \geq \epsilon\} = 0$. A double sequence is said to be row-wise independent if and only if the random variables $X_{n1}, \ldots, X_{nk_n}$ are independent for each n. If $\{\{X_{nk}\}\}$ is a double sequence, then the characteristic and distribution functions of X_{nk} will be denoted by φ_{nk} and F_{nk}, respectively. The normal distribution with mean 0 and variance 1 will be denoted by $N(0, 1)$.

Results Let M be the set of real measures induced by probability distribution functions. Let $\mu \in M$. If μ_F is a sample measure such that F is actually infinite, then

$$\mu_F(\{x\}) = \begin{cases} 0 & ; \ x \notin F \\ 1/\|F\| ; & x \in F \end{cases} \simeq 0.$$

But $\mu(\{x\})$ need not equal 0. Thus, sample measures are not sufficient for approximating arbitrary elements of M. However, if the *finite set F is replaced by a *finite sequence, the result can be proven.

Definition Let $S = \langle x_i \rangle$, $i = 1, \ldots, \omega$, be a *finite sequence, define the sequence measure $\mu_S(A) = \sum_{[x_i \in A]} 1/\omega$, for A an internal subset of *R. If $A \subseteq R$, let $\mu_S(A) = \mu_S(^*A)$.

Theorem 1 *If $\mu \in M$, then there exists a sequence measure μ_S such that for each μ-measurable set A, $\mu_S(A) \simeq \mu(A)$.*

The proof is given in the following two lemmas.

Lemma 1 *There exists a sequence measure μ_S such that $\mu_S(I) \simeq \mu(I)$ for each interval I.*

Proof: Let F be the distribution function of μ, i.e., $F(x) = \mu(\{y \mid y \leq x\})$. Let ω be an infinite natural number. Set $x_k = \inf\{x \mid (2k-1)\}2^{\omega+1} \leq F(x)\}$ for $k = 1, \ldots, 2^\omega$. Then, for $S = \langle x_k \rangle$, $k = 1, \ldots, 2^\omega$, $\mu_S(I) \simeq \mu(I)$ for each interval I.

Lemma 2 *Let μ_S be a sequence measure such that $\mu_S(I) \simeq \mu(I)$ for each interval I; then $\mu_S(A) \simeq \mu(A)$ for each μ-measurable set A.*

Proof: Let a standard $\epsilon > 0$ be given. Let A be μ-measurable; then there exist countable collections of intervals $\{I_i\}$ and $\{J_i\}$ such that $\bigcup_i I_i \supseteq A$, $\bigcup_i J_i \supseteq (R - A)$, and

$$(1) \quad \sum_i \mu(I_i) - \epsilon/4 < \mu(A) = 1 - \mu(R - A) < 1 - \sum_i \mu(J_i) + \epsilon/4.$$

For each $n \in N$,

$$(2) \quad \begin{aligned} &\left|\sum_{i=1}^{n} \mu(J_i) - \sum_{i=1}^{n} \mu_S(J_i)\right| < \epsilon/4, \\ &\left|\sum_{i=1}^{n} \mu(I_i) - \sum_{i=1}^{n} \mu_S(I_i)\right| < \epsilon/4. \end{aligned}$$

Thus, if $T = \{n \in {}^*N \mid \text{condition (2) holds}\}$, then T must contain an infinite natural number ω. Thus,

$$(3)\quad \begin{aligned} &\left|\sum_{i=1}^{\omega} \mu(J_i) - \sum_{i=1}^{\omega} \mu_S(J_i)\right| < \epsilon/4, \\ &\left|\sum_{i=1}^{\omega} \mu(I_i) - \sum_{i=1}^{\omega} \mu_S(I_i)\right| < \epsilon/4. \end{aligned}$$

Furthermore,

$$(4)\quad \begin{aligned} &\sum_{i=1}^{\omega} \mu(J_i) \simeq \sum_{i} \mu(J_i), \\ &\sum_{i=1}^{\omega} \mu(I_i) \simeq \sum_{i} \mu(I_i). \end{aligned}$$

Combining (1), (3), and (4) yields

$$\begin{aligned} \mu(A) - \epsilon/2 &< 1 - \sum_{i} \mu(J_i) - \epsilon/4 \simeq 1 - \sum_{i=1}^{\omega} \mu(J_i) - \epsilon/4 \\ &\leqslant 1 - \sum_{i=1}^{\omega} \mu_S(J_i) \leqslant \mu_S(A) \leqslant \sum_{i=1}^{\omega} \mu_S(I_i) \\ &\leqslant \sum_{i=1}^{\omega} \mu(I_i) + \epsilon/4 \simeq \sum_{i} \mu(I_i) + \epsilon/4 < \mu(A) + \epsilon/2. \end{aligned}$$

Thus $|\mu(A) - \mu_S(A)| < \epsilon$. Since ϵ was arbitrary, $\mu_S(A) \simeq \mu(A)$.

Remark 1 The sample measures μ_F constructed in [1] had the additional properties:

(i) $[0, 1) \subseteq F$; (ii) $F + 1/n = F$ for each $n \in N$; (iii) $\mu_F(A + y) \simeq \mu_F(A)$,

where addition is modulo 1. Using the methods of Lemma 1 a *finite set F can be constructed which satisfies the properties (ii) and (iii); however, property (i) is not obtained by this method.

Remark 2 Just as in [1] if the integral, $\int f d\mu_S$, is defined as $\sum_i f(x_i)/\omega$, then for μ and μ_S as in Theorem 1, $\int f d\mu_S \simeq \int f d\mu$ for each bounded, measurable function f.

