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SINGLE AXIOMS FOR ATOMISTIC AND ATOMLESS MEREOLOGY

ROBERT E. CLAY

It is part of the folk-lore of the subject, that Leśniewski's mereology is neutral with respect to the existence of atoms. It has long been known that one could have a system of atomless mereology by adding the following axiom or its equivalent to any mereological axiom system.

$$[A]:A\,\varepsilon\,A\ .\supset.\ [\exists B].B\,\varepsilon\,\mathsf{pr}(A).$$

Similarly one could have a system of atomistic (completely atomic) mereology by adding

$$[A]::A\,\varepsilon\,A\ .\supset\therefore[\exists B]\therefore B\,\varepsilon\,\mathsf{el}(A):[C]:C\,\varepsilon\,\mathsf{el}(B)\ .\supset.\ C=B.$$

If we define the name "**atm**" by

$$[B]\therefore B\,\varepsilon\,\mathsf{atm}\ .\equiv:\ B\,\varepsilon\,B:[C]:C\,\varepsilon\,\mathsf{el}(B)\ .\supset.\ C=B$$

this simplifies to

$$[A]:A\,\varepsilon\,A\ .\supset.\ [\exists B].B\,\varepsilon\,\mathsf{el}(A).B\,\varepsilon\,\mathsf{atm}.$$

Rickey, *cf.* [4], p. 90, introduced the functor "**at**" defined by

$$[AB]:B\,\varepsilon\,\mathsf{at}(A)\ .\equiv.\ B\,\varepsilon\,\mathsf{el}(A).B\,\varepsilon\,\mathsf{atm}$$

which further reduces the characteristic axiom of atomistic mereology to

$$[A]:A\,\varepsilon\,A\ .\supset.\ [\exists B].B\,\varepsilon\,\mathsf{at}(A).$$

Using Rickey's functor "**at**" Sobociński axiomatized atomistic mereology in [4]. Lejewski gave the first single axioms for atomistic and atomless mereology in [2]. In this paper we shall give shorter single axioms for both systems.

Lejewski's single axiom for atomistic mereology is

L1 $[AB]\because A\,\varepsilon\,\mathsf{at}(B)\ .\equiv::B\,\varepsilon\,B::[CDa]::[E]\therefore E\,\varepsilon\,C\ .\equiv:\ [F]:F\,\varepsilon\,\mathsf{at}(E).$
$\equiv.\ [\exists G].F\,\varepsilon\,\mathsf{at}(G).G\,\varepsilon\,a\therefore D\,\varepsilon\,\mathsf{at}(B).B\,\varepsilon\,a\ .\therefore\supset.\ \mathsf{at}(A)\,\varepsilon\,A.A\,\varepsilon\,\mathsf{at}(C).$

This contains ten occurrences of ε. We shall show that the shorter (nine occurrences of ε) proposition

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$C1$ $[AB] \therefore\!\cdot A \,\varepsilon\, \mathsf{at}(B) \,.\!\equiv\!::\, B \,\varepsilon\, B :: [CDa] :: [E] \therefore E \,\varepsilon\, C \,.\!\equiv\!: [F] : F \,\varepsilon\, \mathsf{at}(E) \,.$
$\equiv\!. \,[\exists G] . F \,\varepsilon\, \mathsf{at}(G) . G \,\varepsilon\, a \therefore D \,\varepsilon\, \mathsf{at}(B) . B \,\varepsilon\, a \therefore\!\supset\!. \,\mathsf{at}(A) \,\varepsilon\, \mathsf{at}(C)$

is also a single axiom for atomistic mereology.

In $L1$ it is easy to derive that if A is an atom of something then A is identical with $\mathsf{at}(A)$. In $C1$ it is easy to derive that if A is an atom of something then $\mathsf{at}(A)$ is an individual, but the difficulty arises in proving that that individual is in fact the individual, A.

We begin with $L1$ and derive $C1$.

