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CREATIVE DEFINITIONS IN PROPOSITIONAL CALCULI

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Leśniewski felt that definitions were most naturally stated as equivalences in the object language and as such a rule of procedure governing their introduction is necessary. This view will be accepted here in our investigation of the role played by definitions in propositional calculi. In this paper* we construct propositional calculi wherein some of the definitions play a creative role; i.e., they do not function as mere abbreviations and are not, even theoretically, superfluous.

A definition will be said to be *creative* for a thesis T in a given presentation of a deductive theory iff T does not contain the defined term (nor any defined via it) and is provable using the definition, but not without it.

The usual approach to definitions is to attempt to prescribe conditions which prevent the creativity of definitions. In trying to understand the role that definitions play in deductive theories we approach the subject from the opposite direction and attempt to construct systems which contain creative definitions. In **3** we give axiomatizations of propositional calculi which contain a single creative definition, a finite number of creative definitions, and also examples which contain an unlimited number of creative definitions.

In **1** the history of the problem is presented as best it is known, including a review of the literature. The rules of procedure for propositional calculi and especially the rule of definition are presented informally in **2** and precisely in the appendix. Several metalogical remarks are presented in **4** including our proof of a hitherto unpublished theorem of A. Lindenbaum which shows that if Cpp is a thesis of a propositional calculus, then that calculus contains no creative definitions.

1 History. Leśniewski recognized that definitions can be creative and, as far as is known, was the first to do so. He also defined this concept and

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introduced the term. Leśniewski discussed creative definitions in his lectures, but he never published his views on this matter. Indeed, there is no mention of creative definitions in his published works, even in his paper [1] which concerns definitions in the propositional calculus. Creative definitions are used frequently in Leśniewski's systems of Protothetic, Ontology, and Mereology.

The first mention of creative definitions in the literature seems to be in a summary of a lecture of Łukasiewicz given February 18, 1928 [3]. Mention is made of several definitions which are creative in certain systems, but no details are given in the summary. At this lecture Leśniewski affirmed his belief in creative definitions and stated that creative definitions should be used as often as possible. Łukasiewicz also mentioned creative definitions in his lecture of March 24, 1928 [4] and referred to them as "hidden axioms."

Łukasiewicz defined what is meant by a creative definition in his 1929 monograph [5] (p. 32 of the English edition). That this is not his concept can be inferred from the preface since it is not listed among the new results he claims.

The first published example of a creative definition is in the paper of Łukasiewicz [6] which was to appear in 1939 in the new journal *Collecteana Logica*. Due to the war this journal never appeared and so the article could be known only thru a review of Heinrich Scholtz [16] until its recent publication in Polish [8] and English [9], [10].

There are scattered references in the literature to creative definitions in the Leśniewskian systems. See for example Słupecki [17], [18]. A detailed study of definitions in these systems is yet to be made. Myhill [13] gives a system of number theory containing infinitely many creative definitions. Popper [14] gives examples of creative definitions in a system of probability, but does so by ignoring a widely accepted rule of definition. Except for occasional remarks this seems to be a complete review of the literature.

2 The Rules of Procedure for Propositional Calculi. We shall be considering two kinds of propositional calculi; those with rules of substitution and detachment and those which have, besides these two, a rule of definition. To specify a propositional calculus is to state its primitive terms, axioms, and rules. Implication (C) or implication and negation (N) will be taken as primitive in the systems we consider although our results hold for other primitives. Axioms must be stated in primitive notation (see **4** for the reason) and are chosen from the (2-valued) tautologies.

If the only rules are substitution and detachment, it is easy to define the set of well formed formulas (wffs) in advance and so the statement of the rule of substitution for propositional variables presents no difficulties. In this case we can think of a propositional calculus as a set of tautologies closed under substitution and detachment. When we speak of, say, *The* Implicational Calculus we speak in this sense. This plays down the role of the axioms, which is sometimes convenient, but it can also lead to misunderstandings, for many times in the sequel the particular axiomatization is important.

The situation becomes more complicated when we have definitions in the system. It is no longer possible to specify the wffs in advance (technically we could, but only in an *ad hoc* and unesthetic way) for there is no way of knowing what definitions might be introduced. Consequently, when we have definitions, the calculus must be viewed, not as a set of theses, but as an axiomatic system which develops step by step. So there is a definite order to the theses.

The rules of substitution and detachment are well known and will not be stated here. The rules for propositional calculi with definitions have been stated precisely by Leśniewski [1] and a simplified version of them is presented in the appendix to this paper. We shall now paraphrase the rule of definition for a propositional calculus which includes equivalence (E) as primitive. The rule of definition allows, but does not force us, to add equivalential theses, called definitions, to the system which are of the form

(1) $E\varphi(p_1, \ldots, p_n)Dp_1 \ldots p_n$

provided that the following conditions are met:

a) The functor D being defined is a new symbol, i.e., it is not equiform to any symbol occurring previously in the system, be it primitive or defined term or variable. (As we are using Łukasiewicz's notation there is no concern about parentheses.)
b) The definiens $\varphi(p_1, \ldots, p_n)$ is a wff containing, besides variables, only primitive and previously defined terms. (We could define by primitive terms alone but this is inconvenient.)
c) $p_1, p_2, \ldots, p_n$ are non-equiform (distinct) variables, where by variable we mean any symbol except the primitive and defined terms (functors).
d) Every variable occurring in the definiens $\varphi(p_1, \ldots, p_n)$ occurs in the definiendum $Dp_1 \ldots p_n$ (freely of course).
e) Every variable in the definiendum occurs also in the definiens.

The most crucial thing to keep in mind here is that the addition of a definition expands the system in two ways: 1) a new symbol is introduced into the vocabulary of the system and hence the stock of well formed formulas is greatly increased, and 2) a new thesis is added to the system and it is eligible to be used in further deductions just like any other thesis of the system.

It is important also to note that the rule of the system adjusts to the last thesis of the system. That is, after a definition has been added to the system, the defined term is eligible for use in further definitions and also well formed formulas involving this term can be substituted for variables in any thesis of the system. Hence when there are definitions in the system the rule of substitution must be formulated in such a way that it permits the use of defined terms in substitutions. Accordingly the *statement* of the rule of substitution must be altered when the system contains definitions. (See the appendix for details.) This interdependence of statement does not imply that one of the rules could be relegated to the status of a derived rule; they remain independent in the usual sense that rules are independent.

Because of this dependence, we shall speak of *the* rule of procedure of the system. This rule consists of three parts which we shall loosely call the rules of detachment, substitution and definition. The rule must adjust to the last thesis of the system since it is not possible to foresee all of the definitions that one might wish to add to the system. Thus the addition of theses to the system according to the rule not only increases the stock of theses but also strengthens the rule. The invention of such self-adjusting rules is due to Leśniewski.

