Notre Dame Journal of Formal Logic
Volume XVI, Number 1, January 1975
NDJFAM

LOGIC OF ANTINOMIES

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Purpose There are essentially three ways of looking at antinomies. The first is to consider them as undesirable anomalies. This is the predominant view, and was Russell's when he discovered the famous antinomy that bears his name. Because of the devastating consequences that result from the presence of a single antinomy in any system based on classical logic, this view has been understandably strong. It is well known, for example, that Frege reacted with extreme and lasting consternation to Russell's discovery. The second view considers antinomies less dramatically, taking them as merely harmless abnormalities. Remarkably enough, this was Cantor's position ([4], pp. 384-5), as well as Wittgenstein's: "If a contradiction were now actually found in arithmetic—that would only prove that an arithmetic with *such* a contradiction in it could render a very good service" ([5], p. 181e). From this second point of view the problem with antinomies is how to confine them, how to prevent their turning every well-formed formula into a theorem without eradicating them and without abandoning or radically altering the system in question. The third and last view is to see antinomies as useful logical entities.[1] According to this position antinomies must be integrated into logical systems starting with the propositional calculus, bearing in mind that while some sentences have only one truth value, others have two. To use an example from ordinary language, a sentence such as "It is raining here now" can only be true or false, but not both. In contrast, "Peter is a good man" is not a single-valued sentence and no attempt should be made to make it one. There are a number of more or less artificial interpretations of propositional semantics whose chief objective is to suppress the antinomies' first obvious meaning, be the antinomies taken from ordinary language or from mathematics. For example, it is against Cantor's original conception of

1. There are several philosophic precedents to this position. Kant, for example, attached to antinomies the positive function of preventing reason from slumbering —apart from their playing a very important demonstrative role in his own philosophy.

Received December 18, 1972

set theory to interpret the multiplicity of all the sets which are not members of themselves as a meaningless notion or as a class that is not a member of any other class. This paper adopts the third viewpoint, at the same time accommodating the second. Our aim is to construct an antinomic logic which is not trivially inconsistent and is complete in the sense that the theorems of the propositional and predicate calculi are, precisely, those formulas which are true or antinomic in each of these calculi. As an application of such calculi, an antinomic set theory is presented, one that in various ways is closer to Cantor's original naive idea of set. We found that in addition to the intrinsic merit of building a set theory with a positive attitude toward antinomies, working with antinomic sets provides a new insight into classical axiomatic set theory.[2]

I. AN ANTINOMIC PROPOSITIONAL CALCULUS

1 Antinomies Semantically Considered There will be two kinds of statement forms, those having only one truth value (true or false) and those called antinomies having two truth values (both true and false). Labelling truth, falsity, and antinomicity with the symbols 0, 1, and 2 respectively, the propositional connectives can be defined by the following truth tables.

$\mathscr{B}_1 \supset \mathscr{B}_2$

$\mathscr{B}_1$ \ $\mathscr{B}_2$	0	1	2
0	0	1	2
1	0	0	0
2	0	1	2

$\mathscr{B}_1 \,\&\, \mathscr{B}_2$

$\mathscr{B}_1$ \ $\mathscr{B}_2$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

$\mathscr{B}_1 \vee \mathscr{B}_2$

$\mathscr{B}_1$ \ $\mathscr{B}_2$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\mathscr{B}_1 \equiv \mathscr{B}_2$

$\mathscr{B}_1$ \ $\mathscr{B}_2$	0	1	2
0	0	1	2
1	1	0	1
2	2	1	2

$\mathscr{B}_1 \equiv^{\circ} \mathscr{B}_2$

$\mathscr{B}_1$ \ $\mathscr{B}_2$	0	1	2
0	0	1	1
1	1	0	1
2	1	1	2

$\neg \mathscr{B}_1$

$\mathscr{B}_1$	$\neg \mathscr{B}_1$
0	1
1	0
2	2

The tables for conjunction, disjunction, and negation are the same as those given in [2], p. 103, where motivation for the construction of these tables is provided. The table for implication differs from that given in [2] for the case in which $\mathscr{B}_1$ has value 2 and $\mathscr{B}_2$ has value 1, a difference that is justified by syntactic reasons. With these tables it is easy to verify that some compound statement forms are antinomic for all possible assignments of truth values to the atomic statement letters, while others are not.

2. The authors wish to thank Professor D. Randolph Johnson for his thoughtful reading of this paper and for his valuable suggestions. Also, it should be noted that portions of a previous version of this paper were used in the second author's doctoral dissertation.

In fact, many classical tautologies still have the value 0 even where value 2 is involved (see examples at the end of this section). The statement form $\mathscr{B}_1 \equiv^\circ \mathscr{B}_2$ can alternatively be defined as an abbreviation of $(\mathscr{B}_1 \equiv \mathscr{B}_2)$ & $(\neg \mathscr{B}_1 \equiv \neg \mathscr{B}_2)$.

Two alphabets will be used for atomic statement letters. The capital Roman $A_1, A_2, \ldots$ are statement letters that take only one of the truth values 0 or 1, while the capital Roman $B_1, B_2, \ldots$ are statement letters that take only the truth value 2. Script $\mathscr{A}_1, \mathscr{A}_2, \ldots$ denote atomic or compound statement forms that take only the truth values 0 or 1, and script $\mathscr{B}_1, \mathscr{B}_2, \ldots$ denote any statement forms, either atomic or compound, whatever their truth values. The formation rules for statement forms follow (we shall let $\mathscr{B}_1 \equiv \mathscr{B}_2$ stand for $(\mathscr{B}_1 \supset \mathscr{B}_2)$ & $(\mathscr{B}_2 \supset \mathscr{B}_1)$).

1. All capital Roman statement letters are statement forms.
2. If $\mathscr{B}_1$ and $\mathscr{B}_2$ are statement forms, then $\mathscr{B}_1 \supset \mathscr{B}_2$, $\mathscr{B}_1$ & $\mathscr{B}_2$, $\mathscr{B}_1 \vee \mathscr{B}_2$, and $\neg \mathscr{B}_1$ are also statement forms (notice that since the $\mathscr{A}$'s are special cases of the $\mathscr{B}$'s, expressions formed using rules 1 and 2 and involving only $\mathscr{A}$'s or combinations of $\mathscr{A}$'s and $\mathscr{B}$'s are also statement forms).
3. Expressions formed according to rules 1 and 2 are the only statement forms (also called formulas, or well-formed formulas-wfs).

Statement forms taking only 0 or 2 for any arbitrary assignment of truth values to their statement letters will be called tautologies. The next metatheorems give some properties of tautologies.

Proposition 1.1. *If $\mathscr{B}_1$ and $\mathscr{B}_1 \supset \mathscr{B}_2$ are tautologies, then $\mathscr{B}_2$ is a tautology.*

Proof: Suppose $\mathscr{B}_2$ takes value 1, since $\mathscr{B}_1$ is a tautology $\mathscr{B}_1 \supset \mathscr{B}_2$ cannot be a tautology, contrary to the hypothesis.

Let us now distinguish between tautologies of type I—those having only 0's as values—and tautologies of type II—those having 0's and 2's as values.

Proposition 1.2. *If $\mathscr{A}_1$ and $\mathscr{A}_1 \supset \mathscr{A}_2$ are both tautologies of type* I, *then $\mathscr{A}_2$ is also a tautology of type* I.

Proof: By Proposition 1.1, $\mathscr{A}_2$ is a tautology. If $\mathscr{A}_2$ were to take value 2, then $\mathscr{A}_1 \supset \mathscr{A}_2$ should also take value 2, contrary to the hypothesis.