$L2$ $[AB] : A \,\varepsilon\, \mathsf{at}(B) \,.\!\supset\!. \,B \,\varepsilon\, B$ [$L1$]
$L3$ $[B] : B \,\varepsilon\, B \,.\!\supset\!. \,[\exists D] . D \,\varepsilon\, \mathsf{at}(B)$ [$L1$, by contradiction]
DL $[Aa] \therefore A \,\varepsilon\, \mathsf{Kl}(a) \,.\!\equiv\!: A \,\varepsilon\, A : [F] : F \,\varepsilon\, \mathsf{at}(A) \,.\!\equiv\!. \,[\exists G] . F \,\varepsilon\, \mathsf{at}(G) . G \,\varepsilon\, a$
$L4$ $[ABa] \therefore B \,\varepsilon\, a : [F] : F \,\varepsilon\, \mathsf{at}(A) \,.\!\equiv\!. \,[\exists G] . F \,\varepsilon\, \mathsf{at}(G) . G \,\varepsilon\, a :\supset\!. \,A \,\varepsilon\, A$
PR $[ABa] \therefore \mathrm{Hp}(2) :\supset\!.$
$[\exists D] .$
3. $D \,\varepsilon\, \mathsf{at}(B) .$ [$L3$; 1]
4. $D \,\varepsilon\, \mathsf{at}(A) .$ [2; 3; 1]
$A \,\varepsilon\, A$ [$L2$; 4]
$L5$ $[Ba] :: B \,\varepsilon\, a \,.\!\supset\!\therefore [A] \therefore A \,\varepsilon\, \mathsf{Kl}(a) \,.\!\equiv\!: [F] : F \,\varepsilon\, \mathsf{at}(A) \,.\!\equiv\!. \,[\exists G] . F \,\varepsilon\, \mathsf{at}(G) .$
$G \,\varepsilon\, a$ [DL; $L4$]
$L6$ $[ABa] : A \,\varepsilon\, \mathsf{at}(B) . B \,\varepsilon\, a \,.\!\supset\!. \,\mathsf{at}(A) \,\varepsilon\, A . A \,\varepsilon\, \mathsf{at}(\mathsf{Kl}(a))$ [$L1$, $C/\mathsf{Kl}(a)$; $L5$]
$L7$ $[Ba] : B \,\varepsilon\, a \,.\!\supset\!. \,\mathsf{Kl}(a) \,\varepsilon\, \mathsf{Kl}(a)$
PR $[Ba] : \mathrm{Hp}(1) \,.\!\supset\!.$
$[\exists D] .$
2. $D \,\varepsilon\, \mathsf{at}(B) .$ [$L3$; 1]
3. $D \,\varepsilon\, \mathsf{at}(\mathsf{Kl}(a)) .$ [$L6$; 2; 1]
$\mathsf{Kl}(a) \,\varepsilon\, \mathsf{Kl}(a)$ [$L2$; 3]
$L8$ $[AB] : A \,\varepsilon\, \mathsf{at}(B) \,.\!\supset\!. \,\mathsf{at}(A) \,\varepsilon\, A$
PR $[AB] : \mathrm{Hp}(1) \,.\!\supset\!.$
2. $B \,\varepsilon\, B .$ [$L2$; 1]
$\mathsf{at}(A) \,\varepsilon\, A$ [$L6$; 1; 2]
$L9$ $[A] : A \,\varepsilon\, A \,.\!\supset\!. \,A \,\varepsilon\, \mathsf{Kl}(A)$ [DL]
$L10$ $[AC] : \mathsf{at}(A) \,\varepsilon\, \mathsf{at}(C) \,.\!\supset\!. \,\mathsf{at}(A) \,\varepsilon\, A . A \,\varepsilon\, \mathsf{at}(C)$
PR $[AC] \therefore \mathrm{Hp}(1) \,.\!\supset\!:$
2. $\mathsf{at}(A) \,\varepsilon\, \mathsf{at}(A) .$ [1]
3. $A \,\varepsilon\, A .$ [$L2$; 2]
4. $\mathsf{at}(\mathsf{at}(A)) \,\varepsilon\, \mathsf{at}(A) .$ [$L8$, $A/\mathsf{at}(A)$; B/A; 2]
5. $\mathsf{at}(\mathsf{at}(A)) = \mathsf{at}(A) :$ [4; 2]
6. $[F] : F \,\varepsilon\, \mathsf{at}(A) \,.\!\equiv\!. \,F \,\varepsilon\, \mathsf{at}(\mathsf{at}(A)) :$ [5]
7. $[F] : F \,\varepsilon\, \mathsf{at}(\mathsf{at}(A)) \,.\!\equiv\!. \,[\exists G] . F \,\varepsilon\, \mathsf{at}(G) . G \,\varepsilon\, \mathsf{at}(A) :$ [2]
8. $[F] : F \,\varepsilon\, \mathsf{at}(A) \,.\!\equiv\!. \,[\exists G] . F \,\varepsilon\, \mathsf{at}(G) . G \,\varepsilon\, \mathsf{at}(A) :$ [6; 7]
9. $A \,\varepsilon\, \mathsf{Kl}(\mathsf{at}(A)) .$ [DL; 3; 8]
10. $\mathsf{at}(A) \,\varepsilon\, \mathsf{Kl}(\mathsf{at}(A)) .$ [$L9$; 2]
11. $\mathsf{Kl}(\mathsf{at}(A)) \,\varepsilon\, \mathsf{Kl}(\mathsf{at}(A)) .$ [$L7$; 2]
12. $\mathsf{at}(A) = A .$ [10; 9; 11]
13. $\mathsf{at}(A) \,\varepsilon\, A .$ [12]

