Notre Dame Journal of Formal Logic
Volume XVI, Number 3, July 1975
NDJFAM

COMPACTNESS IN ABSTRACTIONS OF POST ALGEBRAS

R. BEAZER

Introduction An algebra $\langle A; F\rangle$ is said to be *equationally compact* if the existence of a simultaneous solution of any finite subset of any set Σ of polynomial equations (with constants) in A implies the existence of a simultaneous solution of Σ. An algebra $\langle A; F\rangle$ is said to be *topologically compact* if A is endowed with a compact, Hausdorff topology under which all the operations in F are continuous. The problem of determining the equationally compact algebras and the answer to Mycielski's question (see [15]), "Is every equationally compact algebra a retract of a topologically compact algebra?" for a particular class of algebras, is usually a difficult one. The problem has been solved for semilattices in [11] and [3], for Boolean algebras in [17], and for Post and Post-like algebras in [2]. In recent years, several abstractions of Post algebras have been studied. The purpose of this note is the characterization of the equationally compact algebras in some of these classes.

Preliminaries A *Brouwerian algebra* is an algebra $\langle A; \vee, \wedge, \rightarrow\rangle$, where $\langle A; \vee, \wedge\rangle$ is a lattice and $\rightarrow$ is a binary operation such that $x \wedge y \leq z$ *if and only if* $x \leq y \rightarrow z$. Every Brouwerian algebra is distributive and has a greatest element 1. A *Heyting algebra* is a Brouwerian algebra with a least element 0. In a Heyting algebra A, the element $x \rightarrow 0$ will be denoted by x^* and is the *pseudocomplement* of x in A. The set $S(A) = \{x \in A; x = x^{**}\}$ forms a Boolean algebra; *the algebra of closed elements* of A. The set $D(A) = \{x \in A; x^* = 0\}$ forms a filter; *the filter of dense elements* in A. A bounded, distributive, pseudocomplemented lattice satisfying the identity $x^* \vee x^{**} = 1$ is called a *Stone algebra.* In any Stone algebra A, $S(A)$ coincides with the centre $C(A)$ of A. An L-*algebra* is a Heyting algebra satisfying the identity $(x \rightarrow y) \vee (y \rightarrow x) = 1$. Any L-algebra is a Stone algebra and satisfies the identity $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$. Consequently, the operation $\vee$ can be omitted from the set of fundamental operations. For the connection between L-algebras and logic, the reader is referred to [13].

Received August 29, 1974

A $\mathbf{K}_n$-*algebra* is a Brouwerian algebra satisfying the identity $\bigvee_{i=1}^{n}(x_i \to x_{i+1}) = 1$, $n \geqslant 2$. The class of all $\mathbf{K}_n$-algebras is denoted by $\mathcal{K}_n$. An $\mathbf{L}_n$-*algebra* is a $\mathbf{K}_n$-algebra with least element $\mathbf{0}$. The class of all $\mathbf{L}_n$-algebras is denoted by $\mathcal{L}_n$. The classes $\mathcal{K}_n$ and $\mathcal{L}_n$ were introduced and studied in [12].

A $\mathbf{P}$-*algebra* is an $\mathbf{L}$-algebra whose dual is an $\mathbf{L}$-algebra. Equivalently, a $\mathbf{P}$-algebra is an $\mathbf{L}$-algebra whose dual is a Stone algebra. For other characterizations and applications of $\mathbf{P}$-algebras to logic, the reader is referred to [7] and the references therein. The class of all $\mathbf{P}$-algebras is denoted by $\mathcal{P}$. A $\mathbf{P}^{[n]}$-*algebra* is an $\mathbf{L}_n$-algebra whose dual is an $\mathbf{L}$-algebra. The class of all $\mathbf{P}^{[n]}$-algebras will be denoted by $\mathcal{P}^{[n]}$. Such algebras arise quite naturally. Indeed, it can be shown that the lattice of equational classes of $\mathbf{P}$-algebras is the chain $\mathcal{P}^{[1]} \subset \mathcal{P}^{[2]} \subset \ldots \subset \mathcal{P}^{[n]} \subset \ldots \subset \mathcal{P}$. Furthermore, $\mathcal{P}^{[n]}$ is the class of all $\mathbf{P}$-subalgebras of Post algebras of order n (see [7]).