Proposition 1.3. *If $\mathscr{B}_j$ is a tautology containing as statement letters $A_1, \ldots, A_m, B_1, \ldots, B_n$, and $\mathscr{B}_j'$ arises from $\mathscr{B}_j$ by substituting statement forms $\mathscr{A}_1, \ldots, \mathscr{A}_m$ for $A_1, \ldots, A_m$, and $\mathscr{B}_1, \ldots, \mathscr{B}_n$ for $B_1, \ldots, B_n$, respectively, then $\mathscr{B}_j'$ is a tautology.*

Proof: For a given assignment of truth values to the statement letters of $\mathscr{B}_j'$, the statement forms $\mathscr{A}_1, \ldots, \mathscr{A}_m$ take truth values $x_1, \ldots, x_m$, and $\mathscr{B}_1, \ldots, \mathscr{B}_n$ take truth values $y_1, \ldots, y_n$ (where the x_i's are 0 or 1, and the y_i's are 0, 1, or 2). If the truth values $x_1, \ldots, x_m$ are assigned to $A_1, \ldots, A_m$, and the truth values $y_1, \ldots, y_n$ are assigned to $B_1, \ldots, B_n$, then $\mathscr{B}_j$ takes the same value as $\mathscr{B}_j'$. Since $\mathscr{B}_j$ is a tautology, then so is $\mathscr{B}_j'$.

Proposition 1.4. *If $\mathscr{B}_j'$ arises from $\mathscr{B}_j$ by the substitution of $\mathscr{B}_2$ for one or more occurrences of $\mathscr{B}_1$, then $(\mathscr{B}_1 \equiv^\circ \mathscr{B}_2) \supset (\mathscr{B}_j \equiv^\circ \mathscr{B}_j')$ is a tautology.*

Proof: Consider any assignment of truth values to the statement letters of $\mathscr{B}_j$ and $\mathscr{B}_2$. If either $\mathscr{B}_1$ or $\mathscr{B}_2$ takes value 1 and the other does not, then $\mathscr{B}_1 \equiv^\circ \mathscr{B}_2$ takes value 1. If either $\mathscr{B}_1$ or $\mathscr{B}_2$ takes value 2 and the other takes value 0, then $\mathscr{B}_1 \equiv^\circ \mathscr{B}_2$ takes value 1. So if $\mathscr{B}_1$ and $\mathscr{B}_2$ have different truth values under the given assignment, then $\mathscr{B}_1 \equiv^\circ \mathscr{B}_2$ takes value 1, and $(\mathscr{B}_1 \equiv^\circ \mathscr{B}_2) \supset (\mathscr{B}_j \equiv^\circ \mathscr{B}_j')$ takes value 0. If $\mathscr{B}_1$ and $\mathscr{B}_2$ have the same truth values, then so do $\mathscr{B}_j$ and $\mathscr{B}_j'$, since $\mathscr{B}_j'$ differs from $\mathscr{B}_j$ only in some of those places where $\mathscr{B}_1$ occurs in $\mathscr{B}_j$. Thus, for the given assignment, $\mathscr{B}_j \equiv^\circ \mathscr{B}_j'$ takes value 0 or 2, and $(\mathscr{B}_1 \equiv^\circ \mathscr{B}_2) \supset (\mathscr{B}_j \equiv^\circ \mathscr{B}_j')$ takes value 0 or 2.

The following list of statement forms, all tautologies, is given here for future reference.

(1) $(\mathscr{B}_1 \supset \mathscr{A}_1) \supset ((\mathscr{B}_1 \supset \neg\mathscr{A}_1) \supset \neg\mathscr{B}_1)$
(2) $(\mathscr{B}_1 \supset \mathscr{A}_1) \supset (\neg\mathscr{A}_1 \supset \neg\mathscr{B}_1)$
(3) $\neg(\mathscr{B}_1 \,\&\, \mathscr{B}_2) \equiv^\circ \neg\mathscr{B}_1 \vee \neg\mathscr{B}_2$
(4) $\neg(\mathscr{B}_1 \vee \mathscr{B}_2) \equiv^\circ \neg\mathscr{B}_1 \,\&\, \neg\mathscr{B}_2$
(5) $(\mathscr{B}_1 \supset \mathscr{B}_2) \supset \neg\mathscr{B}_1 \vee \mathscr{B}_2$
(6) $\neg(\mathscr{B}_1 \supset \mathscr{B}_2) \equiv \mathscr{B}_1 \,\&\, \neg\mathscr{B}_2$
(7) $\mathscr{A}_1 \supset \mathscr{B}_1 \equiv^\circ \neg\mathscr{A}_1 \vee \mathscr{B}_1$
(8) $\mathscr{B}_1 \equiv^\circ \neg\neg\mathscr{B}_1$
(9) $((\mathscr{B}_1 \supset \mathscr{B}_2) \supset \mathscr{B}_1) \supset \mathscr{B}_1$
(10) $(\mathscr{B}_1 \supset \mathscr{B}_2) \supset ((\mathscr{B}_1 \supset (\mathscr{B}_2 \supset \mathscr{B}_3)) \supset (\mathscr{B}_1 \supset \mathscr{B}_3))$

The following classical tautologies are not tautologies in the present calculus.

(1) $(\mathscr{A}_1 \supset \mathscr{B}_1) \supset (\neg\mathscr{B}_1 \supset \neg\mathscr{A}_1)$
(2) $(\neg\mathscr{B}_1 \vee \mathscr{B}_2) \supset (\mathscr{B}_1 \supset \mathscr{B}_2)$
(3) $\mathscr{B}_1 \supset (\neg\mathscr{B}_1 \supset \mathscr{A}_1)$
(4) $\neg\mathscr{B}_1 \vee \mathscr{B}_2 \equiv (\mathscr{B}_1 \supset \mathscr{B}_2)$
(5) $(\neg\mathscr{B}_1 \supset \neg\mathscr{B}_2) \supset ((\neg\mathscr{B}_1 \supset \mathscr{B}_2) \supset \mathscr{B}_1)$

2 Antinomies Syntactically Considered and the Completeness Theorem

The same letters used for the various statement forms in the previous section also apply here. We shall call a statement form (or wf) $\mathscr{B}_1$ an antinomy in the syntactic sense if and only if $\mathscr{B}_1$ and $\neg\mathscr{B}_1$ are both theorems. Statement letters $A_1, A_2, \ldots$ will be atomic wfs that are not antinomic (in the syntactic sense), and $B_1, B_2, \ldots$ will be atomic wfs such that both B_i and $\neg B_i$ are provable. The rules of formation for wfs involving all or some of these letters (A's, B's, $\mathscr{A}$'s, or $\mathscr{B}$'s) are the same as those given in the preceding section. In addition, the following closure conditions are in order for the syntactic determination of the $\mathscr{A}$-formulas.

C1a. All statement letters $A_1, A_2, \ldots$ are $\mathscr{A}$-formulas.
C1b. The formulas $A_1 \supset A_2$, $A_1 \,\&\, A_2$, $A_1 \vee A_2$, $\neg A_1$, $B_1 \supset A_1$, and $A_1 \supset (A_1 \vee B_1)$ are all $\mathscr{A}$-formulas.

C2a. If $\mathscr{A}_1$ and $\mathscr{A}_2$ and $\mathscr{A}$-formulas and $\mathscr{B}_1$ is any wf, then $\mathscr{A}_1 \supset \mathscr{A}_2$, $\mathscr{A}_1 \& \mathscr{A}_2$, $\mathscr{A}_1 \vee \mathscr{A}_2$, $\neg\mathscr{A}_1$, $\mathscr{B}_1 \supset \mathscr{A}_1$ and $\mathscr{A}_1 \supset (\mathscr{A}_1 \vee \mathscr{B}_1)$ are $\mathscr{A}$-formulas.
C2b. If $\neg\mathscr{B}_1$ is an $\mathscr{A}$-formula, then $\mathscr{B}_1$ is an $\mathscr{A}$-formula.
C3a. Axiom L3a and Axiom L3b are $\mathscr{A}$-formulas (see axioms below).
C3b. Theorems inferred from $\mathscr{A}$-formulas by *modus ponens* are $\mathscr{A}$-formulas.

It should be noted that under these closure conditions the propositional rule of inference of *modus ponens* plays a basic role in determining the $\mathscr{A}$-formulas. Hence, the class of $\mathscr{A}$-formulas will vary according to the theory T under consideration. If new axioms are added to T, then the class of $\mathscr{A}$-formulas usually increases. If Γ is a sequence of wfs to be added to the axioms of a theory T (presumably Γ would contain some new $\mathscr{A}$-formulas not provable in T), and if $\mathscr{B}_1$ is an $\mathscr{A}$-formula in the theory T′ obtained from T by adding the wfs of Γ as axioms, then we shall say that in the theory T $\mathscr{B}_1$ *is an* $\mathscr{A}$*-formula relative to* Γ.