14. $A\,\varepsilon\,\mathbf{at}(C)\,.$ [1; 12]
$\mathbf{at}(A)\,\varepsilon\,A\,.\,A\,\varepsilon\,\mathbf{at}(C)$ [13; 14]
L11 $[AC]:\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(C)\ .{\equiv}.\ \mathbf{at}(A)\,\varepsilon\,A\,.\,A\,\varepsilon\,\mathbf{at}(C)$ [*L10*]
L12 [= *C1*] [*L1*; *L11*]

Next we derive *L1* from *C1*.

C2 $[AB]:A\,\varepsilon\,\mathbf{at}(B)\ .{\supset}.\ B\,\varepsilon\,B$ [*C1*]
C3 $[B]:B\,\varepsilon\,B\ .{\supset}.\ [\exists D]\,.\,D\,\varepsilon\,\mathbf{at}(B)$ [*C1*]
DC $[Aa]\therefore A\,\varepsilon\,\mathbf{Kl}(a)\ .{\equiv}:\ A\,\varepsilon\,A:[F]:F\,\varepsilon\,\mathbf{at}(A)\ .{\equiv}.\ [\exists G]\,.\,F\,\varepsilon\,\mathbf{at}(G)\,.\,G\,\varepsilon\,a$
C4 $[ABa]\therefore B\,\varepsilon\,a:[F]:F\,\varepsilon\,\mathbf{at}(A)\ .{\equiv}.\ [\exists G]\,.\,F\,\varepsilon\,\mathbf{at}(G)\,.\,G\,\varepsilon\,a\ :{\supset}.\ A\,\varepsilon\,A$ [*C3*; *C2*]
C5 $[Ba]::B\,\varepsilon\,a\ .{\supset}\therefore\ [A]\therefore A\,\varepsilon\,\mathbf{Kl}(a)\ .{\equiv}:\ [F]:F\,\varepsilon\,\mathbf{at}(A)\ .{\equiv}.\ [\exists G]\,.\,F\,\varepsilon\,\mathbf{at}(G)\,.\,G\,\varepsilon\,a$
[*DC*; *C4*]
C6 $[ABa]:A\,\varepsilon\,\mathbf{at}(B)\,.\,B\,\varepsilon\,a\ .{\supset}.\ \mathbf{at}(A)\,\varepsilon\,\mathbf{at}(\mathbf{Kl}(a))$ [*C1*, *C*/$\mathbf{Kl}(a)$; *C5*]
C7 $[Ba]:B\,\varepsilon\,a\ .{\supset}.\ \mathbf{Kl}(a)\,\varepsilon\,\mathbf{Kl}(a)$ [*C6*; *C3*; *C2*]

Note: For proofs of *C4* and *C7* see *L4* and *L7*.

