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CONCRETE COMPUTABILITY

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On the basis of several intuitively obvious properties of computability we show, among other things, that F is a computable partial function on the closure $\mathcal{S}^{\#}$ of an infinite set $\mathcal{S}$ under finite set formation iff there exists a countably infinite subset $\mathcal{U}$ of $\mathcal{S}$ and a finite subset $\mathcal{T}$ of $\mathcal{U}$ such that

(1) if g is any permutation on $\mathcal{S}$ leaving the members of $\mathcal{T}$ fixed, then $Fg^{\#} = g^{\#}F$ where $g^{\#}$ denotes the canonical extension of g to the members of $\mathcal{S}^{\#}$,
(2) if θ is any bijection from $\mathcal{U}$ onto the even numbers then $\overline{\theta}\, F\, \overline{\theta}^{-1}$ is computable on $\mathbf{N}$ where $\overline{\theta}$ is the unique extension of θ such that if $y_1, \ldots, y_n \in \mathcal{U}^{\#}$,

$$\theta(\{y_1, \ldots, y_n\}) = 2(2^{y_1} + \ldots + 2^{y_n}) + 1.$$

1 *Introduction* The *classical* theory of computability is concerned with *finitely* long processes on certain *finitary* or concrete combinations of objects from a *finite* generating set using a *finite* amount of *a priori* information. Church has suggested a certain mathematical (i.e., set theoretic) definition for this abstract concept. This suggestion is called *Church's Thesis*. Various others, e.g., Turing, Post, Markov, have made similar suggestions which have been shown equivalent to that of Church. Interesting generalizations have been obtained by extending any or several of these finitary aspects of computability to the infinite case. The theory of *concrete computability* is that generalization obtained by allowing the generating set to have arbitrary cardinality. This study can be motivated by considering such questions as "In what sense are the rules for first-order logic on uncountably many relation symbols effective?" Many authors have suggested definitions of computability that apply in such cases, e.g., Montague [2] and Moschovakis [3]. In this paper we characterize concrete computability in terms of classical computability, on the basis of several intuitively obvious properties of computability. These results together with Church's Thesis give an absolute characterization of concrete computability.

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2 *Terminology* In order to be precise it is necessary to have a mathematical characterization of the abstract notion of "finitary combination." To this end we suggest the following: x is a *finitary combination* of members of $\mathcal{S}$ iff x is in the closure $\mathcal{S}^\#$ of $\mathcal{S}$ with respect to the formation of finite sets, i.e., $x \epsilon \bigcap\{\mathcal{T}: \mathcal{S} \subset \mathcal{T} \text{ and } (x_1, \ldots, x_n \epsilon\, \mathcal{T} \Rightarrow \{x_1, \ldots, x_n\} \epsilon\, \mathcal{T})\}$. Our faith in this "thesis" is enhanced by the following lemma whose proof is obvious.

Lemma *Let R be a relation whose transitive closure $\overline{R}$ is left-finite on $\mathcal{T}$ (i.e., $\{x: x\bar{R}y\}$ is finite for all y in $\mathcal{T}$). Let f be a function from the R-minimal members of $\mathcal{T}$ into $\mathcal{S}$. Then f has a unique extension $f^\#: \mathcal{T} \to \mathcal{S}^\#$ such that $f^\#(y) = \{f^\#(x): xRy\}$ for all y in $\mathcal{T}$.*

To avoid certain technical difficulties we will restrict our attention to generating sets $\mathcal{S}$ which are *free* in the sense that the members of $\mathcal{S}$ are ϵ-minimal in $\mathcal{S}^\#$ (e.g., $\{\{x, \mathcal{S}\} \mid x \epsilon \mathcal{S}\}$ is always free). It then follows that every function $f: \mathcal{S} \to \mathcal{T}$ has a unique extension to an ϵ-homomorphism $f^\#: \mathcal{S}^\# \to \mathcal{T}^\#$ and that $f^\#$ is one to one (onto) iff f is one to one (onto).

We will identify N with the finite ordinals so that for any $\mathcal{S}$, $\mathsf{N} \subset \emptyset^\# \subset \mathcal{S}^\#$.