The axioms and rule of inference for our propositional calculus L follow.

L1. $\mathscr{B}_1 \supset (\mathscr{B}_2 \supset \mathscr{B}_1)$
L2. $(\mathscr{B}_1 \supset (\mathscr{B}_2 \supset \mathscr{B}_3)) \supset ((\mathscr{B}_1 \supset \mathscr{B}_2) \supset (\mathscr{B}_1 \supset \mathscr{B}_3))$
L3a. $\neg\mathscr{A}_1 \supset (\mathscr{A}_1 \supset \mathscr{B}_1)$
L3b. $\neg\mathscr{A}_1 \supset \neg(\mathscr{A}_1 \& \mathscr{B}_1)$
L4. $(\mathscr{B}_1 \supset \mathscr{B}_2) \supset ((\neg\mathscr{B}_1 \supset \mathscr{B}_2) \supset \mathscr{B}_2)$
L5. $\mathscr{B}_1 \supset (\neg\mathscr{B}_2 \supset \neg(\mathscr{B}_1 \supset \mathscr{B}_2))$
L6. $\mathscr{B}_1 \supset (\mathscr{B}_2 \supset (\mathscr{B}_1 \& \mathscr{B}_2))$
L7a. $\mathscr{B}_1 \supset \neg\neg\mathscr{B}_1$
L7b. $\neg\neg\mathscr{B}_1 \supset \mathscr{B}_1$
L8a. $\mathscr{B}_1 \& \mathscr{B}_2 \supset \mathscr{B}_1$
L8b. $\mathscr{B}_1 \& \mathscr{B}_2 \supset \mathscr{B}_2$
L9a. $\mathscr{B}_1 \supset \mathscr{B}_1 \vee \mathscr{B}_2$
L9b. $\mathscr{B}_2 \supset \mathscr{B}_1 \vee \mathscr{B}_2$
L10. $\neg\mathscr{B}_1 \vee \neg\mathscr{B}_2 \supset \neg(\mathscr{B}_1 \& \mathscr{B}_2)$
L11. $\neg\mathscr{B}_1 \& \neg\mathscr{B}_2 \supset \neg(\mathscr{B}_1 \vee \mathscr{B}_2)$
L12. $\neg(\mathscr{B}_1 \supset \mathscr{B}_2) \supset (\mathscr{B}_1 \& \neg\mathscr{B}_2)$
L13. $\neg B_i \& B_i$

Rule of Inference

(*Modus Ponens*) $\mathscr{B}_1$ and $\mathscr{B}_1 \supset \mathscr{B}_2$ yield $\mathscr{B}_2$.

Proposition 2.1. (Deduction Theorem) *If* Γ *is a set of wfs and* $\mathscr{B}_1$ *and* $\mathscr{B}_2$ *are also wfs, and if* $\Gamma, \mathscr{B}_1 \vdash \mathscr{B}_2$, *then* $\Gamma \vdash \mathscr{B}_1 \supset \mathscr{B}_2$.

The proof involves only axioms L1 and L2, and does not differ from the classical Deduction Theorem. *Cf.* for example [3], p. 32.

Corollary 2.2.

(i) $\mathscr{B}_1 \supset \mathscr{B}_2, \mathscr{B}_2 \supset \mathscr{B}_3 \vdash \mathscr{B}_1 \supset \mathscr{B}_3$
(ii) $\mathscr{B}_1 \supset (\mathscr{B}_2 \supset \mathscr{B}_3), \mathscr{B}_2 \vdash \mathscr{B}_1 \supset \mathscr{B}_3$

Proposition 2.3. *Every theorem is a tautology.*

Proof: All the axioms of L are verifiable tautologies, and by Proposition 1.1 *modus ponens* yields only tautologies from tautologies.

The converse of Proposition 2.3 calls for the following lemma.

Lemma 2.4. *Let* $\mathscr{B}$ *be any wf and let* Γ *be the sequence of statement letters* $A_1, \ldots, A_m, B_1, \ldots, B_k$ *of* $\mathscr{B}$. *For a given assignment of truth values to* $A_1, \ldots, A_m, B_1, \ldots, B_k$ *let* B_i' *be* $\neg B_i \,\&\, B_i$. *Let* A_i' *be* A_i *if* A_i *takes value* 0 *and let* A_i' *be* $\neg A_i$ *if* A_i *takes value* 1. *Let* $\mathscr{B}'$ *be* $\mathscr{B}$ *if* $\mathscr{B}$ *takes value* 0; *let* $\mathscr{B}'$ *be* $\neg\mathscr{B}$ *if* $\mathscr{B}$ *takes value* 1; *let* $\mathscr{B}'$ *be* $\neg\mathscr{B} \,\&\, \mathscr{B}$ *if* $\mathscr{B}$ *takes value* 2. *Let* Γ' *be* $A_1', \ldots, A_m', B_1', \ldots, B_k'$. *Then* $\Gamma' \vdash \mathscr{B}'$ *and when* $\mathscr{B}$ *takes values* 0 *or* 1, *then* $\mathscr{B}$ *is an* $\mathscr{A}$*-formula relative to* Γ' *and is not antinomic relative to* Γ' (*i.e., it is not the case that* $\Gamma' \vdash \mathscr{B}$ *and* $\Gamma' \vdash \neg\mathscr{B}$).

Proof: By induction on the number n of primitive connectives. For the case $n = 0$ we just have the statement letter A_1 where $k = 0$, or B_1 where $m = 0$. In the first case, the lemma reduces to $A_1 \vdash A_1$ or $\neg A_1 \vdash \neg A_1$. Since B_1 takes value 2, then B_1' is $\neg B_1 \,\&\, B_1$ and therefore $\neg B_1 \,\&\, B_1 \vdash \neg B_1 \,\&\, B_1$.

Assume $\mathscr{B}$ takes value 0, then $\mathscr{B}$ is A_1 (Γ' is A_1). A_1 takes value 0 since $\mathscr{B}$ is A_1, so $\mathscr{B}$ is an $\mathscr{A}$-formula by condition C1a for $\mathscr{A}$-formulas. If $\Gamma' \vdash \neg A_1$ and $\Gamma' \vdash A_1$, then $\Gamma' \vdash \neg A_1 \,\&\, A_1$ by Axiom L6. But Γ' is A_1, so by the Deduction Theorem $\vdash A_1 \supset (\neg A_1 \,\&\, A_1)$, a formula which should be a tautology by Proposition 2.3. However, $A_1 \supset (\neg A_1 \,\&\, A_1)$ is not a tautology, so A_1 is not antinomic relative to Γ'.

Assume $\mathscr{B}$ takes value 1. Then $\mathscr{B}$ is A_1 and Γ' is $\neg A_1$. A_1 is an $\mathscr{A}$-formula by conditions C1a and C1b for $\mathscr{A}$-formulas. If $\Gamma' \vdash \neg A_1$ and $\Gamma' \vdash \neg\neg A_1$, then $\Gamma' \vdash A_1$ by Axiom L7b and so $\Gamma' \vdash \neg A_1 \,\&\, A_1$ by Axiom L6. By the Deduction Theorem $\vdash \neg A_1 \supset (\neg A_1 \,\&\, A_1)$, a formula which should be a tautology by Proposition 2.3. However, $\neg A_1 \supset (\neg A_1 \,\&\, A_1)$ is not a tautology, so $\mathscr{B}$ is not antinomic relative to Γ'.

We assume now that the lemma holds for all $j < n$.

Case 1. $\mathscr{B}$ is $\neg\mathscr{B}_1$. Then $\mathscr{B}_1$ has fewer than n occurrences of primitive connectives.

Subcase 1a. Let $\mathscr{B}_1$ take value 0 under the given truth value assignment. Then $\mathscr{B}$ takes value 1. So $\mathscr{B}_1'$ is $\mathscr{B}_1$ and $\mathscr{B}'$ is $\neg\mathscr{B}$. By inductive hypothesis applied to $\mathscr{B}_1$ we have $\Gamma' \vdash \mathscr{B}_1$. Then by Axiom L7a $\Gamma' \vdash \neg\neg\mathscr{B}_1$. But $\neg\neg\mathscr{B}_1$ is $\mathscr{B}'$.