C8 $[AB]:A\,\varepsilon\,\mathbf{at}(B)\ .{\supset}.\ \mathbf{at}(A)\,\varepsilon\,\mathbf{at}(A)$
PR $[AB]:\mathrm{Hp}(1)\ .{\supset}.$
2. $B\,\varepsilon\,B\,.$ [*C2*; 1]
3. $\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(\mathbf{Kl}(B))\,.$ [*C6*; 1; 2]
$\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(A)$ [3]
C9 $[A]:A\,\varepsilon\,A\ .{\supset}.\ A\,\varepsilon\,\mathbf{Kl}(A)$ [*DC*]
C10 $[A]:\mathbf{at}(\mathbf{at}(A))=\mathbf{at}(A)\ .{\supset}.\ \mathbf{at}(A)=A$
PR $[A]\therefore \mathrm{Hp}(1)\ .{\supset}:$
2. $[F]:F\,\varepsilon\,\mathbf{at}(A)\ .{\equiv}.\ F\,\varepsilon\,\mathbf{at}(\mathbf{at}(A)):$ [1]
3. $\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(A)\,.$ [1]
4. $[F]:F\,\varepsilon\,\mathbf{at}(\mathbf{at}(A))\ .{\equiv}.\ [\exists G]\,.\,F\,\varepsilon\,\mathbf{at}(G)\,.\,G\,\varepsilon\,\mathbf{at}(A):$ [3]
5. $[F]:F\,\varepsilon\,\mathbf{at}(A)\ .{\equiv}.\ [\exists G]\,.\,F\,\varepsilon\,\mathbf{at}(G)\,.\,G\,\varepsilon\,\mathbf{at}(A):$ [2; 4]
6. $A\,\varepsilon\,A\,.$ [*L2*; 3]
7. $A\,\varepsilon\,\mathbf{Kl}(\mathbf{at}(A))\,.$ [*DC*; 6; 5]
8. $\mathbf{at}(A)\,\varepsilon\,\mathbf{Kl}(\mathbf{at}(A))\,.$ [*C9*; 3]
9. $\mathbf{Kl}(\mathbf{at}(A))\,\varepsilon\,\mathbf{Kl}(\mathbf{at}(A))\,.$ [*C7*; 3]
$\mathbf{at}(A)=A$ [8; 7; 9]
C11 $[AC]:\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(C)\ .{\supset}.\ \mathbf{at}(A)\,\varepsilon\,A\,.\,A\,\varepsilon\,\mathbf{at}(C)$
PR $[AB]:\mathrm{Hp}(1)\ .{\supset}.$
2. $\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(A)\,.$ [1]
3. $\mathbf{at}(\mathbf{at}(A))\,\varepsilon\,\mathbf{at}(\mathbf{at}(A))\,.$ [*C8*, $A/\mathbf{at}(A)$, B/C; 1]
4. $\mathbf{at}(\mathbf{at}(\mathbf{at}(A)))\,\varepsilon\,\mathbf{at}(\mathbf{Kl}(\mathbf{at}(A)))\,.$ [*C6*, $A/\mathbf{at}(\mathbf{at}(A))$, $B/\mathbf{at}(A)$, $a/\mathbf{at}(A)$; 3; 2]
5. $\mathbf{Kl}(\mathbf{at}(A))\,\varepsilon\,\mathbf{Kl}(\mathbf{at}(A))\,.$ [*C7*; 2]
6. $\mathbf{at}(A)\,\varepsilon\,\mathbf{Kl}(\mathbf{at}(A))\,.$ [*C9*; 2]
7. $\mathbf{at}(A)=\mathbf{Kl}(\mathbf{at}(A))\,.$ [5; 6]
8. $\mathbf{at}(\mathbf{at}(\mathbf{at}(A)))\,\varepsilon\,\mathbf{at}(\mathbf{at}(A))\,.$ [4; 7]
9. $\mathbf{at}(\mathbf{at}(\mathbf{at}(A)))=\mathbf{at}(\mathbf{at}(A))\,.$ [8; 3]
10. $\mathbf{at}(\mathbf{at}(A))=\mathbf{at}(A)\,.$ [*C10*; $A/\mathbf{at}(A)$; 9]
11. $\mathbf{at}(A)=A\,.$ [*C10*; 10]
12. $\mathbf{at}(A)\,\varepsilon\,A\,.$ [11]
13. $A\,\varepsilon\,\mathbf{at}(C)\,.$ [1; 11]
$\mathbf{at}(A)\,\varepsilon\,A\,.\,A\,\varepsilon\,\mathbf{at}(C)$ [12; 13]

$C12$ $[AC]:\mathbf{at}(A)\,\varepsilon\,\mathbf{at}(C)\ .\!\equiv\!.\ \mathbf{at}(A)\,\varepsilon\,A\,.\,A\,\varepsilon\,\mathbf{at}(C)$ [$C11$]
$C13$ $(= L1)$ [$C1$; $C12$]

Since the only definition used, namely that of "**Kl**", is the same in both derivations, the two systems are interderivable. Therefore, *C1* is a single axiom for atomistic mereology.

We now turn to atomless mereology. Lejewski gave the following single axiom for atomless mereology in [2]:

$$[AB] \therefore\!\!\therefore A\,\varepsilon\,\mathbf{pt}(B)\,.\!\equiv\!\therefore\!\therefore\ B\,\varepsilon\,B\,.\,\sim\!\langle B\,\varepsilon\,\mathbf{pt}(A)\rangle \therefore\!\therefore [CDa] :: [E] \therefore E\,\varepsilon\,C\ .\!\equiv\!:: [F] \therefore F\,\varepsilon\,a\ .\!\supset\!:\ E\,\varepsilon\,F\ .\!\vee\!.\ F\,\varepsilon\,\mathbf{pt}(E) :: [F] : F\,\varepsilon\,\mathbf{pt}(E)\ .\!\supset\!.\ [\exists GH]\,.\,G\,\varepsilon\,a\,.\,H\,\varepsilon\,\mathbf{pt}(F)\,.\,H\,\varepsilon\,\mathbf{pr}(G) :: D\,\varepsilon\,\mathbf{pt}(B)\,.\,B\,\varepsilon\,a :: \supset.\ A\,\varepsilon\,\mathbf{pt}(C)\,.$$

This axiom uses part, "**pt**", as primitive and has fourteen occurrences of ε.