If there is a rule of definition in a system but the rule of substitution is not altered, then it is restricted so that one can only substitute wffs which involve only primitive terms. If this is done then every theorem of the system is of one of the following forms: 1) it is a consequence of the axioms by detachment and restricted substitution, 2) it is a restricted substitution instance of a definition, 3) it is a restricted substitution instance of a definiendum, but only when the same restricted substitution instance of the definiens is of type 1). It should be observed that when restricted substitution is used no thesis contains more than one defined term and at most one occurrence of it. Thus restricted substitution is completely unacceptable.

When equivalence is not a primitive term of the system it is impossible to state definitions in the form (1). All of the systems we shall consider will have implication as primitive and hence we shall state definitions as pairs of equivalences

$$(2) \quad \begin{array}{l} C\varphi(p_1, \ldots, p_n)Dp_1 \ldots p_n \\ CDp_1 \ldots p_n\varphi(p_1, \ldots, p_n) \end{array}$$

where the definiens $\varphi(p_1, \ldots, p_n)$ and definiendum $Dp_1 \ldots p_n$ satisfy the same conditions as definitions of type (1). This definitional frame was suggested by Tarski, see [3], and within the full propositional calculus (1) and (2) are equivalent.

Leśniewski would object to this definitional frame as he felt that a definition should be a single statement. When implication is primitive and one wants definitions to be single theses then either the definitional frame of Leśniewski [1] or Lejewski [2] may be used. In this paper we shall only use definitions of type (2). The choice of definitional frame does however affect some of our results (*cf.* the end of **4**).

3 Construction of Propositional Calculi Containing Creative Definitions. As a prototype example consider the C-N propositional calculus whose sole axiom is

$$(1) \quad CCCNuvwCCCCCpqCNrNsrtCCtpCsp$$

If detachment and substitution are the only rules of procedure, then the only consequences of (1) are those obtainable by substitution, i.e., the rule of detachment is dead. For if any detachment is to be made then we need substitution instances of (1) of the form α and $C\alpha\beta$:

(2) $\alpha = CCCNu'v'w'M'$
(3) $C\alpha\beta = CCCNu''v''w''M''$

where the equality sign means that the inscriptions it connects are equiform and M is the conclusion of (1), i.e., *CCCCCpqCNrNsrtCCtpCsp*. The primes are used to denote substitution instances; for example, M' and M'' are substitution instances of M. (2) and the antecedent of (3) both represent α and so must be equiform, i.e.,

$$CCCNu'v'w'M' = CCNu''v''w''$$

Comparing the antecedents we get

$$CCNu'v'w' = CNu''v''$$

and then

$$CNu'v' = Nu''$$

This is impossible since no inscription can begin with both *C* and *N*. Thus there are no substitution instances of the axiom (1) such that one can be detached from the other, i.e., the rule of detachment cannot be used at all. This shows that the only consequences of (1) are those obtainable by substitution. In particular, we cannot derive the full *C*-*N* calculus from (1) using only the rules of substitution and detachment.

If we consider now the propositional calculus with axiom (1) and rules of substitution, detachment and definition then we can use the rule of definition to introduce the definition of alternation into the system:

(4) *CCNuvAuv*
(5) *CAuvCNuv*

Since *Auv* is now a meaningful expression in our system we may substitute it for *w* in (1) to obtain

(6) *CCCNuvAuvCCCCCpqCNrNsrtCCtpCsp*

From this we may detach the first half of the definition (4) to obtain

(7) *CCCCCpqCNrNsrtCCtpCsp*

which is Meredith's [11] axiom for the full *C*-*N* calculus, i.e., from (7) we can deduce all *C*-*N* tautologies by substitution and detachment.

Since the full *C*-*N* calculus can be deduced from (1) by using the rules of substitution, detachment and definition but not by use of substitution and detachment alone, we conclude that the definition of alternation (4)-(5) is creative in the propositional calculus whose sole axiom is (1).

Observe that (4)-(5) is not the only definition which could be used to obtain (7). Any definition with definiens of the form $CN\gamma\delta$ would suffice. However, it follows from later results that only one such definition can be creative in this system.

Observe also that no thesis which begins *CC . . . CN* is self-detachable.

This example shows that the full propositional calculus is *axiomatizable with creative definitions*, i.e., there is an axiom system in which the use of creative definitions allows us to deduce the full propositional calculus. The set of axioms $\mathfrak{A}$ is an *axiomatization with creative definitions* of the propositional calculus $\mathcal{L}$ (and here we think of $\mathcal{L}$ as a set of theorems) iff 1) by using the rules of substitution, detachment and definition we can derive precisely the theses of $\mathcal{L}$ from the axioms $\mathfrak{A}$, but 2) this cannot be done by using only the rules of substitution and detachment. In particular, 1) implies that each axiom of $\mathfrak{A}$ is a thesis of $\mathcal{L}$.

This example makes it clear that we can prove

Theorem 1. *Any propositional calculus $\mathcal{L}$ with C and N among its primitives and containing the thesis CpCqp is axiomatizable with creative definitions.*

Proof: Let A_1, A_2, . . . be axioms for $\mathcal{L}$. Since simplification is a thesis, each of

(8i) $CCCNpqrA_i$ $\qquad i = 1, 2, \ldots$

is provable in $\mathcal{L}$. With these as new axioms then the definition (4)-(5) of alternation is creative.

Since each axiom begins *CCCN* no substitution instance of axiom (8i) can be detached from a substitution instance of axiom (8j). Thus we cannot derive the axioms A_i of $\mathcal{L}$ from these new axioms.

If a rule of definition is available then we can introduce the definition (4)-(5) of alternation and then substitute Apq for r in (8i) to obtain

(9i) $CCCNpqApqA_i$ $\qquad i = 1, 2, \ldots$

Here we tacitly assumed that r does not occur in any A_i. Then detach (4) from (9i) to obtain A_i. Thus the definition of alternation is creative and the axiom system (8i) provides a creative axiomatization of $\mathcal{L}$. This completes the proof.

The annoying hypothesis that simplification be a thesis of $\mathcal{L}$ guarantees that the new axioms (8i) be theses of $\mathcal{L}$. It can be weakened, say, to the thesis $CsCCNpqrs$ or even to the corresponding rule, but little generality is gained by doing this.

