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FORMULAS WITH TWO GENERALIZED QUANTIFIERS

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In this paper we give a partial solution to the two problems Yasuhara presents at the end of [2]. Yasuhara shows that in formal languages having finitary predicate and function symbols and in which "$\wedge$", "$\sim$", and "$\vee$" have their usual meanings and "$(\forall x)$" is equivalent to "$\sim(\exists x)\sim$" and, for some k, "$(\exists x)$" means "there exist at least ω_k elements x such that," the set of closed formulas which are true in all models of cardinality $\geqslant \omega_k$ is the same for each $k \geqslant 0$ and each corresponding interpretation of "$(\exists x)$". He calls this set of formulas **VI**. The set of closed formulas not in **VI** is called **SI**.

For each finite number n, "$(\exists x)$" can be interpreted to mean "there exist at least n elements x such that," and then the set of closed formulas true in all models having at least n elements is called $\mathbf{V}_n$. The set of closed formulas not in $\mathbf{V}_n$ is called $\mathbf{S}_n$. The intersection of all the sets $\mathbf{V}_n$ is called **VF**. If **V** is a set of formulas, then by **V**,2 we mean the set of formulas in **V** having only 2 quantifiers.

Our results are the following:

Theorem 1 $\mathbf{VF},2 \subsetneqq \mathbf{VI},2 \subsetneqq \mathbf{V}_1,2$.
Theorem 2 **VF**,2 *and* **VI**,2 *and* $\mathbf{V}_1$,2 *are recursive.*

Proof of Theorem 1: We first prove $\mathbf{VF},2 \subset \mathbf{VI},2$.

Case 1. If $(\exists x)(\forall y)P(x,y)$ is in **VF**,2, then it is in $\mathbf{V}_1$, by definition. So $(\forall x)(\exists y)\sim P(x,y)$ is not in $\mathbf{S}_1$ and therefore $\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_n,a_1)$ is, for all n, a quantifier-free formula which is not true under any valuation of its atomic formulas, because otherwise $\{a_1, a_2, \ldots, a_n\}$ would be the universe of a model for $(\forall x)(\exists y)\sim P(x,y)$. But this means that if "$(\exists x)$" is given the interpretation "there exist at least ω_0 elements x such that," then $(\forall x)(\exists y)\sim P(x,y)$ is unsatisfiable. Because if $\mathfrak{M}$ were a model for it, then there would be an element a_1 in $\mathfrak{M}$ such that there were infinitely many elements a_2 in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_1,a_2)$. But all but a finite number of these elements a_2 would have infinitely many elements a_3 in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_2,a_3)$. Thus we can find elements a_1, a_2, and a_3 in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_1,a_2) \wedge \sim P(a_2,a_3)$.

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Similarly, for any n, we can find elements $a_1, a_2, a_3, \ldots, a_n$ in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_{n-1},a_n)$. But if n is large enough, there would have to be $j, k < n$ such that $j + 2 < k$ and $A(a_j) \leftrightarrow A(a_k)$ for all atomic formulas $A(x)$ in $P(x,y)$ which have only one free variable. Then, since there are $a_j, \ldots, a_k$ in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_j,a_{j+1}) \wedge \ldots \wedge P(a_{k-1},a_k)$, therefore the formula $\sim P(a_j,a_{j+1}) \wedge \ldots \wedge \sim P(a_{k-1},a_j)$ would have a model, but this is impossible. So $(\forall x)(\exists y) \sim P(x,y)$ is unsatisfiable with the "ω_0-interpretation" of the quantifiers and thus $(\exists x)(\forall y)\, P(x,y)$ is in **VI**,2.

Case 2 If $(\forall x)(\exists y)\, P(x,y)$ is in **VF**,2 then $(\exists x)(\forall y) \sim P(x,y)$ is unsatisfiable for every finite interpretation of the quantifier. Therefore, for all n, $\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_n,a_1)$ is false with every valuation of the atomic formulas in it, because otherwise, with "$(\exists x)$" interpreted as "there exist at least n elements x such that", $\{a_1, a_2, \ldots, a_n\}$ would be the universe of a model for $(\exists x)(\forall y) \sim P(x,y)$. Now if $(\exists x)(\forall y) \sim P(x,y)$ were satisfiable in a model $\mathfrak{M}$ with the "ω_0-interpretation" of the quantifier, then there would be an element a_1 in $\mathfrak{M}$ such that $\sim P(a_1,a_2)$ was satisfied in $\mathfrak{M}$ for all but a finite number of elements a_2 in $\mathfrak{M}$. Thus we could pick one of these elements a_2 which had the property that $\sim P(a_2,a_3)$ was satisfied in $\mathfrak{M}$ for all but a finite number of elements a_3 in $\mathfrak{M}$. In this way, for any n, we could find elements $a_1, a_2, \ldots, a_n$ such that $\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_{n-1},a_n)$ was true in $\mathfrak{M}$. And this is impossible, so $(\exists x)(\forall y) \sim P(x,y)$ is unsatisfiable with the "ω_0-interpretation" of the quantifiers and thus $(\forall x)(\exists y)\, P(x,y)$ is in **VI**,2.