Definition For every set A, we let $|A|$ denote its cardinality.

Definition For any $x \epsilon \mathcal{S}^\#$, $x^\flat$ denotes the smallest subset of $\mathcal{S}$ such that $x \epsilon (x^\flat)^\#$.

Definition For any partial function F on $\mathcal{S}^\#$ we let F_a denote $\lambda x[F(a, x)]$, where $F(a, x)$ denotes the image of the ordered pair $(a, x) \epsilon \mathcal{S}^\#$ under F.

Definition A one to one partial function from a free set $\mathcal{S}$ to a free set $\mathcal{T}$ is called a *change of notation* from $\mathcal{S}$ into $\mathcal{T}$.

Definition If F is a partial function on $\mathcal{S}^\#$ and $\mathcal{U} \subset \mathcal{S}$, then $F_\mathcal{U}$ denotes the largest restriction of F such that $g^\# F_\mathcal{U} = F_\mathcal{U} g^\#$ for every extension g of $\mathrm{id}_\mathcal{U}$ that is a permutation on $\mathcal{S}$. If $F_\mathcal{U} = F$ we say $\mathcal{U}$ *determines* F.

Definition If f is a change of notation from $\mathcal{S}$ into $\mathcal{T}$, then $\Omega_f(F)$ denotes $\bigcup \{g^\# F_{\mathrm{Dom}\, f} g^{\#-1}: g$ is change of notation from $\mathcal{S}$ into $\mathcal{T}$ that extends $f\}$ which is a partial function on $\mathcal{T}^\#$ determined by $\mathrm{Rng}\, f$.

It is easy to show that if there is a coinfinite subset of $\mathcal{S}$ that determines F, then there is a smallest subset $\mathcal{U}$ of $\mathcal{S}$ that determines F, and that for all $x \epsilon \mathrm{Dom}\, F$, $F(x) \epsilon (\mathcal{U} \cup x^\flat)^\#$, for if g is a simple transposition of a member of $F(x)^\flat$ not in $x^\flat \cup \mathcal{U}$ with a member of $\mathcal{S}$ not in $F(x)^\flat \cup x^\flat \cup \mathcal{U}$, then $g^\# F g^{\#-1}(x) = g^\# F(x) \neq F(x)$.

Definition We say that a partial function F on $\mathcal{S}^\#$ is *notationally invariant* iff it is determined by the empty set. If F is notationally invariant then we let $F^\mathcal{T}$ denote $\Omega_f(F)$ where f is the nowhere defined partial function from $\mathcal{S}$ into $\mathcal{T}$.

Definition A one to one function θ from $\mathcal{S}^\#$ into N is said to be a *coding of* $\mathcal{S}$ iff

(1) $\theta[\mathcal{S}]$ is a recursive subset of $\mathbf{N}$,
(2) $\{\langle\theta(x), \theta(y)\rangle : x \in y\}$ is a recursive subset of $\mathbf{N} \times \mathbf{N}$,
(3) $\lambda x[\,|\theta^{-1}(x)|\,]$ is a recursive function on $\mathbf{N}$.

Notation Let e be an object not in $\emptyset^{\#}$, so that $\{e\}$ is free. For example we may let $e = \mathbf{N}$. Let $\mathcal{W}$ denote $\{\{n, e\}: n \in \mathbf{N}\}$ and Ψ denote $\lambda x[2n$ if $x = \{n, e\}$; $2(2^{\Psi(y_1)} + \ldots + 2^{\Psi(y_n)}) + 1$ if $x = \{y_1, \ldots, y_n\} \in \mathcal{W}^{\#} - \mathcal{W}]$.

Clearly $\mathcal{W}$ is a free subset of $\{e\}^{\#}$ and Ψ is a coding of $\mathcal{W}$. Also, if g is a total change of notation from $\mathcal{S}$ into $\mathcal{W}$ such that $\mathrm{Rng}\,\Psi g$ is recursive, then a one to one function $\Gamma: \mathcal{S}^{\#} \to \mathbf{N}$ is a coding of $\mathcal{S}$ iff $\Psi g^{\#}\Gamma^{-1}$ is a one to one partial recursive function having a recursive range and domain.