Subcase 1b. Let $\mathscr{B}_1$ take value 1. Then $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$ and $\mathscr{B}'$ is $\mathscr{B}$. By inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$. But $\neg\mathscr{B}_1$ is $\mathscr{B}'$.

Assume $\mathscr{B}$ takes value 1. Then $\mathscr{B}_1$ takes value 0 and by inductive hypothesis $\mathscr{B}_1$ is not antinomic relative to Γ' and is an $\mathscr{A}$-formula relative to Γ'. By condition C2a for $\mathscr{A}$-formulas $\mathscr{B}$ is an $\mathscr{A}$-formula. If $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \neg\neg\mathscr{B}_1$, then using Axiom L7a $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_1$. But $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Assume $\mathscr{B}$ takes value 0. Then $\mathscr{B}_1$ takes value 1. Since $\mathscr{B}_1$ takes value 1, by inductive hypothesis $\mathscr{B}_1$ is not antinomic relative to Γ' and $\mathscr{B}_1$ is an $\mathscr{A}$-formula. Since $\mathscr{B}_1$ is an $\mathscr{A}$-formula, $\neg\mathscr{B}_1$ (which is $\mathscr{B}$) is an $\mathscr{A}$-formula by condition C2a for $\mathscr{A}$-formulas. If $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \neg\neg\mathscr{B}_1$, then $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_1$ using Axiom L7b. However, by inductive hypothesis, $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ (which is $\neg\mathscr{B}_1$) is not antinomic relative to Γ'.

Subcase 1c. Let $\mathscr{B}_1$ take value 2. Then $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$ & $\mathscr{B}_1$ and $\mathscr{B}'$ is $\neg\mathscr{B}$ & $\mathscr{B}$. By inductive hypothesis applied to $\mathscr{B}_1$ we get $\Gamma' \vdash \neg\mathscr{B}_1$ & $\mathscr{B}_1$. By Axioms L8a and L8b we have $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_1$. By Axiom L7a we get $\Gamma' \vdash \neg\neg\mathscr{B}_1$. Then by Axiom L6 we have $\Gamma' \vdash \neg\neg\mathscr{B}_1$ & $\neg\mathscr{B}_1$ which is $\neg\mathscr{B}$ & $\mathscr{B}$ and which is also $\mathscr{B}'$.

Case 2. $\mathscr{B}$ is $\mathscr{B}_1 \supset \mathscr{B}_2$. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ have fewer occurrences of primitive connectives than $\mathscr{B}$.

Subcase 2a. $\mathscr{B}_1$ takes value 1. Hence $\mathscr{B}$ takes value 0. Then $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$ and $\mathscr{B}'$ is $\mathscr{B}$. So $\Gamma' \vdash \neg\mathscr{B}_1$. By inductive hypothesis $\mathscr{B}_1$ is an $\mathscr{A}$-formula. By Axiom L3a $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$. $\mathscr{B}_1 \supset \mathscr{B}_2$ is $\mathscr{B}$.

Subcase 2b. $\mathscr{B}_2$ takes value 0. Then $\Gamma' \vdash \mathscr{B}_2$ and $\mathscr{B}'$ is $\mathscr{B}$. By Axiom L1 $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$. But $\mathscr{B}_1 \supset \mathscr{B}_2$ is $\mathscr{B}$.

Subcase 2c. $\mathscr{B}_2$ takes value 1 and $\mathscr{B}_1$ takes value 0 (or 2). Then $\mathscr{B}$ takes value 1 and $\mathscr{B}'$ is $\neg\mathscr{B}$. By inductive hypothesis we have $\Gamma' \vdash \mathscr{B}_1$ (or $\Gamma' \vdash \neg\mathscr{B}_1$ & $\mathscr{B}_1$) and $\Gamma' \vdash \neg\mathscr{B}_2$. Thus by Axiom L5 we have $\neg(\mathscr{B}_1 \supset \mathscr{B}_2)$ which is $\mathscr{B}'$.

Assume $\mathscr{B}$ takes value 0. Let $\mathscr{B}_1$ take value 1. By inductive hypothesis $\mathscr{B}_1$ is not antinomic relative to Γ' and $\mathscr{B}_1$ is an $\mathscr{A}$-formula. $\mathscr{B}_1$ to take value 1 implies $\Gamma' \vdash \neg\mathscr{B}_1$. Since $\mathscr{B}_1$ is an $\mathscr{A}$-formula, then by condition C2a for $\mathscr{A}$-formulas $\neg\mathscr{B}_1$ is an $\mathscr{A}$-formula. By condition C3a $\neg\mathscr{B}_1 \supset (\mathscr{B}_1 \supset \mathscr{B}_2)$ is an $\mathscr{A}$-formula relative to Γ'. Since $\neg\mathscr{B}_1$ and $\neg\mathscr{B}_1 \supset (\mathscr{B}_1 \supset \mathscr{B}_2)$ are $\mathscr{A}$-formulas, then $(\mathscr{B}_1 \supset \mathscr{B}_2)$ is an $\mathscr{A}$-formula relative to Γ' by condition C3b and the fact that $\Gamma' \vdash \neg\mathscr{B}_1$. If $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$ and $\Gamma' \vdash \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$, then $\Gamma' \vdash \mathscr{B}_1$ & $\neg\mathscr{B}_2$ and $\Gamma' \vdash \mathscr{B}_1$ by Axiom L12 and Axiom L8a. But $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Assume $\mathscr{B}$ takes value 0. Let $\mathscr{B}_2$ take value 0. By inductive hypothesis $\mathscr{B}_2$ is not antinomic relative to Γ' and $\mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ'. By condition C2a $\mathscr{B}_1 \supset \mathscr{B}_2$ is an $\mathscr{A}$-formula. If $\Gamma' \vdash \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$ and $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$, then $\Gamma' \vdash \mathscr{B}_1$ & $\neg\mathscr{B}_2$, $\Gamma' \vdash \neg\mathscr{B}_2$ and $\Gamma' \vdash \mathscr{B}_1$ by Axiom L12, Axiom L8a and Axiom L8b. Since $\Gamma' \vdash \mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$, then $\Gamma' \vdash \mathscr{B}_2$. However, $\mathscr{B}_2$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Assume $\mathscr{B}$ takes value 1. Let $\mathscr{B}_1$ take value 0 (or 2) and $\mathscr{B}_2$ take value 1. By inductive hypothesis the lemma holds for $\mathscr{B}_2$, so $\Gamma' \vdash \neg\mathscr{B}_2$, $\mathscr{B}_2$ is not antinomic relative to Γ' and $\mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ'. Since $\mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ', then by condition C2a for $\mathscr{A}$-formulas $\mathscr{B}_1 \supset \mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ'. If $\Gamma' \vdash \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$ and $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$, then $\Gamma' \vdash \mathscr{B}_1$ & $\neg\mathscr{B}_2$ and $\Gamma' \vdash \neg\mathscr{B}_2$ by Axiom L12 and Axiom L8a. Since $\Gamma' \vdash \mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_1 \supset \mathscr{B}_2$, then $\Gamma' \vdash \mathscr{B}_2$. However, $\mathscr{B}_2$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Subcase 2d. $\mathscr{B}_2$ takes value 2 and $\mathscr{B}_1$ takes value 0 (or 2). Then $\mathscr{B}_1 \supset \mathscr{B}_2$

takes value 2. Thus $\mathscr{B}_2'$ is $\neg\mathscr{B}_2$ & $\mathscr{B}_2$ and $\mathscr{B}_1'$ is $\mathscr{B}_1$ (or $\mathscr{B}_1$ & $\neg\mathscr{B}_1$) and $\mathscr{B}'$ is $\neg\mathscr{B}$ & $\mathscr{B}$. Therefore by inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_2$ & $\mathscr{B}_2$ and $\Gamma' \vdash \mathscr{B}_1$. Thus by Axiom L8a we deduce $\neg\mathscr{B}_2$ and by Axiom L5 we deduce $\neg(\mathscr{B}_1 \supset \mathscr{B}_2)$ which is $\neg\mathscr{B}$. With the use of Axiom L8b and Axiom L1 we have $\mathscr{B}_1 \supset \mathscr{B}_2$ which is $\mathscr{B}$. Thus it follows that we have $\neg(\mathscr{B}_1 \supset \mathscr{B}_2)$ & $(\mathscr{B}_1 \supset \mathscr{B}_2)$ which is $\neg\mathscr{B}$ & $\mathscr{B}$ or $\mathscr{B}'$. (When $\mathscr{B}_1$ takes value 1, and $\mathscr{B}_2$ takes value 0 (or 2) then Subcase 2a applies.)

