Notre Dame Journal of Formal Logic
Volume XVI, Number 3, July 1975
NDJFAM

A CHARACTERIZATION OF LOGICAL CONSEQUENCE IN QUANTIFICATION THEORY

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In this paper, we characterize the logical consequence relation as the minimal relation in a class of 'adequate' consequence relations for quantification theory. These adequate relations are easily seen to include the usual syntactic relations of deductive consequence and thus our results can, perhaps, be considered an abstract version of the completeness theorem. Our work is closely related to the methods of Hintikka [1] and Smullyan [2], [3]. We adopt [3] as our standard reference and we assume the reader is acquainted with Smullyan's marvellous α, β, γ, δ notation for sentences of quantification theory[1] as well as his Unifying Principle [3], p. 67.

In what follows, $K \vdash X$ will stand for an abstract binary relation between finite sets of sentences K and single sentences X. We allow $K = \emptyset$, the empty set, in which case we write, $\vdash X$. Also if Y is a sentence, we will write $K, Y \vdash X$ for $K \cup \{Y\} \vdash X$, etc.

Definition We say that $\vdash$ is an *adequate consequence relation* (for quantification theory) if it satisfies the following conditions:

I. $K, A, \sim A \vdash X$, *A atomic.*
II. *If* $K, \sim X \vdash X$, *then* $K \vdash X$.
III. *If* $K, \alpha_1 \vdash X$ *or* $K, \alpha_2 \vdash X$, *then* $K, \alpha \vdash X$.
IV. *If* $K, \beta_1 \vdash X$ *and* $K, \beta_2 \vdash X$, *then* $K, \beta \vdash X$.
V. *If* $K, \gamma(a) \vdash X$, *then* $K, \gamma \vdash X$.
VI. *If* $K, \delta(a) \vdash X$, *then* $K, \delta \vdash X$, *if a is new to* K, δ, *and* X.

If $K \models X$ means X is true whenever all sentences in K are true, then

1. By a sentence of quantification theory we mean what Smullyan [3] calls an unsigned sentence, that is, one not preceded by a T or F.

Received July 8, 1973

it is easy to show that $\models$ is adequate. If $K \vdash X$ means X is provable using the sentences of K as hypotheses in one of the usual Hilbert-type axiom systems for quantification theory, it is a routine exercise to verify that $\vdash$ is adequate (at least, after the deduction theorem is proved).

We denote the minimal adequate consequence relation by $\vdash_0$. Clearly $K \vdash_0 X$ if and only if there exists a finite sequence of expressions of the form $K_i \vdash X_i$, $1 \leqslant i \leqslant n$, where $K_n = K$, $X_n = X$, such that each expression is either of form I or follows from one or two previous expressions using II-VI.

Definition Let K be a finite set of sentences. K is *N-inconsistent*[2] *with respect to* $\vdash$ if $K \vdash X$, for every $\sim X \in K$; otherwise K is *N-consistent with respect to* $\vdash$.

Theorem 1 *If* $\vdash$ *is an adequate consequence relation and* K *is N-consistent with respect to* $\vdash$, *then* K *is satisfiable.*

Proof: This follows directly from Smullyan's Unifying Principle by showing that N-consistency with respect to an adequate consequence relation satisfies conditions A_0 - A_4 on p. 66 of [3].[3]

Theorem 2 *If* $\vdash$ *is an adequate consequence relation, then* $K \models X$ *implies* $K \vdash X$, *that is,* $\models$ *is the minimal adequate consequence relation,* $\vdash_0$.

Proof: Suppose $K \models X$, then $L = K \cup \{\sim X\}$ is unsatisfiable. Therefore, by Theorem 1, L is N-inconsistent with respect to $\vdash$. Since $\sim X \in L$, $L \vdash X$, that is, $K, \sim X \vdash X$. Therefore $K \vdash X$.

Theorem 2 can be used to show the various Hilbert-type axiom systems in the literature are complete by showing the adequacy of their deductive consequence relations. Also, because of the identification of $\vdash_0$ and $\models$, the proof sequences for $\vdash_0$, themselves, can be used to 'prove' valid sentences. This system is closely related to the Block Tableaux of Hintikka [1] as formulated by Smullyan [3], p. 101. In fact if X is valid and $\mathcal{T}$ is a closed tableau for X, replace each entry K_i in $\mathcal{T}$ by $K_i \vdash X$, turn the result upside down and we have a proof (in true form) that $\sim X \vdash X$ ($\sim X$ is at the origin of $\mathcal{T}$); by II, we can adjoin $\vdash X$, completing the proof.

Finally, R. Smullyan has pointed out (oral communication) that the notion of N-inconsistency used in Theorem 1 could be replaced by $K \vdash X$, for all X whose parameters appear in K; since any K which is inconsistent in this new sense is N-inconsistent, the proof of Theorem 2 would remain unchanged.

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2. That is, inconsistent for negated sentences.

3. Since N-consistency is a property of finite sets, it is not necessary to show it is of finite character: see [3], p. 69.

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