Definition $\langle s_1, \ldots, s_m\rangle$ denotes that finite change of notation from $\mathcal{W}$ into $\mathcal{S}$ that sends $\{i, e\}$ to s_i for $i = 1, \ldots, m$.

3 *Assumptions* We are now in a position to state our assumptions about computability. Intuitively, when we say that F is a computable partial function (c.p.f.) on $\mathcal{S}^{\#}$, we mean that F has an algorithm describing an effective procedure which a clerk can follow to compute $F(x)$ for any $x \in \mathcal{S}^{\#}$.

Assumption I *If f is a total change of notation from $\mathcal{T}$ onto $\mathcal{S}$ and F is a* c.p.f. *on $\mathcal{S}^{\#}$, then $f^{\#-1}Ff^{\#}$ is computable on $\mathcal{T}^{\#}$.*

Comment When we say that F is computable on $\mathcal{S}^{\#}$ we mean that it is computable using only the membership structure on $\mathcal{S}^{\#}$ and not using any other structure that $\mathcal{S}$ may happen to possess, so computability is preserved by ϵ-isomorphisms. We obtain an algorithm for $f^{\#-1}Ff^{\#}$ by taking an algorithm for F and replacing every mention of a member of $\mathcal{S}^{\#}$ by the corresponding mention of its image under $f^{\#-1}$.

Assumption II *Let F be a* c.p.f. *on $\mathcal{S}^{\#}$. Then there exists a countable set $\mathcal{U} \subset \mathcal{S}$ such that if $\mathcal{U} \subset \mathcal{V} \subset \mathcal{S}$ and $\mathcal{V}^{\#}$ is closed under F then the restriction of F to $\mathcal{V}^{\#}$ is computable on $\mathcal{V}^{\#}$.*

Comment Choose $\mathcal{U}$ so that $x^{\flat} \subset \mathcal{U}$ for every $x \in \mathcal{S}^{\#}$ mentioned in some fixed algorithm for F. This same algorithm then computes the restriction of F to $\mathcal{V}^{\#}$ if $\mathcal{U} \subset \mathcal{V}$ and $\mathcal{V}^{\#}$ is closed under F.

Assumption III *The composition of two* c.p.f.'s *on $\mathcal{S}^{\#}$ is a* c.p.f. *on $\mathcal{S}^{\#}$.*

Assumption IV *The* c.p.f.'s *on* $\mathbf{N}$ *are the closure of the partial recursive functions* (p.r.f.'s) *together with a finite set A of nonrecursive partial functions on* $\mathbf{N}$ *under composition and iteration.* (*The iterate F^{∞} of F is*

$$\bigcap\{G: G = \lambda x[x \text{ if } F(x) = x;\ GF(x) \textit{ otherwise}]\} \text{ i.e., } \lambda x[F^{\mu n[F^{n+1}(x)=F^n(x)]}(x)].)$$

Comment While it is fundamental to the notion of computability to assume that the p.r.f.'s are computable and that the c.p.f.'s are closed under iteration, there may be some question about the existence of a finite generating set. It is shown in [4] that this assumption is equivalent to assuming that the c.p.f.'s contain a universal function which we will henceforth denote by φ (i.e., F is a c.p.f. on $\mathbf{N}$ iff $F = \varphi_a$ for some $a \in \mathcal{S}^\#$). We justify this assumption by the observation that we can effectively number the algorithms for c.p.f.'s on $\mathbf{N}$ and that for all $n,\ m \in \mathbf{N}$, *applying to n the algorithm obtained by decoding m* is an effective procedure. Church's Thesis then is simply the statement that A is empty.

Definition γ denotes $\lambda((n, w_1, \ldots, w_m), x)[\Omega_{\langle w_1,\ldots,w_m\rangle}(\Psi\varphi_n\Psi^{-1})(x)$ if $w_1, \ldots, w_m \in \mathcal{W}]$ which is a notationally invariant partial function on $\mathcal{W}^\#$.