Case 3. $\mathscr{B}$ is $\mathscr{B}_1$ & $\mathscr{B}_2$. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ have fewer occurrences of primitive connectives than $\mathscr{B}$, so by inductive hypothesis $\Gamma' \vdash \mathscr{B}_1'$ and $\Gamma' \vdash \mathscr{B}_2'$.

Subcase 3a. $\mathscr{B}_1$ and $\mathscr{B}_2$ take value 0. Then $\mathscr{B}$ takes value 0 and $\mathscr{B}'$ is $\mathscr{B}$, $\mathscr{B}_1'$ is $\mathscr{B}_1$, and $\mathscr{B}_2'$ is $\mathscr{B}_2$. By inductive hypothesis $\Gamma' \vdash \mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_2$. Then by Axiom L6 we deduce $\mathscr{B}_1$ & $\mathscr{B}_2$ which is $\mathscr{B}'$.

Subcase 3b. $\mathscr{B}_1$ takes value 1. Then $\mathscr{B}$ takes value 1 and $\mathscr{B}'$ is $\neg\mathscr{B}$, $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$. By inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$. By Axiom L9a we deduce $\neg\mathscr{B}_1 \vee \neg\mathscr{B}_2$ and by Axiom L10 we deduce $\neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ which is $\mathscr{B}'$. (When $\mathscr{B}_2$ takes value 1 the argument differs only in the use of Axiom L9b instead of L9a.)

Assume $\mathscr{B}$ takes value 0. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ take value 0. By inductive hypothesis $\mathscr{B}_1$ and $\mathscr{B}_2$ are not antinomic relative to Γ', are $\mathscr{A}$-formulas relative to Γ', and $\Gamma' \vdash \mathscr{B}_1$. By condition C2a $\mathscr{B}_1$ & $\mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ'. By Axiom L3a, $\Gamma' \vdash \neg(\mathscr{B}_1$ & $\mathscr{B}_2) \supset ((\mathscr{B}_1$ & $\mathscr{B}_2) \supset \neg\mathscr{B}_1)$. If $\Gamma' \vdash \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ and $\Gamma' \vdash (\mathscr{B}_1$ & $\mathscr{B}_2)$, then $\Gamma' \vdash \neg\mathscr{B}_1$. However, $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Assume $\mathscr{B}$ takes value 1. Let $\mathscr{B}_1$ take value 1 (the proof is the same if $\mathscr{B}_2$ takes value 1). By inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$, $\mathscr{B}_1$ is not antinomic relative to Γ', and $\mathscr{B}_1$ is an $\mathscr{A}$-formula. By Axiom L3b, $\Gamma' \vdash \neg\mathscr{B}_1 \supset \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$. Since $\Gamma' \vdash \neg\mathscr{B}_1$, then $\Gamma' \vdash \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$. By conditions C3a and C3b $\neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ is an $\mathscr{A}$-formula. By condition C2b $\mathscr{B}_1$ & $\mathscr{B}_2$ is an $\mathscr{A}$-formula. If $\Gamma' \vdash \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ and $\Gamma' \vdash \mathscr{B}_1$ & $\mathscr{B}_2$, then $\Gamma' \vdash \mathscr{B}_1$ by Axiom L8a. However, $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Subcase 3c. $\mathscr{B}_1$ takes value 2 and $\mathscr{B}_2$ takes value 0 (or 2); then $\mathscr{B}'$ is $\neg\mathscr{B}$ & $\mathscr{B}$, $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$ & $\mathscr{B}_1$ and $\mathscr{B}_2'$ is $\mathscr{B}_2$ (or $\neg\mathscr{B}_2$ & $\mathscr{B}_2$). By inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$ & $\mathscr{B}_1$ and $\Gamma' \vdash \mathscr{B}_2$ (or $\Gamma' \vdash \neg\mathscr{B}_2$ & $\mathscr{B}_2$). From $\neg\mathscr{B}_1$ & $\mathscr{B}_1$ and Axiom L8a and Axiom L8b we deduce $\neg\mathscr{B}_1$ and $\mathscr{B}_1$. By Axiom L9a we have $\neg\mathscr{B}_1 \vee \neg\mathscr{B}_2$ and by Axiom L10 we have $\neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ which is $\neg\mathscr{B}$. By Axiom L6 we have $\mathscr{B}_1$ & $\mathscr{B}_2$ which is $\mathscr{B}$ and by use of Axiom L6 again we have $\neg\mathscr{B}$ & $\mathscr{B}$ which is $\mathscr{B}'$. (The proof when $\mathscr{B}_2$ takes value 2 and $\mathscr{B}_1$ takes value 0 (or 2) goes along the same lines.)

Case 4. $\mathscr{B}$ is $\mathscr{B}_1 \vee \mathscr{B}_2$. Then $\mathscr{B}_1$ and $\mathscr{B}_2$ have fewer occurrences of primitive connectives than $\mathscr{B}$, and so by inductive hypothesis we have $\Gamma' \vdash \mathscr{B}_1'$ and $\Gamma' \vdash \mathscr{B}_2'$.

Subcase 4a. $\mathscr{B}_1$ takes value 0. Then $\mathscr{B}$ has value 0, $\mathscr{B}'$ is $\mathscr{B}$ and $\mathscr{B}_1'$ is $\mathscr{B}_1$. Thus by inductive hypothesis $\Gamma' \vdash \mathscr{B}_1$ and by Axiom L9a we have $\mathscr{B}_1 \vee \mathscr{B}_2$ which is $\mathscr{B}'$. (The case where $\mathscr{B}_2$ takes value 0 is similar.)

Subcase 4b. $\mathscr{B}_1$ takes value 1 and $\mathscr{B}_2$ takes value 1. Then $\mathscr{B}$ takes value 1 and $\mathscr{B}'$ is $\neg\mathscr{B}$, $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$ and $\mathscr{B}_2'$ is $\neg\mathscr{B}_2$. So by inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \neg\mathscr{B}_2$. Thus $\neg\mathscr{B}_1$ & $\neg\mathscr{B}_2$ follows from Axiom L6 and by Axiom L11 we obtain $\neg(\mathscr{B}_1 \vee \mathscr{B}_2)$ which is $\neg\mathscr{B}$ or $\mathscr{B}'$.

Assume $\mathscr{B}$ takes value 0. Let $\mathscr{B}_1$ take value 0. By inductive hypothesis $\Gamma' \vdash \mathscr{B}_1$, $\mathscr{B}_1$ is not antinomic relative to Γ', and $\mathscr{B}_1$ is an $\mathscr{A}$-formula relative to Γ'. $\Gamma' \vdash \mathscr{B}_1 \supset (\mathscr{B}_1 \vee \mathscr{B}_2)$ by Axiom L9a. $\mathscr{B}_1 \supset (\mathscr{B}_1 \vee \mathscr{B}_2)$ is an $\mathscr{A}$-formula relative to Γ' by condition C2a for $\mathscr{A}$-formulas, so $\mathscr{B}_1 \vee \mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ' by condition C3b for $\mathscr{A}$-formulas. By condition C2a $\neg(\mathscr{B}_1 \vee \mathscr{B}_2)$ is an $\mathscr{A}$-formula relative to Γ', so $\Gamma' \vdash \neg(\mathscr{B}_1 \vee \mathscr{B}_2) \supset ((\mathscr{B}_1 \vee \mathscr{B}_2) \supset \neg\mathscr{B}_1)$ by Axiom L3a. If $\Gamma' \vdash \neg(\mathscr{B}_1 \vee \mathscr{B}_2)$ and $\Gamma' \vdash \mathscr{B}_1 \vee \mathscr{B}_2$, then $\Gamma' \vdash \neg\mathscr{B}_1$. However, $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'. (The case where $\mathscr{B}_2$ takes value 0 is similar.)