Application Let X be a random variable. Then X induces a measure on R via its distribution function. Let $S = \langle x_i \rangle$ be a *finite sequence, and let μ be the measure induced by X. Then $S \simeq X$ will mean μ and μ_S satisfy Theorem 1. The symbols $\sum^\epsilon$ and $\sum_\epsilon$ will denote the sums over the indices i for which $|x_i| \geqslant \epsilon$ and $|x_i| < \epsilon$, respectively.

Theorem 2 *Suppose the double sequence* $\{\{X_{nk}\}\}$ *is row-wise independent and each* X_{nk} *has mean zero. If there exist *finite sequences* $S_{nk} = \langle x_{nk}^{(i)} \rangle$, $i = 1, \ldots, \omega_{nk}$, *such that* $S_{nk} \simeq X_{nk}$ *and, for every standard* $\epsilon > 0$,

(5a) $\lim_n \sum_k 1/\omega_{nk} \sum^\epsilon (x_{nk}^{(i)})^2 = 1$,

(5b) $\lim_n \sum_k \sum_\epsilon 1/\omega_{nk} = 0$,

(5c) $\lim_n \sum_k 1/\omega_{nk} \sum_i x_{nk}^{(i)} = 0$,

then $\{\{X_{nk}\}\}$ *is infinitesimal and the sequence* $\{\sum_k X_{nk}\}$ *converges weakly to a random variable which is distributed* N(0, 1).

Theorem 3 *Suppose* $\{\{X_{nk}\}\}$ *is a double sequence which is row-wise independent. Suppose there exist *finite sequences* $S_{nk} \simeq X_{nk}$, *satisfying, for every standard* $\epsilon > 0$,

(6a) $\lim_n \sum_k \sum^{\epsilon} 1/\omega_{nk} = 0,$

(6b) $\lim_n \sum_k 1/\omega_{nk} \sum_{\epsilon} (x_{nk}^{(i)})^2 = 1,$

(6c) $\lim_n \sum_k 1/\omega_{nk} |\sum_{\epsilon} x_{nk}^{(i)}| = 0,$

then $\{\{X_{nk}\}\}$ *is infinitesimal and the sequence* $\{\sum_k X_{nk}\}$ *converges weakly to a random variable distributed* N(0, 1).

The proof of Theorem 2 is based on the following result.

Lemma 3 *Let* $\{\{X_{nk}\}\}$ *be an infinitesimal system of random variables with variances* σ_{nk}^2 *and means zero. Let* $\beta_{nk}(t) = \varphi_{nk}(t) - 1$. *If there exists a positive constant* C *such that* $\sum_k \sigma_{nk}^2 < C$, *then* $\lim_n \prod_k \varphi_{nk}(t) = \lim_n \exp[\sum_k \beta_{nk}(t)]$.

This result is proven using the series expansion of log $\varphi_{nk}(t)$, the fact that $\beta_{nk}(t) = \int (\exp[itx] - 1 - itx)\, dF_{nk}$, and $\lim_n \max_k |\beta_{nk}(t)| = 0$. (See [2], pp. 259-260.)

If the expression $\sum_k \beta_{nk}(t)$ is written in terms of the S_{nk} using Remark 2 to replace integrals by sums, and if exp[itx] is expanded in a Taylor series with two terms and remainder, then it becomes apparent that the conditions (5) are sufficient to imply $\lim_n \sum_k \beta_{nk}(t) = -\frac{1}{2}t^2$. But the conditions (5) also imply that for sufficiently large n the X_{nk} satisfy the hypotheses of Lemma 3, and Theorem 2 follows.

The original motivation for this work was a nonstandard proof of the Lindeberg Central Limit Theorem. However, Theorem 2 is not sufficient for this purpose. Theorem 3 is similar to a standard theorem in [2] which is known to be sufficient for the Lindeberg theorem. However, the proof of Theorem 3 depends only on Theorem 2 and elementary results in probability, while the theorem in [2] requires a much stronger central limit theorem whose proof depends on the Lévy-Khintchine representation of infinitely divisible characteristic functions. Furthermore, Theorem 3 has as a corollary the Lindeberg theorem which is stated here for completeness.

Central Limit Theorem (Lindeberg) *Suppose* $\{X_k\}$ *is a sequence of independent random variables with means* α_k *and non-zero variances* σ_k^2. *Let* $\tau_n^2 = \sum_{k=1}^{n} \sigma_k^2$. *If, for every standard* $\epsilon > 0$, $\lim_n 1/\tau_n^2 \sum_k \int_{|x-\alpha_k|<\epsilon} x^2 dF_k = 0$, *then the sequence* $\{\sum_{k=1}^{n} 1/\tau_n (X_k - \alpha_k)\}$ *converges weakly to a random variable which is distributed* N(0, 1).

REFERENCES

[1] Bernstein, A. R., and F. Wattenberg, "Nonstandard measure theory," in *Applications of Model Theory to Algebra, Analysis, and Probability*, Holt, Reinhart, and Winston, New York (1969).

[2] Burrill, C. W., *Measure, Integration, and Probability*, McGraw-Hill Book Company, New York (1972).

[3] Luxemburg, W. A. J., "A general theory of monads," in *Applications of Model Theory to Algebra, Analysis, and Probability*, Holt, Rinehart, and Winston, New York (1969).

[4] Robinson, A., *Non-standard Analysis*, North-Holland Publishing Company, Amsterdam (1966).

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