A *Łukasiewicz algebra of order n* ($n \geqslant 2$) is an algebra $\langle A; \vee, \wedge; \sim, \mathbf{s}_1, \ldots, \mathbf{s}_{n-1}; 1\rangle$, where $\langle A; \vee, \wedge; 1\rangle$ is a distributive lattice with 1 and $\sim$, $\mathbf{s}_1, \ldots, \mathbf{s}_{n-1}$ are unary operations of A satisfying the following conditions:

M1. $\sim\sim x = x$,
M2. $\sim(x \vee y) = \sim x \wedge \sim y$,
M3. $\mathbf{s}_i(x \vee y) = \mathbf{s}_i(x) \vee \mathbf{s}_i(y)$,
M4. $\mathbf{s}_i(x) \vee \sim\mathbf{s}_i(x) = 1$,
M5. $\mathbf{s}_i(\mathbf{s}_j(x)) = \mathbf{s}_j(x)$,
M6. $\mathbf{s}_i(\sim x) = \sim\mathbf{s}_{n-i}(x)$,
M7. $\mathbf{s}_i(x) \vee \mathbf{s}_{i+1}(x) = \mathbf{s}_{i+1}(x)$, $(1 \leqslant i \leqslant n-2)$,
M8. $x \vee \mathbf{s}_{n-1}(x) = \mathbf{s}_{n-1}(x)$,
M9. $(x \wedge \sim\mathbf{s}_i(x) \wedge \mathbf{s}_{i+1}(y)) \vee y = y$, $(1 \leqslant i \leqslant n-2)$.

The class of all Łukasiewicz algebras of order n is denoted by $\mathbf{Łuk}_n$. A systematic study and references to previous work on Łukasiewicz algebras can be found in [4] and [5]. In any Łukasiewicz algebra $\sim 1 = 0$ (the least element) and we have:

$$x \in \mathrm{C}(A) \textit{ if and only if } \mathbf{s}_i(x) = x \ (1 \leqslant i \leqslant n-1).$$

Furthermore, if $x \in \mathrm{C}(A)$ then the complement of x is $\sim x$.

Some abstractions of Post algebras involving a chain of distinguished elements are defined below.

A $\mathbf{P}_0$-*lattice* is a bounded, distributive lattice A which is generated by its centre $\mathrm{C}(A)$ and a finite subchain $0 = \mathbf{e}_0 \leqslant \mathbf{e}_1 \leqslant \ldots \leqslant \mathbf{e}_{n-1} = 1$, called a *chain-base* for A. The *order* of a $\mathbf{P}_0$-lattice A is the smallest integer n such that A has a chain-base with n distinct terms. A $\mathbf{P}_1$-*lattice* is a $\mathbf{P}_0$-lattice with a chain base such that $\mathbf{e}_{i+1} \to \mathbf{e}_i = \mathbf{e}_i$. A $\mathbf{P}_2$-*lattice* is a $\mathbf{P}_1$-lattice $\langle A; \mathbf{e}_0, \ldots, \mathbf{e}_{n-1}\rangle$ such that $!\mathbf{e}_i$ exists, where $!x$ is the largest central element which is $\leqslant x$ and called the *pseudo-supplement* of x in A.

If A is a lattice whose dual is a Stone algebra then $!x$ exists and equals x^{++}, where x^+ is the dual pseudocomplement of x in A.

Post algebras can be defined in various ways. From [9], a Post algebra is a P_2-lattice $\langle A; \mathbf{e}_0, \ldots, \mathbf{e}_{n-1}\rangle$ such that $!\mathbf{e}_{n-2} = 0$. From [4], a Post algebra of order n is a Łukasiewicz algebra of order n having $n - 2$ elements $\mathbf{e}_1, \ldots, \mathbf{e}_{n-2}$ satisfying

$$\mathbf{s}_i(\mathbf{e}_j) = \begin{cases} 0, & i + j < n \\ 1, & i + j \geqslant n \end{cases} .$$

Further definitions may be found in [2] and the references therein.