It might be suspected that the first N in axioms (1) and (8i) is crucial to showing that the axioms are non-detachable without the use of a creative definition. That this is not so is shown by the following creative axiomatization of the full implicational calculus:

(10) $CCuCttCCCpqrCCrpCsp$

To show this implicational thesis is non-detachable assume we have as substitution instances of (10)

(11) $\alpha = CCu'Ct't'CCCp'q'r'CCr'p'Cs'p'$

(12) $C\alpha\beta = CCu''Ct''t''CCCp''q''r''CCr''p''Cs''p''$

To detach the antecedent of (12) must be equiform to (11):

$$Cu''Ct''t'' = CCu'Ct't'CCCp'q'r'CCr'p'Cs'p'$$

Comparing consequences we get

$$t'' = CCp'q'r'$$
$$t'' = CCr'p'Cs'p'$$

and thus

$$Cp'q' = Cr'p'$$
$$r' = Cs'p'$$

Finally we have both $p' = r'$ and $r' = Cs'p'$ which is impossible. Thus the only substitution-detachment consequences of (10) are obtainable by substitution only.

If we have a rule of definition then we can add the definition of verum

(13) $CCtt\mathrm{V}rt$

(14) $C\mathrm{V}rtCtt$.

By substituting Vrt for u in (10) we can detach (14) to obtain

(15) $CCpqrCCrpCsp$

which is Łukasiewicz's axiom [7] for the implicational calculus.

For the implicational calculus the creativity is associated with the second half (14) of the definition. If we try to make (13) the creative part it is natural to take

$$CCCttuCCCpqrCCrpCsp$$

as an axiom. But this is self-detachable and immediately yields (15).

Analogously the positive implicational fragment is creatively axiomatizable by

$$CCuCvvCCCpqrCsCCqCrtCqt$$

where verum (13)-(14) is again the creative definition. There is nothing special about the antecedent $CuCvv$ which precedes Meredith's axiom [12] for the positive implicational calculus. Any antecedent of the form $Cu\alpha$ or $C\alpha u$ which would make the sentence non-detachable (and where α does not contain u) would make the definition

$$C\alpha Dp_1 \ldots p_n$$
$$CDp_1 \ldots p_n\alpha$$

creative (where $p_1, \ldots, p_n$ are all of the variables in α). The reader should by now suspect

Theorem 2. *Any propositional calculus with C among its primitives and containing the thesis CpCqp is axiomatizable with creative definitions.*

Proof: Let $A_1, A_2, \ldots$, none of which contain p or q, be an axiomatization of the calculus. As new axioms take the theses

(16n) $CCCCpppqA_n$ $\qquad n = 1, 2, \ldots$

To show no detachment can be made suppose we have

$$\alpha = CCCCp'p'p'q'A'_n$$
$$C\alpha\beta = CCCCp''p''p''q''A''_m$$

We must then have

$$CCCCp'p'p'q'A'_n = CCCp''p''p''q''$$

and then

$$CCCp'p'p'q' = CCp''p''p''$$

so

$$CCp'p'p' = Cp''p''$$

finally

$$Cp'p' = p''$$
$$p' = p''$$

which is impossible.

Adding the definition of assertium

(17) $CCCpppAsp$
(18) $CAspCCppp$

we can detach (17) from (16n) to obtain A_n. Thus this definition is creative.

Again there is nothing unique about the antecedent $CCCpppq$ of (16n). If B is any implicational wff not containing q such that $CB'q' \neq B''$ for *all* substitution instances B', B'' of B then the axioms $CCCBqA_n$ will do as well as those of the theorem.

All calculi constructed so far have contained only a single creative definition. The following theorem shows how to construct calculi with an arbitrary but finite number of creative definitions.

Theorem 3. *Consider the propositional calculus whose sole axiom is*

(19) $CCN^nCppq_nCCN^{n-1}Cppq_{n-1} \ldots CCNCppq_1A$

where A is any tautology and N^k represents a sequence of k N's. Then the n definitions

(20.k) CN^kCppB_kp
(21.k) CB_kpN^kCpp $\qquad k = 1, 2, \ldots, n$

are all creative.

Proof: The axiom is not self-detachable. If we add the definition of B_n, i.e., (20.n) and (21.n) to the system then B_np can be substituted for q_n in the axiom and (20.n) can be detached to obtain

(22) $CCN^{n-1}Cppq_{n-1} \ldots CCNCppq_1A$,

thereby showing that the definition of B_n is creative.

The system now contains four theses: The axiom (19), the theorem (22) and the definition of B_n: (20.n)-(21.n). We cannot detach from the definition since no thesis begins with N or B_n. Neither can we detach from (22) since no thesis begins $CN^{n-1}C$. (If this last C were not there then (20.n) could be detached again.) Thus no detachment can be made. If we now add the definition of B_{n-1} it will be creative. Induction completes the proof.

Realize that in this theorem the definitions are creative in a certain order. Since we view propositional calculi as systems which *develop* there is always an order imposed on the theses. Here there is more than that. Regardless of the order in which the definitions (20.k)-(21.k), $k = 1, 2, \ldots$, n, are added to the systems the definitions are creative in the order $k = n, \ldots, 2, 1$.

The next theorem removes this order and also shows that there are propositional calculi which contain an unlimited number of creative definitions. One is tempted to say "infinitely many" but this is an abuse of language.

Theorem 4. *Let* $A_1, A_2, \ldots$ *be any sequence of distinct tautologies beginning with C and not containing q. Then in the propositional calculus whose axioms are*

(23n) $CCN^{2n+1}CppqN^{2n}A_n$, $\qquad n = 1, 2, \ldots$

the definitions

(24n) $CN^{2n+1}CppD_np$
(25n) $CD_npN^{2n+1}Cpp$ $\qquad n = 1, 2, \ldots$

are all creative.

Proof: If we add the definitions (24n_i)-(25n_i), $i = 1, 2, \ldots, l$ of $D_{n_1}, D_{n_2}, \ldots, D_{n_l}$ to the system we can detach (24n_i) from (23n_i) to obtain

(26n_i) $N^{2n_i}A_{n_i}$ $\qquad i = 1, 2, \ldots, l$

To see that no other detachments can be made one need only examine all the possible cases:

a) To detach from an axiom we need a thesis beginning $CN^{2m+1}C$ and the only such theses are the l definitions already introduced into the system. The corresponding detachments have already been made so no more detachments can be made from the axioms.
b) Nothing can be detached from any of the definitions already in the system. To detach from the first part of a definition we need a thesis beginning with an odd number of N's. The only theses beginning with N are the (26n_i) and they all begin with an even number of N's. We cannot detach from the second part of a definition as no thesis begins with a defined term.
c) Nothing can be detached from the theses (26n_i) as they do not begin with C.

Now if we add the definition of D_m, $m \neq n_1, n_2, \ldots, n_l$, then we can deduce the new thesis $N^{2m}A_m$ and so this definition is creative. Thus each of the definitions is creative. Observe that the order in which these definitions are added to the system is immaterial; they are creative in any order.