The new axiom, namely,

$E1$ $$[AB] :: A\,\varepsilon\,\mathbf{ex}(B)\ .\!\equiv\!\therefore\ [f] \therefore [Ca] :: C\,\varepsilon\,f(a)\ .\!\equiv\!\therefore\ C\,\varepsilon\,C \therefore [D] \therefore D\,\varepsilon\,\mathbf{ex}(C)\ .\!\equiv\!:\ [E] : E\,\varepsilon\,a\ .\!\supset\!.\ D\,\varepsilon\,\mathbf{ex}(E) :: \supset :: [F] :: A\,\varepsilon\,F\ .\!\vee\!.\ B\,\varepsilon\,F\ .\!\vee\!\therefore [\exists b] \therefore A\,\varepsilon\,b\ .\!\vee\!.\ B\,\varepsilon\,b : [d] : F\,\varepsilon\,d\ .\!\supset\!.\ \sim\!\langle f(b)\,\varepsilon\,f(d)\rangle$$

uses exterior, "**ex**", as primitive. It is shorter (twelve occurrences of ε), but it has the added complexity of requiring quantification over a semantical category of functors in addition to quantification over names. *E1* is a modification of the shortest known single axiom for mereology, which is due to Lejewski [1] and appears as *X1* in the following axiom system for atomless mereology.

$X1$ $$[AB] :: A\,\varepsilon\,\mathbf{ex}(B)\ .\!\equiv\!\therefore\ [f] \therefore [Ca] :: C\,\varepsilon\,f(a)\ .\!\equiv\!\therefore\ C\,\varepsilon\,C \therefore [D] \therefore D\,\varepsilon\,\mathbf{ex}(C)\ .\!\equiv\!:\ [E] : E\,\varepsilon\,a\ .\!\supset\!.\ D\,\varepsilon\,\mathbf{ex}(E) :: \supset :: [F] :: [\exists b] \therefore A\,\varepsilon\,b\ .\!\vee\!.\ B\,\varepsilon\,b : [d] : F\,\varepsilon\,d\ .\!\supset\!.\ \sim\!\langle f(b)\,\varepsilon\,f(d)\rangle$$

$DX2$ $[AB] \therefore A\,\varepsilon\,\mathbf{el}(B)\ .\!\equiv\!:\ A\,\varepsilon\,A\,.\,B\,\varepsilon\,B : [D] : D\,\varepsilon\,\mathbf{ex}(B)\ .\!\supset\!.\ D\,\varepsilon\,\mathbf{ex}(A)$
$X15$ $[A] : A\,\varepsilon\,A\ .\!\supset\!.\ [\exists F]\,.\,F\,\varepsilon\,\mathbf{el}(A)\,.\,\sim\!\langle A\,\varepsilon\,F\rangle$

This rather curious numbering is designed to facilitate the second half of the proof of the equivalence of the two systems.

First we shall derive *X1*, *DX2*, and *X15* from *E1*.

$E2$ $[A]\,.\,\sim\!\langle\wedge\,\varepsilon\,\mathbf{ex}(A)\rangle$ [$[a]\,.\,\sim\!\langle\wedge\,\varepsilon\,a\rangle$]
$E3$ $[A] : \wedge\,\varepsilon\,\mathbf{ex}(A)\ .\!\equiv\!.\ A\,\varepsilon\,\mathbf{ex}(A)$ [$X1[b]\,.\,\sim\!\langle\wedge\,\varepsilon\,b\rangle$]
$E4$ $[A]\,.\,\sim\!\langle A\,\varepsilon\,\mathbf{ex}(A)\rangle$ [$E2$; $E3$]
$DE1$ $[Aa] :: A\,\varepsilon\,\mathbf{Kl}(a)\ .\!\equiv\!\therefore\ A\,\varepsilon\,A \therefore [D] \therefore D\,\varepsilon\,\mathbf{ex}(A)\ .\!\equiv\!:\ [E] : E\,\varepsilon\,a\ .\!\supset\!.\ D\,\varepsilon\,\mathbf{ex}(E)$
$E5$ $[f] \therefore [Ca] :: C\,\varepsilon\,f(a)\ .\!\equiv\!\therefore\ C\,\varepsilon\,C \therefore [D] \therefore D\,\varepsilon\,\mathbf{ex}(C)\ .\!\equiv\!:\ [E] : E\,\varepsilon\,a\ .\!\supset\!.\ D\,\varepsilon\,\mathbf{ex}(E) :: \supset.\ f \circ \mathbf{Kl}$
PR $[f] \therefore \mathrm{Hp}(1) :: \supset:$
2. $[Ca] : C\,\epsilon\,f(a)\ .\!\equiv\!.\ C\,\epsilon\,\mathbf{Kl}(a) :$ [$DE1$; 1]
3. $[a]\,.\,f(a) \circ \mathbf{Kl}(a)\,.$ [2]
$f \circ \mathbf{Kl}$ [3]