A *Stone-lattice of order* n $(n \geqslant 2)$ is an L-algebra A in which there exists a chain $0 = \mathbf{e}_0 < \mathbf{e}_1 < \ldots < \mathbf{e}_{n-1} = 1$ such that $\mathbf{e}_{i+1}$ is the smallest dense element in the interval $[\mathbf{e}_i, 1]$. A systematic study of Stone-lattices of order n was made in [14].

1 *Compactness in* $\mathsf{P}^{[n]}$*-algebras* The following results, the proofs of which are in [12] and [9] respectively are crucial in characterizing the equationally compact $\mathsf{P}^{[n]}$-algebras.

Lemma 1 *If A is a Heyting algebra then $A \in \mathcal{L}_n$ $(n \geqslant 2)$ if and only if $\mathsf{C}(A)$ is a subalgebra of A and $\mathsf{D}(A) \in \mathcal{K}_{n-1}$.*

Lemma 2 *A is a P_2-algebra of order n if and only if it is a Stone lattice of order n whose dual is a Stone algebra.*

Theorem 1 *If $A \in \mathcal{P}^{[n]}$ then the following are equivalent:*

(i) *A is equationally compact.*
(ii) *A is a direct product of finitely many complete Post algebras of order at most n.*
(iii) *A is a retract of a topologically compact algebra in $\mathcal{P}^{[n]}$.*

Proof: (i) $\Longrightarrow$ (ii). Let $A \in \mathcal{P}^{[n]}$ be equationally compact and let $\mathbf{e} \in A - \{1\}$. Then, since A is a Heyting algebra, the interval $[\mathbf{e}, 1]$ is a pseudocomplemented lattice; $x \rightarrow \mathbf{e}$ being the pseudocomplement of x in $[\mathbf{e}, 1]$. Let $\mathsf{D}_\mathbf{e}$ denote the dense filter in $[\mathbf{e}, 1]$. The set Σ of equations

$$x \wedge \mathbf{e} = \mathbf{e}$$
$$x \rightarrow \mathbf{e} = \mathbf{e}$$
$$\{x \wedge d = x;\ d \in \mathsf{D}_\mathbf{e}\}$$

is easily seen to be finitely solvable and, therefore, by compactness, is solvable. Clearly, a solution of Σ is the smallest dense element in $[\mathbf{e}, 1]$. Thus, we can produce an ascending chain E: $0 = \mathbf{e}_0 \leqslant \mathbf{e}_1 \leqslant \ldots$, where $\mathbf{e}_{i+1}$ is the smallest dense element in $[\mathbf{e}_i, 1]$. Since $A \in \mathcal{L}_n$, we conclude from Lemma 1 that E has cardinality at most n. Therefore, A is a Stone lattice of order at most n whose dual is a Stone algebra. Equivalently, by Lemma 2, A is a P_2-algebra. According to [8], A must be a direct product of finitely many Post algebras of order at most n. Finally, A, being equationally compact as a lattice, is complete and therefore each factor in the direct decomposition is complete.

(ii) $\Rightarrow$ (iii). Let $A = \prod_{i=1}^{r} A_i$, where each A_i is a complete Post algebra of order $n_i \leq n$. Then from [2], there exist cardinals $\mathfrak{M}_i$ and retractions $\rho_i: \underset{\sim}{n}_i^{\mathfrak{M}_i} \to A_i$ preserving $\vee, \wedge, \to$ and its dual. Clearly, the family $\{\rho_i: 1 \leq i \leq r\}$ induces a retraction $\prod \rho_i: \prod_{i=1}^{r} \underset{\sim}{n}_i^{\mathfrak{M}_i} \to \prod_{i=1}^{r} A_i$ and $\prod_{i=1}^{r} \underset{\sim}{n}_i^{\mathfrak{M}_i}$ is a topologically compact algebra in $\mathcal{P}^{[n]}$ when each chain algebra $\underset{\sim}{n}_i$ is endowed with the discrete topology.

(iii) $\Rightarrow$ (iv). This is well-known (see [17]).

Prior to characterizing the topologically compact algebras in $\mathcal{P}^{[n]}$, we recall some definitions.

A subset F of a **P**-algebra A is said to be a **P**-*filter* if it is a lattice filter closed under !.