Assume $\mathscr{B}$ takes value 1. Let $\mathscr{B}_1$ and $\mathscr{B}_2$ therefore take value 1. By inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$ and $\Gamma' \vdash \neg\mathscr{B}_2$, $\mathscr{B}_1$ and $\mathscr{B}_2$ are not antinomic relative to Γ', and $\mathscr{B}_1$ and $\mathscr{B}_2$ are $\mathscr{A}$-formulas relative to Γ'. Since $\mathscr{B}_1$ and $\mathscr{B}_2$ are $\mathscr{A}$-formulas relative to Γ', then by condition C2a for $\mathscr{A}$-formulas $\mathscr{B}_1 \vee \mathscr{B}_2$ is an $\mathscr{A}$-formula relative to Γ'. By Axiom L3a $\Gamma' \vdash \neg(\mathscr{B}_1 \vee \mathscr{B}_2) \supset ((\mathscr{B}_1 \vee \mathscr{B}_2) \supset \mathscr{B}_1)$. If $\Gamma' \vdash \neg(\mathscr{B}_1 \vee \mathscr{B}_2)$ and $\Gamma' \vdash \mathscr{B}_1 \vee \mathscr{B}_2$, then $\Gamma' \vdash \mathscr{B}_1$. However, $\mathscr{B}_1$ is not antinomic relative to Γ', so $\mathscr{B}$ is not antinomic relative to Γ'.

Subcase 4c. $\mathscr{B}_1$ takes value 2 and $\mathscr{B}_2$ takes value 1 (or 2). Then $\mathscr{B}$ takes value 2. So $\mathscr{B}'$ is $\neg\mathscr{B}$ & $\mathscr{B}$, $\mathscr{B}_1'$ is $\neg\mathscr{B}_1$ & $\mathscr{B}_1$ and $\mathscr{B}_2'$ is $\neg\mathscr{B}_2$ (or $\neg\mathscr{B}_2$ & $\mathscr{B}_2$). By inductive hypothesis $\Gamma' \vdash \neg\mathscr{B}_1$ & $\mathscr{B}_1$ and $\Gamma' \vdash \neg\mathscr{B}_2$ (or $\Gamma' \vdash \neg\mathscr{B}_2$ & $\mathscr{B}_2$); it then follows that $\neg\mathscr{B}_1$ is obtained from Axiom L8a, $\neg\mathscr{B}_1$ & $\neg\mathscr{B}_2$ is obtained from Axiom L6 and finally from Axiom L11 we get $\neg(\mathscr{B}_1 \vee \mathscr{B}_2)$ which is $\neg\mathscr{B}$. $\mathscr{B}_1$ follows from Axiom L8b and by Axiom L9a we have $\mathscr{B}_1 \vee \mathscr{B}_2$ which is $\mathscr{B}$. Therefore by Axiom L6 we obtain $\neg\mathscr{B}$ & $\mathscr{B}$ which is $\mathscr{B}'$. (For the case when $\mathscr{B}_1$ takes value 1 (or 2) and $\mathscr{B}_2$ takes value 2 the argument is similar.)

Proposition 2.5. *If a wf $\mathscr{B}$ is a tautology, then $\mathscr{B}$ is a theorem.*

Proof: Let $\mathscr{B}$ be a tautology and let $A_1, \ldots, A_m, B_1, \ldots, B_k$ be the statement letters occurring in $\mathscr{B}$. For any assignment of truth values to the statement letters $A_1, \ldots, A_m, B_1, \ldots, B_k$ occurring in $\mathscr{B}$ we have by Lemma 2.4 that $A_1', \ldots, A_m', B_1', \ldots, B_k' \vdash \mathscr{B}'$. But since $\mathscr{B}$ is a tautology, for any given assignment of truth values to the statement letters occurring in $\mathscr{B}$, the tautology $\mathscr{B}$ will take value 0 or 2. Thus $\mathscr{B}'$ is $\mathscr{B}$ or $\mathscr{B}'$ is $\neg\mathscr{B}$ & $\mathscr{B}$; in any case, $A_1', \ldots, A_m', B_1', \ldots, B_k' \vdash \mathscr{B}$ for any assignment of truth values to the statement letters occurring in $\mathscr{B}$. B_k' is $\neg B_k$ & B_k and $A_1', \ldots, A_m'$, $B_1', \ldots, \neg B_k$ & $B_k \vdash \mathscr{B}$. By application of Proposition 2.1 (Deduction Theorem) it follows that $A_1', \ldots, A_m', B_1', \ldots, B_{k-1}' \vdash \neg B_k$ & $B_k \supset \mathscr{B}$. By Axiom L13 we obtain $\vdash \neg B_k$ & B_k. Thus we obtain $A_1', \ldots, A_m', B_1', \ldots, B_{k-1}' \vdash \mathscr{B}$. By repeating this argument k-times we eliminate the B_i's. If A_m takes value 0 for a given assignment of truth values to the statement letters, then A_m' is A_m and $A_1', \ldots, A_{m-1}', A_m \vdash \mathscr{B}$. If A_m takes value 1 for another assignment with the same truth value for the A_i's up to A_{m-1}', then $A_1', \ldots, A_{m-1}'$, $\neg A_m \vdash \mathscr{B}$. Applying the Deduction Theorem is both cases it follows that $A_1', \ldots, A_{m-1}' \vdash A_m \supset \mathscr{B}$ and $A_1', \ldots, A_{m-1}' \vdash \neg A_m \supset \mathscr{B}$. By Axiom L4 we have

$A_1', \ldots, A_{m-1}' \vdash \mathscr{B}$. Repeating this argument m-times we eliminate the A_i's. Thus we obtain $\vdash \mathscr{B}$.

Proposition 2.6. *A wf $\mathscr{B}$ is a tautology if and only if it is a theorem of the theory* L.

Proof: Proposition 2.3 and Proposition 2.5.

Corollary 2.7. *The theory* L *is absolutely consistent (i.e., not all wfs are provable).*

Proof: There are wfs which are not tautologies.

It should perhaps be remarked that the metatheory employed so far uses the entire classical propositional logic—not an antinomic one. In fact, nowhere in the remainder of the paper will antinomic proof methods be extended to the metatheory.

II. AN ANTINOMIC PREDICATE CALCULUS

3 Antinomic Predicate Formulas Semantically Considered In anticipation of the formal definition below, we can say that an antinomic formula $\mathscr{B}_i(x_1, \ldots, x_n)$ will be one that for all n-tuples $x_1, \ldots, x_n$, both $\mathscr{B}_i(x_1, \ldots, x_n)$ and $\neg\mathscr{B}_i(x_1, \ldots, x_n)$ hold. Accordingly, $(x_1) \ldots (x_n)\, \mathscr{B}_i(x_1, \ldots, x_n)$ will be an antinomic sentence if $\mathscr{B}_i(x_1, \ldots, x_n)$ is an antinomic predicate formula. Let us use Roman capital $A_1, A_2, \ldots$ for predicate letters which for no n-tuple $x_1, \ldots, x_n$ it is the case that both $A_i(x_1, \ldots, x_n)$ and $\neg A_i(x_1, \ldots, x_n)$ hold, and let us use Roman capital $B_1, B_2, \ldots$ for predicate letters for which there exist at least one n-tuple $x_1, \ldots, x_n$ such that both $B_i(x_1, \ldots, x_n)$ and $\neg B_i(x_1, \ldots, x_n)$ hold. Let us also use the predicate letters $B_1^*, B_2^*, \ldots$ to denote that subcollection of the B's for which $B_i^*(x_1, \ldots, x_n)$ and $\neg B_i^*(x_1, \ldots, x_n)$ both hold for all n-tuples. (n indicates the number of arguments—or rank—of the predicate letters A_i or B_i, which is different in general for each i.) Let script $\mathscr{A}_1, \mathscr{A}_2, \ldots$ denote predicate formulas which for no n-tuple $x_1, \ldots, x_n$ it is the case that both $\mathscr{A}_i(x_1, \ldots, x_n)$ and $\neg\mathscr{A}_i(x_1, \ldots, x_n)$ hold, and let script $\mathscr{B}_1, \mathscr{B}_2, \ldots$ denote predicate formulas for which there may be n-tuples $x_1, \ldots, x_n$ such that both $\mathscr{B}_i(x_1, \ldots, x_n)$ and $\neg\mathscr{B}_i(x_1, \ldots, x_n)$ hold (obviously, the $\mathscr{A}$'s are particular cases of the $\mathscr{B}$'s). Let us use $x_1, x_2, \ldots$ for individual variables, $a_1, a_2, \ldots$ for individual constants, the same symbols used before for the propositional connectives (including $\equiv^\circ$), and the symbol (x_i) for universal quantification (existential quantification being defined in the usual way). The rules of formation for terms and formulas follow.