In a similar way we can prove:

Theorem 5. *Let* $A_1, A_2, \ldots$ *be an infinite independent axiomatization of a propositional calculus containing the thesis* $CpCqp$ *and such that each* A_i *begins with* C *and does not contain* q. *Then this system can be creatively axiomatized using the axioms*

$$CCN^kCppqA_k \qquad k = 1, 2, \ldots$$

where each of the definitions

$$\begin{matrix} CN^kCppF_kp \\ CF_kpN^kCpp \end{matrix} \qquad k = 1, 2, \ldots$$

is creative.

All examples presented so far have been such that the rule of detachment is rendered useless by the structure of the axioms. Even when this is not the case the system can still contain creative definitions. As an example, take the system with single axiom

$$CCuvCCvu\mathsf{M}$$

An unlimited number of detachments can be made but still the definition

$$\begin{matrix} Cu\mathrm{As}u \\ C\mathrm{As}uu \end{matrix}$$

of assertium is creative.

We have constructed propositional calculi that contain 1, n and an unlimited number of creative definitions, but to obtain an unbounded number of creative definitions we have always used an infinite axiom system. This leaves the interesting

Open Question: Is there a finitely *axiomatized* propositional calculus containing an unlimited number of ("infinitely many") creative definitions?

4 Metalogical Considerations. The following theorem gives a condition sufficient to guarantee that a propositional calculus contains no creative definitions. It was recollected by Sobociński and he attributes it to A. Lindenbaum. We know of no reference to it in the literature. The proof is mine.

Theorem 6. (A. Lindenbaum). *Let* $\mathcal{L}$ *be a propositional calculus with* C *among its primitives and having as rules substitution, detachment and definition (stated as pairs of implications). Also let* Cpp *be provable using only substitution and detachment. Then no definition is creative in this system.*

Proof: Let T be a theorem of $\mathcal{L}$ which does not contain the term D defined by

(1) $C\varphi(p_1, \ldots, p_n)Dp_1 \ldots p_n$
(2) $CDp_1 \ldots p_n\varphi(p_1, \ldots, p_n)$

Let $A_1, A_2, \ldots, A_m$ be a proof of T. This means that A_m is T and that each step A_i of the proof has one of the following justifications:

a) A_i is an axiom of $\mathcal{L}$.
b) A_i results from $A_k = CA_jA_i$ by detaching A_j, where $k, j < i$.
c) A_i results from substitution in A_j, where $j < i$.
d) A_i is one half of a definition.

To say that the definition of D is *not* creative for T means that there is a proof of T no line of which contains the defined term D. This means that not only is reason d) never used, but also substitutions are restricted to those which do not involve D or terms defined using D. To obtain such a proof replace the defined terms in the proof $A_1, A_2, \ldots, A_m$ by their definiens and preface the whole by (a proof of) Cpp to obtain

$$A_0' \text{ (= } Cpp \text{ or a proof of it)}, A_1', A_2', \ldots, A_m'$$

To be precise we define A' inductively as follows:

(i) if A is a variable then $A' = A$
(ii) if $A = C\alpha\beta$ then $A' = C\alpha'\beta'$
(iii) if $A = N\alpha$ then $A' = N\alpha'$
(iv) if $A = D\alpha_1 \ldots \alpha_n$ then $A' = \varphi(\alpha_1', \ldots, \alpha_n')$

(If $\mathcal{L}$ contains other primitives besides C and N, or does not contain N, or the proof uses several definitions, the changes needed in this proof are easy.) Observe that A' is uniquely determined for every A.

We now show that $A_0', A_1', A_2', \ldots, A_m'$ is a proof of T which does not involve the defined term D:

a) If A is an axiom of $\mathcal{L}$ then A_i is expressed in primitive notation and so $A_i' = A_i$.
b) If A_i results from $A_k = CA_jA_i$ by detaching A_j where $k, j < i$ then A_i' results from $A_k' = CA_j'A_i'$ by detaching A_i'.
c) Let A_i result from $A_j (j < i)$ by substitution. More precisely let

$$A_i = A_j(p_1/\alpha_1, \ldots, p_l/\alpha_l)$$

where $p_1, p_2, \ldots, p_l$ are the variables of A_j.

By a straightforward induction one can prove

$$A_i' = A_j'(p_1/\alpha_1', \ldots, p_l/\alpha_l')$$

i.e., A_i' results from A_j' by substitution.
d) If A_i is one half of the definition of D then A_i' is

(3) $C\varphi(p_1, \ldots, p_n)\varphi(p_1, \ldots, p_n)$

and this results from Cpp (i.e., A_0') by substituting $\varphi(p_1, \ldots, p_n)$, which does not contain D, for p. Thus line A_i' of the proof is justified.

Hence $A_0', A_1', \ldots, A_m'$ is a proof of T which uses only the rules of substitution and detachment. Hence the definition of D is *not* creative in the proof of thesis T.

Examination of the proof indicates that the hypothesis that Cpp is provable is too strong. We can easily prove

Corollary 7. *Let $\mathcal{L}$ be a propositional calculus with C among its primitives. Then the definition* (1)-(2) *is* not *creative in this propositional calculus if we can prove* (3) *by using only the rules of substitution and detachment.*

There are calculi which contain no substitution instance of Cpp and which still have no creative definitions, for example, the propositional calculus with sole axiom NNA, A a tautology.

Theorem 8. *Let $\mathcal{L}$ be a propositional calculus. Then $\mathcal{L}$ is consistent if the rules are substitution and detachment. Moreover, if we add a rule of definition then the system is still consistent.*

Proof: The usual truth tables give a consistency proof for $\mathcal{L}$ when the rules are substitution and detachment since, by convention, all axioms of $\mathcal{L}$ are classical tautologies. Since the rule of definition allows us to add definitions only one at a time it suffices to show that the addition of a single definition preserves the consistency of $\mathcal{L}$. Let the definition be

$$C\varphi(p_1, \ldots, p_n)Dp_1 \ldots p_n$$
$$CDp_1 \ldots p_n\varphi(p_1, \ldots, p_n)$$

then construct a truth table for D by setting $Dp_1 \ldots p_n$ = true iff $\varphi(p_1, \ldots, p_n)$ = true. Clearly both halves of the definition of D are verified under this interpretation. The validity of the other rules is not effected by this interpretation of D.