Case 3 If $(\exists x)(\exists y)\, P(x,y)$ is in **VF**,2, then $P(a,b)$ must be a tautology, because if some valuation makes $\sim P(a,b)$ true, then $\{a,b\}$ could be the universe for a model of $(\forall x)(\forall y) \sim P(x,y)$ with the interpretation "there exist at least 2 elements x such that" for "$(\exists x)$". Therefore $(\exists x)(\exists y)\, P(x,y)$ is in **VI**.

Case 4 If $(\forall x)(\forall y)\, P(x,y)$ is in **VF**,2, then $P(a,b)$ must be a tautology so $(\forall x)(\forall y)\, P(x,y)$ is in **VI**.

Thus we have shown **VF**,2 $\subset$ **VI**,2. To see that **VF**,2 $\neq$ **VI**,2, consider the sentence $(\exists x)(\forall y)(P(x) \leftrightarrow \sim P(y))$. If "$(\exists x)$" is given the interpretation "there exist at least two x such that" and the model has $\{1,2\}$ as its universe and $P(1)$ and $\sim P(2)$ are satisfied, we see that the sentence is in $\mathbf{S}_2$ (and thus its negation is not in **VF**,2). But it is not in **SI** (and thus its negation is in **VI**,2), because any model satisfying $(\forall y)(P(a) \leftrightarrow \sim P(y))$ for the "ω_0-interpretation" of the quantifiers would have either all but a finite number of elements in P or all but a finite number of elements outside P and in neither case could $(\exists x)(\forall y)(P(x) \leftrightarrow \sim P(y))$ be true.

Yasuhara proved **VI** $\subset$ $\mathbf{V}_1$ in [1], so it only remains for us to prove **VI**,2 $\neq$ $\mathbf{V}_1$,2 to finish the proof of Theorem 1. We give the example

$$(\exists x)(\exists y)((P(x,y) \wedge P(y,x)) \vee (\sim P(x,y) \wedge \sim P(y,x))).$$

It is obviously in $\mathbf{V}_1$,2. But if we consider any model $\mathfrak{M}$ in which

$P(x,y) \leftrightarrow \sim P(y,x)$ is always true for $y \neq x$, we see that $\mathfrak{M}$ does not satisfy the formula with the "ω_0-interpretation" of the quantifiers, so that the formula is not in **VI**,2. Q.E.D.

Proof of Theorem 2: The set of formulas of the forms $(\forall x)(\forall y)\,P(x,y)$ and $(\exists x)(\exists y)\,P(x,y)$ in **VF**,2 is clearly recursive, since $P(x,y)$ must be a tautology.

If $(\forall x)(\exists y)\,P(x,y)$ is in **VF**,2, then there is no sequence $a_1, \ldots, a_n$ such that $\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_n,a_1)$ has a valuation of its atomic formulas which makes it true, because otherwise $\{a_1, \ldots, a_n\}$ would be the universe for a model of $(\exists x)(\forall y){\sim}P(x,y)$ with the interpretation "there exist at least n x's such that" for "$(\exists x)$" so that $(\forall x)(\exists y)\,P(x,y)$ would not be in $\mathbf{V}_n$. But by an argument in the proof of Theorem 1, there is such a sequence if $(\forall x)(\exists y)\,P(x,y)$ is not in **VF**,2. So there is a decision procedure for testing formulas of the form $(\forall x)(\exists y)\,P(x,y)$ for membership in **VF**,2, because for any $P(x,y)$ there is a mechanical way of choosing N such that if there is no such sequence such that $n \leqslant N$, then there is no such sequence at all. The same decision procedure for membership in **VF**,2 applies to formulas of the form $(\exists x)(\forall y)\,P(x,y)$. So **VF**,2 is recursive.