1. Individual variables and individual constants are terms.
2. If $t_1, \ldots, t_n$ are terms, then $A_i(t_1, \ldots, t_n)$ and $B_j(t_1, \ldots, t_n)$ are atomic formulas (assuming that both A_i and B_j are of the same given rank n).
3. Atomic formulas are wfs (well-formed formulas).
4. If $\mathscr{B}_1$ and $\mathscr{B}_2$ are wfs (in particular, if $\mathscr{B}_1$ is $\mathscr{A}_1$ and $\mathscr{B}_2$ is $\mathscr{A}_2$), then so are $\mathscr{B}_1 \supset \mathscr{B}_2$, $\mathscr{B}_1 \,\&\, \mathscr{B}_2$, $\mathscr{B}_1 \vee \mathscr{B}_2$, $\neg\mathscr{B}_1$, and $(x_i)\, \mathscr{B}_1(x_i)$.

5. If $\mathcal{A}_1$ and $\mathcal{B}_1$ are wfs, then so are $\mathcal{A}_1 \supset \mathcal{B}_1$, $\mathcal{B}_1 \supset \mathcal{A}_1$, $\mathcal{A}_1$ & $\mathcal{B}_1$, $\mathcal{B}_1$ & $\mathcal{A}_1$, $\mathcal{A}_1 \vee \mathcal{B}_1$, and $\mathcal{B}_1 \vee \mathcal{A}_1$.
6. These are all the terms and wfs.

Wfs have meaning only when an interpretation is given to the formal language just described. An interpretation for us shall consist of the following items.

1. A non-empty set D called the domain of the interpretation.
2. An assignment to each predicate letter A_i of rank n of an n-place relation $A_i^{\prime}$ in D.
3. An assignment to each predicate letter B_j of rank n of a pair of non-disjoint n-place relations $B_j^{\prime}$ and $B_j^{\prime\prime}$ in D such that the union of $B_j^{\prime}$ and $B_j^{\prime\prime}$ is the whole cartesian product D^n.
4. An assignment of a fixed element of D to each individual constant.

The notions of satisfiability, truth, and antinomicity shall be made precise in the following way. Let an interpretation with non-empty domain D be given. Let Σ be the set of all denumerable sequences of D. We shall define what it means for a sequence $s = (b_1, b_2, \ldots)$ in Σ to satisfy a wf $\mathcal{A}_i$ or $\mathcal{B}_i$ under the given interpretation. Let s^* be a function of one argument with values in D such that

(1) If t is x_i, then $s^*(t)$ is b_i.
(2) If t is an individual constant, then $s^*(t)$ is the interpretation in D of this constant.

Now we define the notion of satisfiability by induction.

1a. If $\mathcal{B}_j$ is an atomic wf of the form $A_i(t_1, \ldots, t_n)$ and $A_i^{\prime}$ is the corresponding relation in the given interpretation, then the sequence s satisfies $\mathcal{B}_j$ if and only if $A_i^{\prime}(s^*(t_1), \ldots, s^*(t_n))$ (i.e., iff the n-tuple $(s^*(t_1), \ldots, s^*(t_n))$ is in the relation $A_i^{\prime}$).
1b. If $\mathcal{B}_j$ is an atomic wf of the form $B_i(t_1, \ldots, t_n)$, then s satisfies $\mathcal{B}_j$ if and only if $B_i^{\prime}(s^*(t_1), \ldots, s^*(t_n))$.
2a. If $\mathcal{B}_j$ is an atomic wf of the form $A_i(t_1, \ldots, t_n)$, then s satisfies $\neg\mathcal{B}_j$ if and only if s does not satisfy $A_i(t_1, \ldots, t_n)$.
2b. If $\mathcal{B}_j$ is an atomic wf of the form $B_i(t_1, \ldots, t_n)$, then s satisfies $\neg\mathcal{B}_j$ if and only if $B_j^{\prime\prime}(s^*(t_1), \ldots, s_n^*(t\ ))$.
2c. If $\mathcal{B}_j$ is a wf (atomic or not), then s satisfies $\neg\neg\mathcal{B}_j$ if and only if s satisfies $\mathcal{B}_j$.

If $\mathcal{B}_j$ and $\mathcal{B}_k$ are wfs, then

3a. s satisfies $\mathcal{B}_j \supset \mathcal{B}_k$ if and only if s does not satisfy $\mathcal{B}_j$ or s satisfies $\mathcal{B}_k$.
3b. s satisfies $\neg(\mathcal{B}_j \supset \mathcal{B}_k)$ if and only if s satisfies $\mathcal{B}_j$ and s satisfies $\neg\mathcal{B}_k$.
4a. s satisfies $(\mathcal{B}_j$ & $\mathcal{B}_k)$ if and only if s satisfies $\mathcal{B}_j$ and s satisfies $\mathcal{B}_k$.
4b. s satisfies $\neg(\mathcal{B}_j$ & $\mathcal{B}_k)$ if and only if s satisfies $\neg\mathcal{B}_j$ or s satisfies $\neg\mathcal{B}_k$.
5a. s satisfies $\mathcal{B}_j \vee \mathcal{B}_k$ if and only if s satisfies $\mathcal{B}_j$ or s satisfies $\mathcal{B}_k$.
5b. s satisfies $\neg(\mathcal{B}_j \vee \mathcal{B}_k)$ if and only if s satisfies $\neg\mathcal{B}_j$ and s satisfies $\neg\mathcal{B}_k$.

6a. s satisfies $(x_i)\,\mathscr{B}_j$ if and only if every sequence of Σ which differs from s in at most the i'th component satisfies $\mathscr{B}_j$.
6b. s satisfies $\neg(x_i)\,\mathscr{B}_j$ if and only if there is a sequence s' in Σ which differs from s in at most the i'th component such that s' satisfies $\neg\mathscr{B}_j$.

The definitions of true, false, and antinomic formulas are as follows.

D1. A wf $\mathscr{B}_j$ is said to be true (for a given interpretation) if and only if every sequence in Σ satisfies $\mathscr{B}_j$.
D2. A wf $\mathscr{B}_j$ is said to be false (for a given interpretation) if and only if every sequence in Σ satisfies $\neg\mathscr{B}_j$.
D3. A wf $\mathscr{B}_j$ is said to be antinomic (for a given interpretation) if and only if $\mathscr{B}_j$ is both true and false.
D4. An interpretation is said to be a model for a set Γ of wfs if and only if every wf in Γ is either true or antinomic for that interpretation.

The following properties can be verified from the preceding definitions plus definitions D5 to D8, which can be found after properties P10, P16, and P18.