Our notion of a propositional calculus makes this theorem trivial. When we allow arbitrary wffs as axioms this theorem can be improved to

Theorem 9. *Consider the system with C among its primitives, any set of wffs as axioms, and rules of substitution and detachment. If this system is consistent with Cpp then the system obtained by adding a rule of definition is also consistent.*

The assumption that the system be consistent with Cpp is necessary for consider the system whose axioms are $NCpp$ and $CCpqCCqrCpr$. This system is consistent but a rule of definition allows us to derive Cpp by using the definition of assertium. Thus the addition of definitions makes the system negation inconsistent, although it is still absolutely consistent.

It is sometimes felt that the Leśniewskian style definitions serve as *additional axioms*. Whether or not this view is adopted, it is meaningful to consider the *independence* of the definitions from the axioms of the theory. To do this, however, we consider the system not as containing a rule of definition, but rather as a propositional calculus with an additional primitive term and two new axioms. With this view we state

Theorem 10. *Let $\mathcal{L}$ be a propositional calculus with any set of C-N tautologies as axioms. Moreover let D be another* primitive term *of $\mathcal{L}$ and let*

(4) $C\varphi(p_1, \ldots, p_n)Dp_1 \ldots p_n$
(5) $CDp_1 \ldots p_n\varphi(p_1, \ldots, p_n)$

be additional axioms which satisfy the conditions imposed on definitions. Let substitution and detachment to be the only rules and suppose that under the usual interpretation of the system $\varphi(X_1, \ldots, X_n) = 1$ *and* $\varphi(X_1', \ldots, X_n') = 0$ *where* 1 *and* 0 *are the designated and undesignated values respectively and* $X_1, \ldots, X_n, X_1', \ldots, X_n'$ *are some fixed elements of the interpretion. Then the axioms* (4) *and* (5) *are independent.*

Proof: Assign $X_1, \ldots, X_n$ to $p_1, \ldots, p_n$ respectively and interpret D as falsum. Then

$$CDX_1 \ldots X_n\varphi(X_1, \ldots, X_n) = C01 = 1$$

while

$$C\varphi(X_1, \ldots, X_n)DX_1 \ldots X_n = C10 = 0$$

and all of the other axioms, being tautologies, are verified. Thus (4) is independent. The independence of (5) is obtained by assigning $X_1', \ldots, X_n'$ to $p_1, \ldots, p_n$ and interpreting D as verum.

This is as good a theorem as we can get, for if $\varphi(p_1, \ldots, p_n)$ $[N\varphi(p_1, \ldots, p_n)]$ is a tautology then (5) [(4)] may be provable, depending on what axioms $\mathcal{L}$ has. In no case however can both (4) and (5) be dependent. This shows that the creativity of a definition has nothing to do with its independence.

Łukasiewicz insisted that the axioms of a propositional calculus be stated in primitive terms. This was done for "not only esthetic, but also theoretical" reasons [19]. We conjecture that one reason that this condition is imposed on axiom systems is to avoid some creative definitions. Consider the following example: Take C and N as primitive and let M be Meredith's axiom. Before stating any axioms define alternation by

(6) $CCNpqApq$
(7) $CApqNpq$

and then give the single axiom

(8) $CCCNpqApq\mathsf{M}$

By detaching (6) from (8) we get M and hence the full C-N calculus. However when we "eliminate" the defined term from (8) we get

$$CCCNpqCNpqM$$

which is non-detachable. So in this system the definition plays a creative role. We know of no such calculi which have been inadvertently published but suspect there are some.

Throughout this paper, pairs of implications have been used as the definitional frame. This form of writing definitions would be unacceptable to Leśniewski who felt that definitions should be single expressions. In his paper [1] he stated definitions in the form

(9) $NCC\alpha\beta NC\beta\alpha$

where α was the definiendum and β the definiens. This is a legitimate form of writing definitions since it is equivalent to $E\beta\alpha$ in the full propositional calculus. Moreover it is the shortest definitional frame for the C-N calculus where definitions are expressed as single theses.

If definitions are written in this way then the propositional calculus with axiom

(10) $CNCCpqNCqpA$

contains creative definitions. In fact the first definition added to the system is creative regardless of what it is. Thus any definition, written in Leśniewski's definitional frame, can be creative.

It is important to realize that the results of this paper are dependent on the definitional frame used. In substance they remain correct but many details must be altered if a different definitional frame is used. For example, if in the propositional calculus with axiom (10), we add Cpp as another axiom, then there are still creative definitions. Thus Lindenbaum's theorem is dependent on the definitional frame used. If form (9) is used then the Cpp of Lindenbaum's theorem must be replaced by $NCCppNCpp$.

Tarski suggested the definitional frame

$$CCC\alpha\beta CC\beta\alpha rr$$

Negation is not used here and one can get $C\alpha\beta$ and $C\beta\alpha$ by using simplification alone. This still admits creative definitions, for if we take the axiom

$$CCCCpqCCqprrA$$

then the first definition, written in Tarski's frame, is creative. In fact every definitional frame we know admits creative definitions.

5 Conclusion. Ever since Russell wrote in *Principia Mathematica* that definitions are "theoretically superfluous" it has been widely held that definitions are never creative. Although our view of definitions is different than Russell's the results of this paper should provide ample warning that more care is needed when definitions are used.

In a certain sense it is not surprising that definitions can be creative for: (1) Whenever a definition is added to a system, not only is a new symbol introduced but also new theses are introduced. Thus the expressive strength of the language is increased and the stock of theorems available for use in later proofs is increased. (2) The system contains an additional rule, so it is natural that it enables us to derive new theses. It is only slightly surprising that we are able to derive new theses involving only primitive terms.

APPENDIX

A FORMALIZATION OF THE RULES OF PROCEDURE FOR PROPOSITIONAL CALCULI.

We intend to state—in the most precise way that we know—the deductive rules for a C-N calculus when the rules of the system are

(A) Substitution and Detachment.
(B) Substitution, Detachment and Definition.

We do both of these to show that the formulation of the rules of substitution and definition are interconnected. When going from (A) to (B), the formulation of the rule of substitution must be adjusted.