E6 $[AB] :\cdot A \varepsilon \mathbf{ex}(B) .\equiv:: [F] :: A \varepsilon F .\vee. B \varepsilon F .\vee\therefore [\exists b] \therefore A \varepsilon b .\vee. B \varepsilon b :$
$[d] : F \varepsilon d .\supset. \sim\langle \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)\rangle$ [*E1*; *DE1*; *E5*]

E7 $[AB] :\cdot \sim\langle A \varepsilon \mathbf{ex}(B)\rangle .\equiv:: [\exists F] :: \sim\langle A \varepsilon F\rangle . \sim\langle B \varepsilon F\rangle \therefore [b] \therefore A \varepsilon b .\vee B \varepsilon b :$
$\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$ [*E6*]

E8 $[A] \therefore [\exists F] \therefore \sim\langle A \varepsilon F\rangle : [b] : A \varepsilon b .\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$
[*E7*, *B*/*A*; *E4*]

E9 $[Aa] : A \varepsilon a .\supset. \mathbf{Kl}(a) \varepsilon \mathbf{Kl}(a)$

PR $[Aa] : \mathrm{Hp}(1) .\supset.$
$[\exists d] .$

2. $\mathbf{Kl}(a) \varepsilon \mathbf{Kl}(d) .$ [*E8*, *b*/*a*; 1]

$\mathbf{Kl}(a) \varepsilon \mathbf{Kl}(a)$ [2]

DE2 $[AB] \therefore A \varepsilon \mathbf{el}(B) .\equiv: A \varepsilon A . B \varepsilon B : [D] : D \varepsilon \mathbf{ex}(B) .\supset. D \varepsilon \mathbf{ex}(A)$

E10 $[ACDa] : A \varepsilon \mathbf{Kl}(a) . D \varepsilon a . C \varepsilon \mathbf{ex}(A) .\supset. C \varepsilon \mathbf{ex}(D)$

PR $[ACDa] \therefore \mathrm{Hp}(3) .\supset:$

4. $[E] : E \varepsilon a .\supset. C \varepsilon \mathbf{ex}(E) :$ [*DE1*; 1; 3]

$C \varepsilon \mathbf{ex}(D) .$ [4; 2]

E11 $[ADa] : A \varepsilon \mathbf{Kl}(a) . D \varepsilon a .\supset. D \varepsilon \mathbf{el}(A)$

PR $[ADa] \therefore \mathrm{Hp}(2) .\supset:$

3. $[C] : C \varepsilon \mathbf{ex}(A) .\supset. C \varepsilon \mathbf{ex}(D) :$ [*E10*; 1; 2]

$D \varepsilon \mathbf{el}(A)$ [*DE2*; 2; 1; 3]

E12 $[A] : A \varepsilon A .\supset. A \varepsilon \mathbf{Kl}(A)$ [*DE1*]

E13 $[AB] \therefore A \varepsilon A . B \varepsilon B : [D] : D \varepsilon \mathbf{ex}(A) .\equiv. D \varepsilon \mathbf{ex}(B) :\supset. A = B$

PR $[AB] \therefore \mathrm{Hp}(3) :\supset.$

4. $A \varepsilon \mathbf{Kl}(A) .$ [*E12*; 1]

5. $B \varepsilon \mathbf{Kl}(A) .$ [*DE1*; 2; 3]

6. $\mathbf{Kl}(A) \varepsilon \mathbf{Kl}(A) .$ [*E9*; 1]

$A = B$ [4; 5; 6]

E14 $[AB] : A \varepsilon \mathbf{el}(B) . B \varepsilon \mathbf{el}(A) .\supset. A = B$

PR $[AB] \therefore \mathrm{Hp}(2) .\supset:$

3. $[D] : D \varepsilon \mathbf{ex}(B) .\supset. D \varepsilon \mathbf{ex}(A) :$ [*DE2*; 1]

4. $[D] : D \varepsilon \mathbf{ex}(A) .\supset. D \varepsilon \mathbf{ex}(B) :$ [*DE2*; 2]

5. $[D] : D \varepsilon \mathbf{ex}(A) .\equiv. D \varepsilon \mathbf{ex}(B) :$ [3; 4]

$A = B$ [*E13*; 1; 2; 5]

E15 $[A] : A \varepsilon A .\supset. [\exists F] . F \varepsilon \mathbf{el}(A) . \sim\langle A \varepsilon F\rangle$

PR $[A] : \mathrm{Hp}(1) .\supset.$
$[\exists Fd] .$

2. $\sim\langle A \varepsilon F\rangle .$ }
3. $F \varepsilon d .$ } [*E8*; 1]
4. $\mathbf{Kl}(A) \varepsilon \mathbf{Kl}(d) .$ }