Notice that for every $\mathcal{S}$, $\gamma^{\mathcal{S}}$ is notationally invariant and that $\gamma^{\mathcal{S}}{}_{(n, s_1, \ldots, s_m)} = \Omega_{\langle s_1,\ldots,s_m\rangle}(\Psi\varphi_n\Psi^{-1})$ which is a partial function on $\mathcal{S}^\#$ determined by $\{s_1, \ldots, s_m\}$ and hence, if defined, $\gamma^{\mathcal{S}}((n, s_1, \ldots, s_m), x) \in (x^\flat \cup \{s_1, \ldots, s_m\})^\#$. It follows that $\gamma^{\mathcal{S}}{}_{(n, s_1, \ldots, s_m)}(x) = y$ iff $y = \theta^{-1}\varphi_n\theta(x)$ for all and at least one bijection θ from the transitive closure of $\{x, y\}$ into $\mathbf{N}$ such that

(1) $\theta(s_i) = 2i$ if $s_i \in \operatorname{Dom}\theta$,
(2) $\theta(\{y_1, \ldots, y_k\}) = 2(2^{y_1} + \ldots + 2^{y_k}) + 1$ if $\{y_1, \ldots, y_k\} \in \operatorname{Dom}\theta$,
(3) $\theta[x^\flat - \{s_1, \ldots, s_m\}] = \{2(m+1), \ldots, 2(m+j)\}$ where $j = |x^\flat - \{s_1, \ldots, s_m\}|$.

It should be noted that there are at most $j!$ such θ's. This suggests the following assumption.

Assumption V *F is computable on $\mathcal{S}^\#$ if F has a $\gamma^{\mathcal{S}}$-index in the sense that $F = \gamma^{\mathcal{S}}{}_{(n, s_1, \ldots, s_m)}$ for some $n \in \mathbf{N}$ and $s_1, \ldots, s_m \in \mathcal{S}$.*

Comment If $n \in \mathbf{N}$ and $s_1, \ldots, s_m \in \mathcal{S}$ and $\gamma^{\mathcal{S}}((n, s_1, \ldots, s_m), x)$ has a value y, then one can determine y by computing $f^\#\ \Psi^{-1}\ \varphi_n\ \Psi f^{\#^{-1}}(x)$ for all changes of notation f from $\mathcal{S}$ into $\mathcal{W}$ such that

(a) $\operatorname{Rng} f = \{s_1, \ldots, s_m\} \cup x^\flat$,
(b) $\operatorname{Dom} f$ is an initial segment of $\mathcal{W}$ under the ordering: $\{i, e\} \le \{j, e\}$ iff $i \le j$,
(c) $f(\{i, e\}) = s_i$ for $i = 1, \ldots, m$.

Notice that there are at most $|x^\flat|!$ such f's. Let f satisfy (a), (b), and (c). Let θ denote $\Psi f^{\#^{-1}}$. To compute $\theta^{-1}\varphi_n\theta$, one begins by computing θ on the members of $(\{s_1, \ldots, s_m\} \cup x^\flat)^\#$ in order of rank until $\theta(x)$ is computed. One then computes $\varphi_n\theta(x)$ and then continues computing the values of θ until $\varphi_n\theta(x)$ occurs in the range of θ. This will eventually happen provided $x \in \operatorname{Dom}\gamma_{(n, s_1, \ldots, s_m)}$ since $\gamma_{(n, s_1, \ldots, s_m)}$ is determined by $\{s_1, \ldots, s_m\}$. This procedure fails to terminate if $x \notin \operatorname{Dom}\gamma_{(n, s_1, \ldots, s_m)}$. If the outcome of this procedure is y for every possible f, then $\gamma^{\mathcal{S}}{}_{(n, s_1, \ldots, s_m)}(x) = y$. It is not hard to show that $\gamma^{\mathcal{S}}$ has a $\gamma^{\mathcal{S}}$-index.

Assumption VI *If a partial function on* $\mathbf{N}$ *is computable on* $\{e\}^\#$ *then it is computable on* $\mathbf{N}$.