If $\mathcal{K}$ is a class of similar algebras then $A \in \mathcal{K}$ is said to be *profinite* if it is an inverse limit of finite algebras in $\mathcal{K}$.

Theorem 2 *If $A \in \mathcal{P}^{[n]}$ then the following are equivalent:*

(i) *A is topologically compact.*
(ii) *$C(A)$ is a topologically compact Boolean algebra.*
(iii) *$C(A)$ is complete and completely distributive.*
(iv) *A is profinite.*
(v) *A is complete and completely distributive.*
(vi) *A is a direct product of finitely many complete and completely distributive Post algebras of order at most n.*

Proof: (i) $\Rightarrow$ (ii). Let A be a topologically compact algebra in $\mathcal{P}^{[n]}$. Then $C(A)$ is a topologically compact Boolean algebra; since $C(A)$ is a Boolean subalgebra of A and the continuous image of A under the mapping $x \to x^{**}$.

(ii) $\Rightarrow$ (iii). This is proved in [16], Proposition 3.

(iii) $\Rightarrow$ (iv). In any **P**-algebra A there is a one-to-one correspondence between the **P**-congruences Φ and the **P**-filters F (see [7]). Under this correspondence $F = \{x \in A;\ \langle x, 1\rangle \in \Phi\}$ and $\Phi = \{\langle x, y\rangle \in A^2;\ (x \to y) \wedge (y \to x) \in F\}$. It is easy to see that a principal (lattice) filter in A is a maximal **P**-filter if and only if it is generated by an atom in $C(A)$. Now, following the proof of Theorem 5.2 in [1], let $\mathcal{A}$ be the set of atoms of $C(A)$ and let $\mathcal{F} = \{f_\alpha;\ \alpha \in \Lambda\}$ be the set of finite joins of members of $\mathcal{A}$. Then $\mathcal{F}$ is a sublattice of $C(A)$ with $\vee f_\alpha = 1$. Partially ordering the index set Λ by requiring the $\beta \leq \alpha$ in Λ if and only if $f_\beta \leq f_\alpha$ in A insures that Λ is a down-directed set indexing the **P**-congruences $\Phi_\alpha = \{\langle x, y\rangle \in A^2;\ (x \to y) \wedge (y \to x) \geq f_\alpha\}$ in such a way that $\Phi_\alpha \subseteq \Phi_\beta$ whenever $\beta \leq \alpha$. Consequently, the quotient algebras $A_\alpha = A/\Phi_\alpha$ and homomorphisms $\phi_{\alpha\beta}: A_\alpha \to A_\beta$ defined by $\phi_{\alpha\beta}([x]_\alpha) = [x]_\beta$ whenever $\beta \leq \alpha$ (where $[x]_\alpha$ is the congruence class modulo Φ_α containing x) form an inverse system. Now, if $f_\alpha = \bigvee_{i=1}^{r(\alpha)} a_{\alpha_i}$, where $a_{\alpha_i} \in \mathcal{A}$, and $\Phi_{\alpha_i} = \{\langle x, y\rangle \in A^2;\ (x \to y) \wedge$