5. $A \varepsilon \mathbf{Kl}(d) .$ [*E12*; 1; 4]

6. $E \varepsilon \mathbf{el}(A) .$ [*E11*; 5; 3]

$[\exists F] . F \varepsilon \mathbf{el}(A) . \sim\langle A \varepsilon F\rangle$ [6; 2]

E16 $[DFGa] \therefore [E] : E \varepsilon a .\supset. D \varepsilon \mathbf{ex}(E) : F \varepsilon a . G \varepsilon \mathbf{el}(F) :\supset: [E] : E \varepsilon a \cup G .$
$\supset. D \varepsilon \mathbf{ex}(E)$

PR $[DFGa] \therefore \mathrm{Hp}(3) :\supset:$

4. $D \varepsilon \mathbf{ex}(F) .$ [1; 2]

5. $D \varepsilon \mathbf{ex}(G):$ [$DE2$; 3; 4]
$[E]:E \varepsilon a \cup G .\supset. D \varepsilon \mathbf{ex}(E):$ [1; 5; 3]

$E17$ $[FGa]:F \varepsilon a . G \varepsilon \mathbf{el}(F) .\supset. \mathbf{Kl}(a) \circ \mathbf{Kl}(a \cup G)$ [$DE1$; $E16$]

$E18$ $[AF] \therefore A \varepsilon A :[b]:A \varepsilon b .\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d) :\supset. F \varepsilon \mathbf{el}(A)$

PR $[AF] \therefore$ Hp(2) $:\supset.$
$[\exists d].$
3. $F \varepsilon d.$
4. $\mathbf{Kl}(A) \varepsilon \mathbf{Kl}(d).$ [2; 1]
5. $A \varepsilon \mathbf{Kl}(d).$ [$E12$; 1; 4]
$F \varepsilon \mathbf{el}(A)$ [$E11$; 5; 3]

$E19$ $[ABF] :: A \varepsilon A . B \varepsilon B \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d) ::\supset$
$::[\exists G] :: \sim\langle A \varepsilon G\rangle . \sim\langle B \varepsilon G\rangle \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . G \varepsilon d.$
$\mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$

PR $[ABF] ::$ Hp(3) $::\supset::$
4. $F \varepsilon \mathbf{el}(A).$ [$E18$; 1; 3]
5. $F \varepsilon \mathbf{el}(B) \therefore$ [$E18$; 3; 2]
$[\exists G] \therefore$
6. $G \varepsilon \mathbf{el}(F).$
7. $\sim\langle F \varepsilon G\rangle \therefore$ [$E15$; 4]
8. $A \varepsilon G .\mathsf{v}. B \varepsilon G :\supset: A = G .\mathsf{v}. B = G \therefore$ [6]
9. $A \varepsilon G .\mathsf{v}. B \varepsilon G :\supset. F \varepsilon \mathbf{el}(G) \therefore$ [8; 4; 5]
10. $A \varepsilon G .\mathsf{v}. B \varepsilon G :\supset. F = G \therefore$ [$E14$; 9; 6]
11. $\sim\langle A \varepsilon G .\mathsf{v}. B \varepsilon G\rangle \therefore$ [10; 7]
12. $[b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d \cup G) \therefore$ [3; 6; $E17$]
13. $[b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . G \varepsilon d \cup G . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d \cup G) \therefore$ [12; 6]
14. $[b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . G \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d) \therefore$ [13]
$[\exists G] :: \sim\langle A \varepsilon G\rangle . \sim\langle B \varepsilon G\rangle \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . G \varepsilon d.$
$\mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$ [11; 14]

$E20$ $[AB] :: \sim\langle A \varepsilon A\rangle . B \varepsilon B .\supset:: [\exists G] :: \sim\langle A \varepsilon G\rangle . \sim\langle B \varepsilon G\rangle \therefore [b] \therefore A \varepsilon b.$
$\mathsf{v}. B \varepsilon b :\supset. [\exists d] . G \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$

PR $[AB] ::$ Hp(2) $.\supset::$
$[\exists G] ::$
3. $G \varepsilon \mathbf{el}(B).$
4. $\sim\langle B \varepsilon G\rangle.$ [$E15$; 2]
5. $\sim\langle A \varepsilon G\rangle \therefore$ [1]
6. $[b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. B \varepsilon b . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(b) \therefore$ [$E9$; 1]
7. $[b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. G \varepsilon b \cup G . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(b \cup G) \therefore$ [6; $E17$; 3]
$[\exists G] :: \sim\langle A \varepsilon G\rangle . \sim\langle B \varepsilon G\rangle \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d].$
$G \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$ [5; 4; 7]