The rules are formulated using (an extension of) the inscriptional syntax **M** developed in [15] which is based on Leśniewski's Ontology. An acquaintance with [15] is assumed. Before formulating rules (A) and (B) we give a series of Terminological Explanations (*T.E.*'s) which define the terms used in the statement of the rules. The following is a dictionary of terms which we shall use from **M**:

$A\,\varepsilon\,a$	A is an a (ε is the primitive of Ontology)
$a \propto b$	a is equinumerous with b
$a \subset b$	a is contained in b
$A\,\varepsilon \sim[b]$	A is not b
$A\,\varepsilon\,\mathbf{vrb}(B)$	A is a word of B
$A\,\varepsilon\,\mathbf{cnf}(B)$	A is equiform with B
$A\,\varepsilon\,\mathbf{expr}$	A is an expression
$A\,\varepsilon\,\mathbf{ingr}(B)$	A is an ingredient of B
$A\,\varepsilon\,\mathbf{1vrb}(B)$	A is the first word of B (similarly for $\mathbf{2vrb}(B)$, . . .)
$A\,\varepsilon\,\mathbf{Cmpl}(a)$	A is the complex of the a's
$A\,\varepsilon\,\mathbf{Concat}(BC)$	A is the concatenation of B and C (similarly for $\mathbf{Concat}(BCD)$)
$A\,\varepsilon\,\mathbf{scd}(B)$	A follows B
$A\,\varepsilon\,\mathbf{prcd}(B)$	A precedes B

We shall also use two new primitive terms which pertain to the propositional calculus in question:

$A\,\varepsilon\,\mathrm{var}(B)$	A is a variable in expression B
$A\,\varepsilon\,\mathrm{th}(B)$	A is a thesis of the system occurring before B (or equal to B)

The first of these terms allows us to define a variable by

$$[A]:A\,\varepsilon\,\mathrm{var}\,.\!\equiv\!.\,[\exists B].\,A\,\varepsilon\,\mathrm{var}(B)$$

The rules shall be formulated for a propositional calculus whose sole axiom is the following axiom of Meredith [11].

$\mathfrak{Ax}$ $\qquad$ *CCCCCpqCNrNsrtCCtpCsp*

In the *T.E.*'s below we shall explicitly refer to this inscription by its name $\mathfrak{Ax}$. The quantifiers below range over names of inscriptions.

TE1 $\quad [AB] : A\ \varepsilon\, \mathsf{Neg}(B)\ .\!\equiv\!.$
$\qquad A\ \varepsilon\, \mathbf{Concat(1vrb}(A),\ B).$
$\qquad \mathbf{1vrb}(A)\ \varepsilon\ \mathbf{cnf}\ (\mathbf{9vrb}(\mathfrak{Ax}))$

A is the negation of *B*.

A will be the negation of *B* if it consists of a negation symbol (*N*) followed by *B*, i.e.,

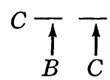
A:

Such diagrams and the informal explanations which are given are intended to aid the reader and are not to be considered part of the formulation of the rule.

TE2 $\quad [ABC] : A\ \varepsilon\, \mathsf{Imp}(BC)\ .\!\equiv\!.$
$\qquad A\ \varepsilon\, \mathbf{Concat(1vrb}(A),\ B,\ C).$
$\qquad \mathbf{1vrb}(A)\ \varepsilon\, \mathbf{cnf(1vrb}(\mathfrak{Ax}))$

A is *B* implies *C*.

A:

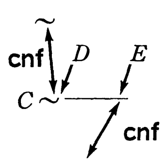


TE3 $\quad [ABCDE] : A\ \varepsilon\, \mathsf{cndet}(BCDE)\ .\!\equiv\!.$
$\qquad B\ \varepsilon\, \mathsf{Imp}(DE)\ .$
$\qquad C\ \varepsilon\, \mathbf{cnf}(D)\ .$
$\qquad A\ \varepsilon\, \mathbf{cnf}(E)$

C:

B:

A:



TE4 $\quad [ABC] : A\ \varepsilon\, \mathsf{cndet}(BC)\ .\!\equiv\!.\ [\exists DE]\ .\ A\ \varepsilon\, \mathsf{cndet}(BCDE)$

A is a consequence of *B* by detaching *C*.

Regardless of what other rules we might have in the system the rule of detachment can always be *formulated* independently of the others.

The crucial step in formulating a rule of substitution is to specify what the well formed formulas are. We do this now when *C* and *N* are the only functors, i.e., when there are no definitions.

TE5 $[Aa] :: A \,\varepsilon\, \mathsf{wff}(a) .\equiv:.$
$[B] : B \,\varepsilon\, \mathsf{var}(A) .\supset. B \,\varepsilon\, a : A \,\varepsilon\, A :$
$[BC] : B \,\varepsilon\, \mathsf{Neg}(C) . B \,\varepsilon\, \mathbf{ingr}(A) . C \,\varepsilon\, a .\supset. B \,\varepsilon\, a :$
$[BCD] : B \,\varepsilon\, \mathsf{Imp}(CD) . B \,\varepsilon\, \mathbf{ingr}(A) . C \,\varepsilon\, a . D \,\varepsilon\, a .\supset. B \,\varepsilon\, a :$
$[C] : C \,\varepsilon\, a .\supset. C \,\varepsilon\, \mathbf{ingr}(A) .$
$[\exists C] . C \,\varepsilon\, \mathsf{var}(A)$

TE6 $[A] :. A \,\varepsilon\, \mathsf{wff} .\equiv:$
$A \,\varepsilon\, \mathbf{expr} :$
$[a] : A \,\varepsilon\, \mathsf{wff}(a) .\supset. A \,\varepsilon\, a$

A is a well formed formula of the C-N calculus.

In *TE5* a is a (name of a distributive) collection of expressions in A which contains all the variables of A (*TE5.1*) of which there is at least one (*TE5.5*) and which is closed under the formation of negations (*TE5.2*) and implications (*TE5.3*). *TE6* says that A is well formed if it is in the intersection of all such a's. This is the classical Frege technique for converting recursive definitions to explicit ones. The clause *TE5.4* is added so that the number of possible a's is at most $2^{n(n+1)/2}$ where n is the number of words in A. This guarantees that the intersection in *TE6* is finite and so wff is "primitive recursive" not just "recursive."

TE7 $[ABa] :\cdot: A \,\varepsilon\, \mathsf{sub}(Ba) .\equiv::$
$\mathbf{vrb}(B) \propto a .$
$A \,\varepsilon\, \mathbf{Cmpl}(a) :$
$[C] : C \,\varepsilon\, a .\supset. C \,\varepsilon\, \mathsf{wff} :.$
$[CD] :. C \,\varepsilon\, \mathbf{vrb}(B) . D \,\varepsilon\, a . (a \cap \mathbf{prcd}(D)) \propto (\mathbf{vrb}(B) \cap \mathbf{prcd}(C))$
$.\supset: C \,\varepsilon\, \mathsf{var}(B) .\mathrm{v}. C \,\varepsilon\, \mathbf{cnf}(D) :.$
$[CDEF] : C \,\varepsilon\, \mathbf{vrb}(B) . D \,\varepsilon\, \mathbf{vrb}(B) . C \,\varepsilon\, \mathbf{cnf}(D) . E \,\varepsilon\, a . F \,\varepsilon\, a . (a \cap \mathbf{prcd}(E))$
$\propto (\mathbf{vrb}(B) \cap \mathbf{prcd}(C)) . (a \cap \mathbf{prcd}(F)) \propto (\mathbf{vrb}(B) \cap \mathbf{prcd}(D))$
$.\supset. E \,\varepsilon\, \mathbf{cnf}(F)$

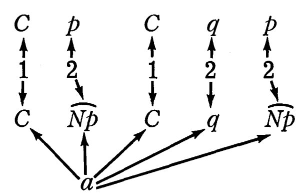
TE8 $[AB] : A \,\varepsilon\, \mathsf{sub}(B) .\equiv. [\exists a] . A \,\varepsilon\, \mathsf{sub}(Ba)$

A is a substitution instance of B.