$(y \to x) \geq a_{\alpha_i}\}$ then, following the proof of Theorem 5.2 in [1], we see that A_α can be embedded in $\prod_{i=1}^{r(\alpha)} A/\Phi_{\alpha_i}$. The maximality of Φ_{α_i} implies that A/Φ_{α_i} is a simple algebra in $\mathcal{P}^{[n]}$. According to [7], a simple P-algebra must be a chain. However, the identity $\bigvee_{i=1}^{n}(x_i \to x_{i+1}) = 1$ holds only in those chains of cardinality at most n. Consequently, each A_α is finite. Clearly, $\bigcap \Phi_\alpha$ is the smallest congruence on A and so the correspondence $\phi: A \to \underleftarrow{\mathrm{Lim}}\, A_\alpha$ defined by $[\phi(x)](\alpha) = [x]_\alpha$ is an embedding of A into a profinite algebra. The proof that ϕ is surjective can be lifted verbatim from Theorem 5.2 in [1].
(iv) $\Rightarrow$ (i). Trivial.
(iv) $\Rightarrow$ (v). If $A = \underleftarrow{\mathrm{Lim}}\, A_\alpha$ is profinite then A is a complete sublattice of the complete and completely distributive lattice $\prod A_\alpha$. Therefore A is completely distributive.
(v) $\Rightarrow$ (iii). It suffices to show that $C(A)$ is a complete sublattice of A. Let $\{b_i;\ i \in I\} \subseteq C(A)$ and $b = \wedge_A\{b_i;\ i \in I\}$, where $\wedge_A$ indicates that the meet is taken in A. Then $b^{**} \leq b_i^{**} = b_i$, for all $i \in I$, so that $b^{**} \leq b$. Since $b^{**} \geq b$, it follows that $b^{**} = b \in C(A)$. A dual argument shows that $\vee_A\{b_i;\ i \in I\} \in C(A)$. Thus, $C(A)$ is a complete sublattice of A.
(i) $\Rightarrow$ (vi). If $A \in \mathcal{P}^{[n]}$ is topologically compact then it is equationally compact and therefore $A = \prod_{i=1}^{r} A_i$, where each A_i is a complete Post algebra of order at most n. From Proposition 4 in [16], $C\left(\prod_{i=1}^{r} A_i\right)$ is complete and completely distributive. Since $C\left(\prod_{i=1}^{r} A_i\right) = \prod_{i=1}^{r} C(A_i)$, it follows that each $C(A_i)$ is complete and completely distributive. Therefore, by results in [6], each A_i is complete and completely distributive and so is $\prod_{i=1}^{r} A_i$.
(vi) $\Rightarrow$ (v). Trivial.

2 *Compactness in* P_0*-lattices* The following was proved in [9]:

Lemma 3 *Any* P_1*-lattice of order* n *is a Stone lattice of order* n.

Theorem 3 *If* $\langle A; e_0, \ldots, e_{n-1}\rangle$ *is a* P_0*-lattice then the following are equivalent:*

(i) *A is an equationally compact lattice.*
(ii) *A is a direct product of finitely many complete Post algebras of order at most n.*
(iii) *A is a (lattice) retract of a topologically compact* P_0*-lattice.*

Proof: It suffices to prove that (i) $\Rightarrow$ (ii). Let A be an equationally compact P_0-lattice. Let $a, b \in A$, $U = \{u \in A;\ a \wedge u \leq b\}$ and consider the set Σ of equations

$$b \vee (a \wedge x) = b$$
$$\{x \wedge u = u;\ u \in U\}.$$

Clearly, Σ is finitely solvable and, therefore, by compactness, is solvable. Any solution is the largest $x \in A$ satisfying $a \wedge x \leqslant b$. Consequently, A is a Heyting algebra. A dual argument shows that the dual of A is also a Heyting algebra. According to [8], a P_0-lattice which is also a Heyting algebra must be an L-algebra in which there exists a chain-base $0 = \mathbf{f}_0 \leqslant \ldots \leqslant \mathbf{f}_{n-1} = 1$ such that $\langle A; \mathbf{f}_0, \ldots, \mathbf{f}_{n-1}\rangle$ is a P_1-lattice. Now, since the dual of A is a P_0-lattice and a Heyting algebra, it follows that the dual of A is an L-algebra. Consequently, A is a P_2-lattice; since it is a Stone lattice of order at most n whose dual is a Stone algebra. Therefore A is a direct product of finitely many Post algebras of order at most n each of which is complete; by virtue of the completeness of A.

Corollary 1 *A* P_0*-lattice* $\langle A; \mathbf{e}_0, \ldots, \mathbf{e}_{n-1}\rangle$ *is topologically compact if and only if it is a direct product of finitely many complete and completely distributive Post algebras of order at most n.*

3 *Compactness in Łukasiewicz algebras* The proofs of the following results can be found in [9] and [4] (Theorem 6.1) respectively.

Lemma 4 *If A is a bounded, distributive lattice then the dual of A is a Stone algebra if and only if* $!x$ *exists and* $!(x \vee y) = !x \vee !y$, *for all* $x, y \in A$.