$E21$ $[AB] :: \sim\langle A \varepsilon A\rangle . \sim\langle B \varepsilon B\rangle .\supset\therefore \sim\langle A \varepsilon \wedge\rangle . \sim\langle B \varepsilon \wedge\rangle \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset.$
$\wedge \varepsilon \wedge . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(\wedge)$ [Ontology]

$E22$ $[AB] :: [\exists F] \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d) ::\equiv:: [\exists F] ::$
$\sim\langle A \varepsilon F\rangle . \sim\langle B \varepsilon F\rangle \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . F \varepsilon d . \mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$
[$E19$; $E20$; $E20$; $E21$]

$E23$ $[AB] :: \sim\langle A \varepsilon \mathbf{ex}(B)\rangle .\equiv:: [\exists F] \therefore [b] \therefore A \varepsilon b .\mathsf{v}. B \varepsilon b :\supset. [\exists d] . F \varepsilon d.$
$\mathbf{Kl}(b) \varepsilon \mathbf{Kl}(d)$ [$E7$; $E22$]

E24 $[AB] :\cdot: A\,\varepsilon\,\mathbf{ex}(B)\ .\equiv::\ [F] \therefore [\exists b] \therefore A\,\varepsilon\,b\ .\mathrm{v}.\ B\,\varepsilon\,b : [d] : F\,\varepsilon\,d\ .\supset.$
$\sim\langle \mathbf{Kl}(b)\,\varepsilon\,\mathbf{Kl}(d)\rangle$ [*E23*]
E25 $(= X1)$ [*E24*; *E5*]

E25, *DE2*, *E15* are *X1*, *DX2*, *X15* respectively. Therefore atomless mereology is derivable from *E1*.

Next we derive *E1* from *X1*, *DX2*, *X15*. The proof closely parallels the one we have just completed. Let *X2* through *X25* be identical to *E2* through *E25* with the following modifications:

X6 $[AB] :: A\,\varepsilon\,\mathbf{ex}(B)\ .\equiv\therefore [F] \therefore [\exists b] \therefore A\,\varepsilon\,b\ .\mathrm{v}.\ B\,\varepsilon\,b : [d] : F\,\varepsilon\,d\ .\supset.$
$\sim\langle \mathbf{Kl}(b)\,\varepsilon\,\mathbf{Kl}(d)\rangle$
X7 $[AB] :: \sim\langle A\,\varepsilon\,\mathbf{ex}(B)\rangle\ .\equiv\therefore\ [\exists F] \therefore [b] \therefore A\,\varepsilon\,b\ .\mathrm{v}.\ B\,\varepsilon\,b\ :\supset.\ [\exists d]\,.\,F\,\varepsilon\,d\,.$
$\mathbf{Kl}(b)\,\varepsilon\,\mathbf{Kl}(d)$
X8 $[A] \therefore [\exists F] : [b] : A\,\varepsilon\,b\ .\supset.\ [\exists d]\,.\,F\,\varepsilon\,d\,.\,\mathbf{Kl}(b)\,\varepsilon\,\mathbf{Kl}(d)$
X23 $[AB] :\cdot: \sim\langle A\,\varepsilon\,\mathbf{ex}(B)\rangle\ .\equiv::\ [\exists F] :: \sim\langle A\,\varepsilon\,F\rangle\,.\sim\langle B\,\varepsilon\,F\rangle \therefore [b] \therefore A\,\varepsilon\,b\ .\mathrm{v}.\ B\,\varepsilon\,b :\supset.$
$[\exists d]\,.\,F\,\varepsilon\,d\,.\,\mathbf{Kl}(b)\,\varepsilon\,\mathbf{Kl}(d)$
X24 $[AB] :\cdot: A\,\varepsilon\,\mathbf{ex}(B)\ .\equiv::\ [F] :: A\,\varepsilon\,F\ .\mathrm{v}.\ B\,\varepsilon\,F\ .\mathrm{v}\therefore [\exists b] \therefore A\,\varepsilon\,b\ .\mathrm{v}.\ B\,\varepsilon\,b : [d] :$
$F\,\varepsilon\,d\ .\supset.\ \sim\langle \mathbf{Kl}(b)\,\varepsilon\,\mathbf{Kl}(d)\rangle$
X25 $(= E1)$.

Replacement of *E* theses by the corresponding *X* theses yields proofs of *X2* through *X25* except for *X15* which is an axiom and so has no need of proof. The definitions used in both systems are identical so *E1* is derivable from atomless mereology and the two systems are interderivable. Therefore *E1* is a single axiom for atomless mereology.

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University of Notre Dame
Notre Dame, Indiana