Comment Embedding $\mathbf{N}$ into $\{e\}^\#$, clearly, gives no new information or structure from which to compute additional partial functions on $\mathbf{N}$.

Assumption VII *If F is a* c.p.f. *on $\mathcal{W}^\#$ then F is computable on $\{e\}^\#$.*

Comment To compute $F(x)$ for any $x \in \{e\}^{\#}$, first apply to x the characteristic function for $\mathcal{W}^{\#}$ which is computable on $\{e\}^{\#}$ since it clearly has a γ-index. If one determines that $x \in \mathcal{W}^{\#}$, then one applies to x an algorithm for computing F on $\mathcal{W}^{\#}$.

4 *Computable Partial Functions*

Theorem *Let F be a partial function on $\mathcal{S}^{\#}$. The following are equivalent:*

(1) *F is computable on $\mathcal{S}^{\#}$.*
(2) *F is locally computable in the sense that if Γ is a coding of a subset $\mathcal{V}$ of $\mathcal{S}$ such that $\mathcal{V}^{\#}$ is closed under F, then $\Gamma F \Gamma^{-1}$ is a* c.p.f. *on* $\mathbf{N}$.
(3) *F is finitely determined and $\mathcal{S}$ has a subset $\mathcal{U}$ with cardinality* $\min(\aleph_0, |\mathcal{S}|)$ *that determines F and that has a coding Γ such that $\Gamma F \Gamma^{-1}$ is a* c.p.f. *on* $\mathbf{N}$.
(4) *F has a $\gamma^{\mathcal{S}}$-index in the sense that $F = \gamma_a^{\mathcal{S}}$ for some $a \in \mathcal{S}^{\#}$.*

Proof: $(4 \Rightarrow 1)$ by Assumption V.

$(1 \Rightarrow 2)$. Notice that Ψ and Ψ^{-1} are computable on $\{e\}^{\#}$ since they obviously have γ-indices. Now suppose that F is computable on $\mathcal{S}^{\#}$. Let $\mathcal{U}$ be as in Assumption II, and let $\mathcal{V}$ be a subset of $\mathcal{S}$ such that $\mathcal{V}^{\#}$ is closed under F. We will suppose that $\mathcal{V}$ is infinite for otherwise (2) visibly holds. Let g be a total change of notation from $\mathcal{V}$ onto a coinfinite subset of $\mathcal{W}$ whose image under Ψ is recursive. Let $\mathcal{T}$ be the smallest subset of $\mathcal{S}$ such that $\mathcal{U} \cup \mathcal{V} \subset \mathcal{T}$ and $\mathcal{T}^{\#}$ is closed under F. Let h be a total one to one extension of g from $\mathcal{T}$ onto $\mathcal{W}$. By II, the restriction of F to $\mathcal{T}^{\#}$ is computable on $\mathcal{T}^{\#}$. So, by I, $h^{\#}Fh^{\#-1}$ is computable on $\mathcal{W}^{\#}$. So, by VII, it is computable on $\{e\}^{\#}$. Hence, by III, $\Psi h^{\#}Fh^{\#-1}\Psi^{-1}$ which is a partial function on $\mathbf{N}$ is a c.p.f. on $\{e\}^{\#}$. So, by VI, it is computable on $\mathbf{N}$. But then its restriction to $\mathrm{Rng}\, \Psi g^{\#}$, $\Psi g^{\#}Fg^{\#-1}\Psi^{-1}$, is computable since $\mathrm{Rng}\, \Psi g^{\#}$ is recursive. This finishes the proof since every coding Γ of $\mathcal{V}$ is of the form $\delta \Psi g^{\#}$ where δ is a one to one partial recursive function.

$(2 \Rightarrow 3)$. Suppose that (2) holds and F is not finitely determined. Then clearly $\mathcal{S}$ is infinite.

Given a partial function ξ on $\mathbf{N}$, each finite change of notation f from $\mathcal{S}$ into $\mathcal{W}$ has an extension f' such that $\Psi f'^{\#} F \neq \xi \Psi f'^{\#}$ for otherwise $F = \Omega_{f^{-1}}(\Psi^{-1} \xi \Psi)$. Obviously we can take f' to be finite.