Lemma 5 *If* $\langle A; \vee, \wedge; \sim, \mathbf{s}_1, \ldots, \mathbf{s}_{n-1}; 1\rangle \in \mathbf{Luk}_n$ *and* $b \in \mathrm{C}(A)$ *then* $\langle (b]; \vee, \wedge; -, \mathbf{s}_1, \ldots, \mathbf{s}_{n-1}; b\rangle \in \mathbf{Łuk}_n$, *where* $-x = \sim x \wedge b$ *whenever* $x \in (b]$, *and* $h: A \to (b]$ *defined by* $h(x) = x \wedge b$ *is a surjective (Łukasiewicz) homomorphism.*

Theorem 4 *If* $\langle A; \vee, \wedge; \sim, \mathbf{s}_1, \mathbf{s}_2; 1\rangle \in \mathbf{Łuk}_3$ *then the following are equivalent:*

(i) *A is equationally compact.*
(ii) *A is complete and has a smallest dense element.*
(iii) *A is the direct product of a complete Boolean algebra and a complete Post algebra of order* 3.
(iv) *A is a (Łukasiewicz) retract of a topologically compact algebra in* $\mathbf{Łuk}_3$.

Proof: (i) (ii). Let $A \in \mathbf{Łuk}_3$ be equationally compact. Clearly, $\mathbf{s}_1(x)$ is the largest central element in A which is $\leqslant x$ and so $\mathbf{s}_1(x) = !x$. It follows from M3 and Lemma 4 that A is a dual Stone algebra; $\sim\mathbf{s}_1(x)$ being the dual pseudocomplement x^+ of x in A. Similarly, since $\mathbf{s}_2(x)$ is the smallest central element in A which is $\geqslant x$ and $\mathbf{s}_i(x \wedge y) = \mathbf{s}_i(x) \wedge \mathbf{s}_i(y)$ holds in any Łukasiewicz algebra, A is a Stone algebra; $\sim\mathbf{s}_2(x)$ being the pseudocomplement x^* of x in A. Now, the set Σ of equations

$$\sim\mathbf{s}_2(x) = 0$$
$$\{x \wedge f = x;\ f \in \mathrm{D}(A)\}$$

is finitely solvable and, therefore, by compactness, is solvable. Any solution of Σ is the smallest dense element in A. A, being equationally compact as a lattice, is complete.

(ii) $\Rightarrow$ (iii). Let A be complete, $\mathbf{d}$ be the smallest dense element in A and $a = \mathbf{d}^{++}$. Then, by Lemma 5, (a] and $(a^*]$ are Łukasiewicz algebras of

order 3 and $h_1 : A \to (a]$, $h_2 : A \to (a^*]$ defined by $h_1(x) = a \wedge x$, $h_2(x) = a^* \wedge x$ respectively, are surjective (Łukasiewicz) homomorphisms. Clearly, since a and a^* are complementary, the mapping $h: A \to (a] \times (a^*]$ defined by $h(x) = \langle h_1(x), h_2(x)\rangle$ is a (Łukasiewicz) isomorphism. Now, the pseudocomplement of $b \in (a]$ is $b^0 = a \wedge b^*$ and so $b \vee b^0 = (b \vee a) \wedge (b \vee b^*) \geqslant a$; since $b \vee b^* \in \mathrm{D}(A)$ implies that $b \vee b^* \geqslant d \geqslant d^{++} = a$. But $b \vee b^0 \leqslant a$ and so $b \vee b^0 = a$. Hence $(a]$ is a complete Boolean algebra. Next, the dual pseudocomplement of $b \in (a^*]$ is $b^\infty = b^+ \wedge a^*$ and, since A is a dual Stone algebra, $b^{\infty\infty} = b^{++} \wedge a^*$. Defining $e_1 = d \wedge a^* \in (a^*]$, we see that $\mathbf{s}_1(e_1) = (d \wedge d^+)^{\infty\infty} = (d \wedge d^+)^{++} \wedge a^* = (d^{++} \wedge d^+) \wedge a^* = 0 \wedge a^* = 0$. Similarly, $\mathbf{s}_2(e_1) = (d \wedge a^*)^{**} \wedge a^* = (d^{**} \wedge a^*) \wedge a^* = 1 \wedge a^* = a^*$. Thus, $(a^*]$ is a complete Post algebra of order 3.
(iii) $\Rightarrow$ (iv). Trivial.
(iv) $\Rightarrow$ (i). Well-known.