The proof for **VI**,2 is more difficult. We claim that any formula of the form $(\exists x)(\forall y)\,P(x,y)$ is in **SI** if and only if there is a valuation for each atomic formula in $P(a,b)$ such that $A(a) \leftrightarrow A(b)$ for each atomic $A(a)$ and $A(b)$ in $P(a,b)$, and $P(a,b)$ is true. If there is such a valuation, then we can take a set of symbols $\{s_1, s_2, s_3, \ldots\}$ which is closed under each function symbol F in $P(a,b)$ as the universe of a model and, for each predicate A in $P(a,b)$, give each atomic formula $A(s_j, s_k)$ such that $A(a,b)$ is in $P(a,b)$ and $k > j$, or $A(s_j)$, where $A(a)$ or $A(b)$ is in $P(a,b)$, the same truth value given to $A(a,b)$ or $A(a)$ or $A(b)$, respectively, in $P(a,b)$. This is a consistent valuation and therefore has a model $\mathfrak{M}$ which has $\{s_1, s_2, s_3, \ldots\}$ as its universe and thus satisfies $(\exists x)(\forall y)\,P(x,y)$ since $P(s_j,s_k) \leftrightarrow P(a,b)$ for $k > j$.

Conversely, if $(\exists x)(\forall y)\,P(x,y)$ is in **SI**, then the condition is satisfied, because there has to be an infinite set of true formulas $(\forall y)\,P(a,y)$ for elements a in $\mathfrak{M}$ and there must be some element a such that the valuation of all atomic formulas of the form $A(a)$ matches the valuations of the corresponding atomic formulas for an infinite set of other elements and for any element a with this valuation there has to be an infinite set of elements b which have the same valuation and such that $P(a,b)$ is true. So there is a decision procedure for deciding whether any formula of the form $(\forall x)(\exists y)\,P(x,y)$ is in **VI**,2.

We note that $(\forall x)(\forall y)\,P(x,y)$ has the same condition for membership in **SI** as $(\exists x)(\forall y)\,P(x,y)$ had in the above, so it is decidable whether any formula of the form $(\exists x)(\exists y)\,P(x,y)$ is in **VI**,2.

We now show that $(\exists x)(\forall y)\,P(x,y)$ and $(\forall x)(\forall y)\,P(x,y)$ are in **VI**,2 precisely if they are in **VF**,2, and the proof of the part of Theorem 2 concerning **VI**,2 will then be complete.

If a formula of the form $(\exists x)(\forall y)\,P(x,y)$ is not in **VF**,2, then there is some model for $(\forall x)(\exists y)\sim P(x,y)$ with some finite interpretation of "$(\exists x)$" so that by an argument in Theorem 1 there is some sequence $a_1, a_2, \ldots, a_n$ such that $\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_{n-1},a_n) \wedge \sim P(a_n,a_1)$ is satisfiable, so that $\{a_1, a_2, \ldots, a_n\}$ is the universe of a model for $(\forall x)(\exists y)\sim P(x,y)$ with the "1-interpretation" of "$(\exists x)$", so $(\exists x)(\forall y)\,P(x,y)$ is not in $\mathbf{V}_1,2$ and therefore not in $\mathbf{VI},2$ either. So

$$(\exists x)(\forall y)\,P(x,y) \in \mathbf{VF},2 \leftrightarrow$$
$$(\exists x)(\forall y)\,P(x,y) \in \mathbf{VI},2 \leftrightarrow (\exists x)(\forall y)\,P(x,y) \in \mathbf{V}_1,2.$$

Similarly

$$(\forall x)(\forall y)\,P(x,y) \in \mathbf{VF},2 \leftrightarrow$$
$$(\forall x)(\forall y)\,P(x,y) \in \mathbf{VI},2 \leftrightarrow (\forall x)(\forall y)\,P(x,y) \in \mathbf{V}_1,2,$$

since the condition for membership in $\mathbf{VF},2$ and $\mathbf{V}_1,2$ is that $P(a,b)$ must be a tautology.

For $\mathbf{V}_1,2$ we note that $(\exists x)(\exists y)\,P(x,y)$ is in it precisely if $P(a,a)$ is a tautology, $(\forall x)(\exists y)\,P(x,y)$ is in it precisely if $P(a,a)$ is a tautology, $(\forall x)(\forall y)\,P(x,y)$ is in it precisely if $P(a,b)$ is a tautology and $(\exists x)(\forall y)\,P(x,y)$ is in it precisely if

$$\sim(\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_{n-1},a_n) \wedge \sim P(a_n,a_1))$$

is a tautology for all n, which can be seen by noting that $(\forall x)(\exists y)\sim P(x,y)$ is in $\mathbf{S}_1$ precisely if $(\exists x)(\forall y)\,P(x,y)$ is not in $\mathbf{V}_1,2$, and by recalling previous arguments from this paper. Q.E.D.

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