We construct a chain $\emptyset = f_0 \subset f_1 \subset \ldots \subset \bigcup_{i=1}^{\infty} f_i = f$ of finite changes of notation from $\mathcal{S}$ into $\mathcal{W}$ and an ω^2-sequence $a_1, a_2, \ldots, a_\omega, \ldots, a_{2\omega}, \ldots$ of members of $\mathcal{S}^{\#}$ by taking f_k to be any one to one finite extension of f_{k-1} such that

(1) $\{k, e\} \in \mathrm{Rng}\, f_k$,
(2) $\Psi f_k^{\#} F \neq \varphi_k \Psi f_k^{\#}$,
(3) $F(a_{n\omega+m}) \in (\mathrm{Dom}\, f_k)^{\#}$, where k is the image of (n, m) under some fixed pairing function,

and $a_{k\omega}, a_{k\omega+1}, \ldots$, to be such that

$$F[(\mathrm{Dom}\, f_k)^{\#}] - (\mathrm{Dom}\, f_k)^{\#} \subsetneq \{a_{k\omega}, a_{k\omega+1}, \ldots\}.$$

Then $(\mathrm{Dom}\, f)^{\#}$ is closed under F and $\Psi f^{\#}$ is a coding of $\mathrm{Dom}\, f$. But for all $n \in \mathbf{N}$, $\Psi f^{\#} F f^{\#-1} \Psi^{-1} \neq \varphi_n$; and, hence, it is not a c.p.f. on $\mathbf{N}$, contrary to our assumption that (2) holds. Thus, F is finitely determined. The rest of (3) is an obvious consequence of (2).

$(3 \Rightarrow 4)$ Suppose that (3) holds and that $\{s_1, \ldots, s_m\}$ determines F. Let f be a one to one extension of $\langle s_1, \ldots, s_m \rangle$ from an initial segment of $\mathcal{W}$ onto $\mathcal{U}$. Then $\Psi f^{\#-1}$ is a coding of $\mathcal{U}$ and hence $f^{\#} \Psi^{-1} F \Psi f^{\#-1}$ is a computable partial function on $\mathbf{N}$, say φ_n. Thus $F = \Omega_f(\Psi \varphi_n \Psi^{-1}) = \overset{\mathcal{S}}{\gamma}_{(n, s_1, \ldots, s_m)}$. Q.E.D.

Notice that while every c.p.f. on $\mathcal{W}^{\#}$ is a c.p.f. on $\{e\}^{\#}$, the converse is not true since, for example, $\lambda\{n, e\}[\{n+1, e\}]$ is computable on $\{e\}^{\#}$ but not on $\mathcal{W}^{\#}$ since on $\mathcal{W}^{\#}$ it is not finitely determined.

5 *Computable Sets*

Definition We let W_a denote $\mathrm{Dom}\, \gamma_a$. A set $S \subset \mathcal{S}^{\#}$ is *computable* on $\mathcal{S}^{\#}$ iff it has a *W-index*, i.e., it is the domain of a c.p.f. on $\mathcal{S}^{\#}$. S is bicomputable on $\mathcal{S}^{\#}$ iff both S and $\mathcal{S}^{\#} - S$ are computable on $\mathcal{S}^{\#}$.

To get a reasonable theory of computable sets we make

Assumption VIII *There exist a complexity measure h on* $\mathrm{Dom}\, \varphi$, i.e., h: $\mathrm{Dom}\, \varphi \to \mathbf{N}$ *and the graph $\mathcal{G}_h$ of h is bicomputable.*

Comment We justify this assumption by letting $h(i, x)$ denote the number of steps taken to compute $\varphi_i(x)$ under some fixed algorithm for φ. It should be noted that this assumption is equivalent to assuming that the c.p.f.'s on $\mathbf{N}$ are the partial functions that are Turing reducible to some total function on $\mathbf{N}$ (i.e., the members of A are total); for in such a case, by Kleene's normal-form theorem, a complexity measure for $\mathrm{Dom}\, \varphi$ exists, while if such a measure exists then we may let $A = \{g\}$ where $g = \lambda\, n, x, m\, [\varphi_n(x)$ if $h(n, x) = m$; else $0]$ so that $\varphi_n(x) \cong g(n, x, h(n, x))$.