Corollary 2 *If* $\langle A; \vee, \wedge; \sim, \mathbf{s}_1, \mathbf{s}_2; 1\rangle \in \mathbf{Luk}_3$ *then it is topologically compact if and only if it is the direct product of a complete and completely distributive Boolean algebra and a complete and completely distributive Post algebra of order* 3.

Cignoli [4], calls a lattice filter in a Łukasiewicz algebra A a *Stone filter* if it is closed under the unary operation $\mathbf{s}_1$. The isomorphism of the lattice of (Łukasiewicz) congruences Φ on A with the lattice of Stone filters F under the correspondence $F = \{x \in A; \langle x, 1\rangle \in \Phi\}$, $\Phi = \{\langle x, y\rangle \in A^2; x \wedge f = y \wedge f$ for some $f \in F\}$ was established in [4], Theorem 3.10. However, the coincidence of Stone filters and $\mathbf{P}$-filters in A, coupled with the known equivalence of $x \wedge f = y \wedge f$ and $(x \to y) \wedge (y \to x) \geqslant f$ in any Heyting algebra, shows that the above correspondence coincides with its counterpart in $\mathcal{P}^{[n]}$ (see Theorem 2). Furthermore, the simple algebras in $\mathbf{Luk}_n$ are finite (Corollary 5.6, [4]). Bearing these facts in mind, a simple modification of the proof of Theorem 2 shows that the equivalence of (i) through (v) in the statement of Theorem 2 holds for any Łukasiewicz algebra.

REFERENCES

[1] Beazer, R., "Some remarks on Post algebras," *Colloquium Mathematicum*, vol. 24 (1974), pp. 167-178.

[2] Beazer, R., "Post-like algebras and injective Stone algebras," *Algebra Universalis* (to appear).

[3] Bulman-Fleming, S., "On equationally compact semilattices," *Algebra Universalis*, vol. 2 (1972), pp. 146-151.

[4] Cignoli, R., *Moisil algebras*, *Notas de Lôgica Matemâtica*, vol. 27 (1970), published by Universidad Nacional del Sur, Bahia Blanca.

[5] Cignoli, R., "Representation of Łukasiewicz and Post algebras by continuous functions," *Colloquium Mathematicum*, vol. 24 (1972), pp. 127-138.

[6] Epstein, G., "The lattice theory of Post algebras," *Transactions of the American Mathematical Society*, vol. 95 (1960), pp. 300-317.

[7] Epstein, G., and A. Horn, "P-algebras, an abstraction from Post algebras," *Algebra Universalis*, vol. 4 (1974), pp. 195-206.

[8] Epstein, G., and A. Horn, "Chain-based lattices," *Notices of the American Mathematical Society*, vol. 20 (1973), p. 610.

[9] Epstein, G., and A. Horn, "Chain-based lattices" (preprint).

[10] Grätzer, G., *Lattice Theory*, W. H. Freeman and Co., San Francisco (1971).

[11] Grätzer, G., and H. Lakser, "Equationally compact semilattices," *Colloquium Mathematicum*, vol. 20 (1969), pp. 27-30.

[12] Hecht, T., and T. Katriňák, "Equational classes of relative Stone algebras," *Notre Dame Journal of Formal Logic*, vol. XIII (1972), pp. 248-254.

[13] Horn, A., "Logic with truth values in a linearly ordered Heyting algebra," *The Journal of Symbolic Logic*, vol. 34 (1969), pp. 395-408.

[14] Katriňák, T., and A. Mitschke, "Stonesche Verbände der ordnung *n* und Postalgebren," *Mathematischen Annalen*, vol. 199 (1972), pp. 13-30.

[15] Mycielski, J., "Some compactifications of general algebras," *Colloquium Mathematicum*, vol. 13 (1964), pp. 1-9.

[16] Rema, P. S., "On compact topological lattices," *Mathematica Japonicae*, vol. 9 (1964), pp. 93-98.

[17] Weglorz, B., "Equationally compact algebras I," *Fundamenta Mathematicae*, vol. 59 (1966), pp. 289-298.

University of Glasgow
Glasgow, Scotland