Leśniewski cleverly solves the problem of formulating the rule of substitution by not saying what is substituted. Instead, A (the result of the substitution) is decomposed into parts (the same number as there are words in B) in such a way that 1) functors in B correspond to equiform functors in A, and such that 2) variables in B correspond to expressions in A. For example

B:

A:



TE7.3 dictates that we can only substitute wffs, *TE7.4* that we can only

substitute for variables, and *TE7.5* that we must substitute the same wff for every instance of a variable.

We are now in a position to state

(A) *The rule of procedure for a propositional calculus with substitution and detachment.*

If B is the last thesis of the propositional calculus with $\mathfrak{Ax}$ *as single axiom then we can add inscription A as a new thesis only if at least one of the following conditions is satisfied:*

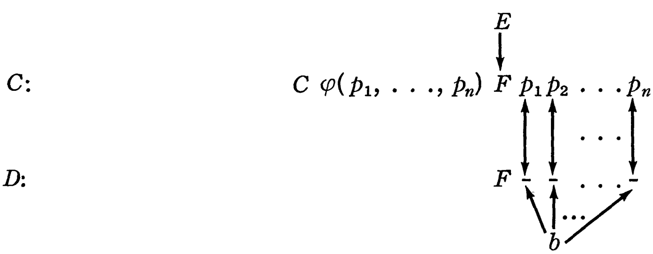
1. $[\exists C] . A\, \varepsilon\, \mathsf{sub}(C) . C\, \varepsilon\, \mathsf{th}(B)$
2. $[\exists CD] . A\, \varepsilon\, \mathsf{cndet}(CD) . C\, \varepsilon\, \mathsf{th}(B) . D\, \varepsilon\, \mathsf{th}(B)$.

Now we shall indicate what changes are to be made in the *T.E.*'s if a rule of definition is included in the system. We add primes to the *T.E.*'s to distinguish them. *TE1'-TE4'* are the same as *TE1-TE4.*

In describing well formed formulas we need closure not just under implication and negation but also under every functor which is introduced into the system by definition. We do not have a list of the possible defined functors (and even if we did it would be infinite) and so we cannot just tack individual clauses onto the old *TE5.* Instead we must refer each defined functor back to its definition to see if it has the appropriate number of arguments. This is the purpose of

TE4A' $[CDEb] :: D\, \varepsilon\, \mathsf{sub}(EbC) .\equiv\therefore$
$D\, \varepsilon\, \mathbf{Concat}(\mathbf{1vrb}(D), \mathbf{Cmpl}(b))$.
$\mathbf{1vrb}(D)\, \varepsilon\, \mathbf{cnf}(E)$.
$b \propto (\mathbf{vrb}(C) \cap \mathbf{scd}(E))$.
$E\, \varepsilon\, \mathbf{vrb}(C)$.
$E\, \varepsilon \sim\![\mathsf{var}]$:
$[F] : F\, \varepsilon\, \mathbf{vrb}(C) . F\, \varepsilon\, \mathbf{scd}(E) .\supset. F\, \varepsilon\, \mathsf{var}$:
$[FG] : F\, \varepsilon\, \mathbf{vrb}(G) . G\, \varepsilon\, \mathsf{th}(C) .\supset. F\, \varepsilon \sim\![\mathbf{cnf}(E)]$

In discussing this *T.E.* we refer to the following diagram:



E is a word of thesis *C* (*TE4A'.4*) which is not a variable (*TE4A'.5*) and is thus a functor which has never occurred in the system before (*TE4A'.7*). Since *E* is followed by a string of variables (*TE4A'.6*) thesis *C* is the first half of the definition which introduces the functor "*F*" in the diagram. Once functor *E* is in the system all expressions *D* consisting of a symbol equiform to E (*TE4A'.2*) and containing the appropriate number of well formed formulas as arguments (*TE4A'.3*) should also be well formed formulas. This is guaranteed by *TE5'.4* below

TE5' $[ABa]::A\ \varepsilon\ \mathrm{wffd}(aB)\ .\equiv\therefore$
$[C]:C\ \varepsilon\ \mathrm{var}(A)\ .\supset.\ C\ \varepsilon\ a:A\ \varepsilon\ A:$
$[CD]:C\ \varepsilon\ \mathrm{Neg}(D)\ .\ C\ \varepsilon\ \mathbf{ingr}(A)\ .\ D\ \varepsilon\ a\ .\supset.\ C\ \varepsilon\ a:$
$[C]:C\ \varepsilon\ a\ .\supset.\ C\ \varepsilon\ \mathbf{ingr}(A):$
$[CDEb]:C\ \varepsilon\ \mathrm{th}(B)\ .\ D\ \varepsilon\ \mathbf{ingr}(A)\ .\ b\subset a\ .\ D\ \varepsilon\ \mathrm{sub}(EbC)\ .\supset.\ D\ \varepsilon\ a$

TE6' $[AB]:.A\ \varepsilon\ \mathrm{wffd}(B)\ .\equiv:$
$A\ \varepsilon\ \mathbf{expr}:$
$[a]:A\ \varepsilon\ \mathrm{wffd}(aB)\ .\supset.\ A\ \varepsilon\ a$

A is a well formed formula with respect to the last thesis B of a system which contains definitions. We cannot define a well formed formula without reference to the last thesis B of the system since as definitions are added to the system the stock of well formed formulas increases.

In *TE5'* a is a collection of expressions in A (*TE5'.3*) containing all the variables of A (*TE5'.1*) and closed under the formation of negations (*TE5'.2*). It is not necessary to say that a is closed under the formation of implications (*cf.*, *TE5.3*) since the axiom ends "*Csp*" and hence $\mathfrak{A}\mathfrak{x}$ serves as the introductory thesis (the C of *TE4A'* and *TE5'.4*) for implication.