Notice that $\lambda((n, s_1, \ldots, s_m), x)\, [h(n, x)$ if $\gamma((n, s_1, \ldots, s_m), x)$ is defined$]$ is a notationally invariant measure for $\mathrm{Dom}\, \gamma$.

Lemma *For every partial function F on $\mathcal{S}^{\#}$, F is computable on $\mathcal{S}^{\#}$ iff its graph $\mathcal{G}_F$ is computable on $\mathcal{S}^{\#}$.*

Proof: Suppose that $\mathcal{G}_F$ is computable on $\mathcal{S}^{\#}$. Then $\mathcal{G}_F$ is locally computable. It follows by the classical theory that F is locally computable. Hence F is on $\mathcal{S}^{\#}$. Conversely, if F is computable on $\mathcal{S}^{\#}$ then $\lambda(x, y)\, [(x, y)$ if $F(x) = y]$ is computable on $\mathcal{S}^{\#}$ and hence its domain $\mathcal{G}_F$ has a W-index. Q.E.D.

Henceforth we identify partial functions with their graphs.

Definition Let $\mathfrak{L}$ be the first order language having two binary predicates "$\in$" and "$=$" and having the members of $\mathcal{S} \cup A$ as constants. We consider $\mathcal{S}^{\#}$ to be an $\mathfrak{L}$-structure in the obvious way. A set $S \subset \mathcal{S}^{\#}$ is Σ-*definable in A* iff S is of the form $\rho[R]$ where ρ denotes $\lambda(x, y)\, [x]$ and R is Δ_0-*definable in A* in the sense that $R = \{(x, y) \in \mathcal{S}^{\#}: \mathcal{S}^{\#} \models \mathcal{A}(x, y)\}$ for some formula $\mathcal{A}$ in the closure Δ_0 of the atomic formulas under negation, conjunction, and bounded quantification of the forms "$\exists v_i \in y_j \ldots$"

Theorem *The following are equivalent:*

(1) *S is computable.*
(2) *S is Σ-definable in A.*
(3) *S is the range of a* c.p.f. *having a bicomputable domain.*
(4) *S is the range of a* c.p.f.

Proof: ($1 \Rightarrow 2$) The partial functions Σ-definable in A include φ and hence γ and hence γ_a since the p.r.f.'s and the members of A are Σ-definable in A and the partial functions Σ-definable in A are closed under composition and iteration. (Of course the domain of any Σ-definable partial function is Σ-definable since if R is Σ-definable in A so is $\rho[R]$.)

($2 \Rightarrow 3$) $S = \mathrm{Rng}\ \rho | R$ and R is bicomputable if $S = \rho[R]$ and R is Δ_0-definable in A.

($3 \Rightarrow 4$) Obviously.

($4 \Rightarrow 1$) Suppose that S is the range of a c.p.f. F and that Γ is a coding of a set $\mathcal{U}$ of cardinality $\min(\aleph_0, |\mathcal{S}|)$ that determines F. Then $\Gamma[S] = \mathrm{Rng}\ \Gamma F = \mathrm{Rng}\ \Gamma F \Gamma^{-1}$. So $\Gamma[S]$ is a computable subset of $\mathbf{N}$, so $\Gamma\ \mathrm{id}_S\ \Gamma^{-1}$ is computable. But clearly $\mathcal{U}$ determines id_S and hence by our main theorem id_S is computable on $\mathcal{S}^{\#}$. Thus $S = \mathrm{Dom}\ \mathrm{id}_S$ is computable on $\mathcal{S}^{\#}$. Q.E.D.

6 *Conclusion* From the results of Gordon [1] and our last theorem we see that, when the definitions of computability proposed by Moschovakis [3] and Montague [2] are relativized to A, they correspond to the intuitive notion of computability. Thus Church's Thesis ("$A = \emptyset$") is equivalent to the statement that their absolute form corresponds to intuitive computability.

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