TE5' contains no clause analogous to *TE5.5* since one can obtain

$$[AB]:A\ \varepsilon\ \mathrm{wffd}(B)\ .\supset.\ [\exists C]\ .\ C\ \varepsilon\ \mathrm{var}(A)$$

from the lemmas

$$[DEC]\ .\sim(D\ \varepsilon\ \mathrm{sub}(E\wedge C))$$
$$[AB]:A\ \varepsilon\ A\ .\sim([\exists C]\ .\ C\ \varepsilon\ \mathrm{var}(A))\ .\supset.\ \sim(A\ \varepsilon\ \mathrm{wffd}(\wedge B))$$

where $\wedge$ is the empty name.

As in *TE5*, *TE5'.3* is superfluous (but desirable). To see this let $\mathrm{wffd}^*(aB)$ be defined as in *TE5'* except that *TE5'.3* is deleted. Thus

$$[ABa]:A\ \varepsilon\ \mathrm{wffd}(aB)\ .\equiv.\ A\ \varepsilon\ \mathrm{wffd}^*(aB)\ .\ a\subset \mathbf{ingr}(A)$$

Using this and

$$[ABa]:A\ \varepsilon\ \mathrm{wffd}^*(aB)\ .\supset.\ A\ \varepsilon\ \mathrm{wffd}(a\cap \mathbf{ingr}(A),\ B)$$

one can prove

$$[AB]:A\ \varepsilon\ \mathrm{wffd}(B)\ .\equiv.\ A\ \varepsilon\ \mathrm{wffd}^*(B)$$

where $\mathrm{wffd}^*(B)$ is defined like $\mathrm{wffd}(B)$ except that $\mathrm{wffd}^*(aB)$ is used instead of $\mathrm{wffd}(aB)$.

TE7'[*TE8'*] is just like *TE7*[*TE8*] except that $\mathrm{wffd}(B)$ $[\mathrm{subd}(Ba)]$ is used instead of $\mathrm{wff}[\mathrm{sub}(Ba)]$. Call this $\mathrm{subd}(Ba)$ $[\mathrm{subd}(B)]$.

TE9' $[ABC]\therefore A\ \varepsilon\ \mathrm{def}(CB)\ .\equiv:$
$C\ \varepsilon\ \mathrm{wffd}(B):A\ \varepsilon\ A:$

$$\begin{aligned}
&\mathbf{1vrb}(A)\,\varepsilon\,\mathsf{N}[\mathsf{var}]:\\
&[DE]:E\,\varepsilon\,\mathsf{th}(B)\,.\,D\,\varepsilon\,\mathbf{vrb}(E)\ .\supset.\ \mathbf{1vrb}(A)\,\varepsilon\,\mathsf{N}[\mathbf{cnf}(D)]:\\
&[D]:D\,\varepsilon\,\mathsf{var}(C)\ .\supset.\ [\exists E]\,.\,E\,\varepsilon\,\mathbf{vrb}(A)\,.\,D\,\varepsilon\,\mathbf{cnf}(E):\\
&[F]:F\,\varepsilon\,\mathbf{vrb}(A)\,.\,F\,\varepsilon\,\mathbf{scd}(\mathbf{1vrb}(A))\ .\supset.\ [\exists E]\,.\,E\,\varepsilon\,\mathbf{cnf}(F)\,.\,E\,\varepsilon\,\mathsf{var}(C):\\
&[DE]:D\,\varepsilon\,\mathbf{vrb}(A)\,.\,E\,\varepsilon\,\mathbf{vrb}(A)\,.\,D\,\varepsilon\,\mathbf{cnf}(E)\ .\supset.\,D=E
\end{aligned}$$

A is eligible as a definiendum with definiens C with respect to the last thesis B of the system.

A must be of the shape

$$Fp_1 \ldots p_n$$

where "F" is a new symbol (*TE9'.3*) which is not a variable (*TE9'.2*). The variables $p_1, \ldots, p_n$ are all distinct (*TE9'.6*) and are precisely those variables which occur in the definiens C(*TE9'.4, 5*) which is a well formed formula with respect to B(*TE9'.1*). We can now state

(B) *The rule of procedure for a propositional calculus with substitution, detachment and definition.*

If B is the last thesis of the propositional calculus with axiom 𝔄𝔯 *then we can add new theses to the system only under one of the following conditions:*

1. *We can add a new thesis A by substitution if*

$$[\exists C]\,.\,A\,\varepsilon\,\mathsf{subd}(C)\,.\,C\,\varepsilon\,\mathsf{th}(B)$$

2. *We can add a new thesis A by detachment if*

$$[\exists CD]\,.\,A\,\varepsilon\,\mathsf{cndet}(CD)\,.\,C\,\varepsilon\,\mathsf{th}(B)\,.\,D\,\varepsilon\,\mathsf{th}(B)$$

3. *We can add new theses A and A' as a definition (of the first word of C) if*

$$\begin{aligned}&[\exists CDC'D']\,.\,C\,\varepsilon\,\mathsf{def}(DB)\,.A\,\varepsilon\,\mathsf{Imp}(DC)\,.A'\,\varepsilon\,\mathsf{Imp}(C'D')\,.A\,\varepsilon\,\mathbf{prcd}(A')\,.\\&\quad C'\,\varepsilon\,\mathbf{cnf}(C)\,.\,D'\,\varepsilon\,\mathbf{cnf}(D)\end{aligned}$$

A definition has the form

$$A{:}\qquad C\overbrace{\varphi(p_1,\ldots,p_n)}^{D}\ \overbrace{Fp_1\ldots p_n}^{C}$$

$$A'{:}\qquad \overbrace{CFp_1\ldots p_n}^{C'}\ \overbrace{\varphi(p_1,\ldots,p_n)}^{D'}$$

It is important that A precede A' for if it did not then *TE4A'* would not serve its proper function. In particular *TE4A'.6* and *7* would fail.

The rules (A) and (B) refer explicitly to the axiom 𝔄𝔯 of Meredith. If the system to be described contains different axioms or primitive terms certain canonical changes need to be made: If there are different axioms *TE1.2* must be change to $\mathbf{1vrb}(A)\,\varepsilon\,\mathbf{cnf}(X)$ where X is the name of an "N" in one of the axioms. *TE2.2* must be changed if no axiom begins with "C".

TE5' needs modification if no axiom ends "*Cpq*". If the calculus is implicational delete, *TE1*(*1'*) and *TE5.2* (*5'.2*). If there are other primitives, *TE5* and *TE5'* need additional clauses to guarantee that *a* is closed under these functors.

The Terminological Explanations given here are adoptions and simplifications of those given by Leśniewski [1]. Moreover, we have presented them using system **M** of [15], whereas he used ordinary language.

Throughout this appendix our informal comments have resorted, in the interest of perspicuousness, to the usual platonistic language and use-mention ambiguities. The Terminological Explanations themselves however are stated in an inscriptional syntax so as to satisfy the most demanding nominalist.

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