P1. If a wf $\mathscr{B}_1$ is an $\mathscr{A}_j$ (for a given interpretation), then $\mathscr{B}_1$ cannot be both true and false (for that interpretation).
P2. If $\mathscr{B}_1$ and $\mathscr{B}_1 \supset \mathscr{B}_2$ are true and not antinomic (for a given interpretation), then so is $\mathscr{B}_2$ (for that interpretation).
P3. If $\mathscr{B}_1$ and $\mathscr{B}_1 \supset \mathscr{B}_2$ are antinomic (for a given interpretation), then so is $\mathscr{B}_2$ (for that interpretation).
P4. If $\mathscr{B}_1$ is true but not antinomic and $\mathscr{B}_2$ is antinomic (for a given interpretation), then $\mathscr{B}_1 \supset \mathscr{B}_2$ is antinomic (for that interpretation).
P5. If $\mathscr{B}_1$ is antinomic and $\mathscr{B}_2$ is true but not antinomic (for a given interpretation), then $\mathscr{B}_1 \supset \mathscr{B}_2$ is true and not antinomic (for that interpretation).
P6. If $\mathscr{B}_1$ is true or antinomic and $\mathscr{B}_2$ is false but not antinomic (for a given interpretation), then $\mathscr{B}_1 \supset \mathscr{B}_2$ is false and not antinomic (for that interpretation).
P7. A sequence s satisfies $\mathscr{B}_1 \equiv \mathscr{B}_2$ if and only if s satisfies both $\mathscr{B}_1 \supset \mathscr{B}_2$ and $\mathscr{B}_2 \supset \mathscr{B}_1$.
P8. A sequence s satisfies $\mathscr{B}_1 \equiv^\circ \mathscr{B}_2$ if and only if s satisfies both $\mathscr{B}_1 \equiv \mathscr{B}_2$ and $\neg\mathscr{B}_1 \equiv \neg\mathscr{B}_2$.
P9. A sequence s satisfies $(\exists x_i)\,\mathscr{B}_1$ if and only if there is a sequence s' which differs from s in at most the i'th place such that s' satisfies $\mathscr{B}_1$.
P10. $\mathscr{B}_1$ is true (for a given interpretation) if and only if $(x_i)\,\mathscr{B}_1$ is true (for that interpretation).

D5. By the closure of $\mathscr{B}_i$ we mean the closed wf obtained by prefixing universal quantifiers that quantify those variables which are free in $\mathscr{B}_i$. If $\mathscr{B}_i$ has no free variables, then the closure of $\mathscr{B}_i$ is defined to be $\mathscr{B}_i$ itself.

P11. $\mathscr{B}_1$ is true (for a given interpretation) if and only if its closure is true (for that interpretation).
P12. If $\mathscr{B}_1$ is antinomic (for a given interpretation), then the closure of $\mathscr{B}_1$ is both true and false (for that interpretation).

P13. Every wf $\mathscr{B}_1$ obtained by substitution of wfs for the statement letters of a tautology is either true or antinomic for any interpretation.
P14. If $(x_i)\mathscr{B}_1(x_i)$ is satisfied by a sequence s, then so is $\mathscr{B}_1(t)$, where t is an arbitrary term. Hence $(x_i)\mathscr{B}_1(x_i) \supset \mathscr{B}\ (t)$ is satisfied by all sequences in any given interpretation.
P15. If $\mathscr{B}_1$ does not contain x_i free, then $(x_i)(\mathscr{B}_1 \supset \mathscr{B}_2) \supset (\mathscr{B}_1 \supset (x_i)\ \mathscr{B}_2)$ is satisfied by all sequences in any given interpretation.
P16. $\neg\mathscr{B}_1(x_i) \supset \neg(x_i)\mathscr{B}_1(x_i)$ is satisfied by all sequences in any given interpretation.

D6. A wf $\mathscr{B}_1$ is said to be logically valid if and only if it is true or antinomic for every interpretation.
D7. $\mathscr{B}_1$ is said to logically imply $\mathscr{B}_2$ if and only if in any interpretation every sequence which satisfies $\mathscr{B}_1$ satisfies $\mathscr{B}_2$.

P17. $\mathscr{B}_1$ logically implies $\mathscr{B}_2$ if and only if $\mathscr{B}_1 \supset \mathscr{B}_2$ is logically valid.
P18. If $\mathscr{B}_1$ logically implies $\mathscr{B}_2$ and $\mathscr{B}_1$ is satisfied by every sequence in a given interpretation, then $\mathscr{B}_2$ is satisfied by every sequence in that interpretation.

D8. $\mathscr{B}_1$ is a logical consequence of a set Γ of wfs if and only if in any interpretation every sequence which satisfies every formula in Γ also satisfies $\mathscr{B}_1$.

P19. If $\mathscr{B}_1$ is a logical consequence of a set Γ of wfs and all wfs in Γ are satisfied by every sequence in a given interpretation, then $\mathscr{B}_1$ is satisfied by every sequence in that interpretation.
P20. A closed wf $\mathscr{B}_1$ is antinomic for a given interpretation if and only if $\neg\mathscr{B}_1$ is antinomic for that interpretation.
P21. A closed wf $\mathscr{B}_1$ is true but not antinomic for a given interpretation if and only if $\neg\mathscr{B}_1$ is false but not antinomic for that interpretation.
P22. Let $(x_i)\mathscr{B}_1(x_i)$ be a closed wf, then this wf is true for a given interpretation and there is a term t for which both $\mathscr{B}_1(t)$ and $\neg\mathscr{B}_1(t)$ are true for that interpretation, if and only if $(x_i)\mathscr{B}_1(x_i)$ is antinomic for that interpretation.

4 Antinomic Predicate Formulas Syntactically Considered and the Completeness Theorem A well-formed predicate formula $\mathscr{B}_i$ will be said to be antinomic (in the syntactic sense) if and only if both $\mathscr{B}_i$ and $\neg\mathscr{B}_i$ are provable. We now introduce an axiomatic system K for a predicate calculus which will include antinomies (in the syntactic sense). The symbols used for the language of K are as follows. The same symbols introduced in the preceding section for the propositional connectives, individual variables and constants, and universal quantification will apply. Roman capital $A_1, A_2, \ldots$ will be used for predicate letters which for no n-tuple of terms $t_1, \ldots, t_n$ it is the case that both $A_i(t_1, \ldots, t_n)$ and $\neg A_i(t_1, \ldots, t_n)$ are provable. Roman capital $B_1, B_2, \ldots$ will be used for predicate letters which for some n-tuple of terms $t_1, \ldots, t_n$ both $B_i(t_1, \ldots, t_n)$ and $\neg B_i(t_1, \ldots, t_n)$ are provable. The symbols $B_1^*, B_2^*, \ldots$ will denote that subsequence of the B's which for all n-tuples of terms $t_1, \ldots, t_n$ it is the case that both $B_i^*(t_1, \ldots, t_n)$ and $\neg B_i^*(t_1, \ldots, t_n)$ are provable. (In each

case, n indicates the rank of the specific letter A_i or B_j under consideration.) The rules of formation for terms and well-formed formulas are also the same as those given in the preceding section (with script $\mathscr{B}$'s again denoting well-formed formulas in general), plus the following rule: *If* $\mathscr{B}_1$ *is a wf, then* $(x_i)\,\mathscr{B}_1$ *is also a wf.* In addition, we shall let script $\mathscr{A}_1, \mathscr{A}_2, \ldots$ denote a subcollection of the set of $\mathscr{B}$'s determined by the following closure conditions.

1a. All atomic wfs of the form $A_i(t_1, \ldots, t_n)$ are $\mathscr{A}$-formulas.
1b. If $\mathscr{A}_1$ denotes $A_i(t_1, \ldots, t_n)$, $\mathscr{A}_2$ denotes $\mathscr{A}_j(s_1, \ldots, s_m)$, and $\mathscr{B}_1$ denotes $B_k(r_1, \ldots, r_l)$, then $\mathscr{A}_1 \supset \mathscr{A}_2$, $\mathscr{A}_1 \,\&\, \mathscr{A}_2$, $\mathscr{A}_1 \vee \mathscr{A}_2$, $\neg\mathscr{A}_1$, $\mathscr{A}_1 \supset (\mathscr{A}_1 \vee \mathscr{B}_1)$, and $\mathscr{B}_1 \supset \mathscr{A}_1$ are all $\mathscr{A}$-formulas (where the t's, s's, and r's are all terms).
2a. If $\mathscr{A}_1$ and $\mathscr{A}_2$ are $\mathscr{A}$-formulas and $\mathscr{B}_1$ is any wf, then $\mathscr{A}_1 \supset \mathscr{A}_2$, $\mathscr{A}_1 \,\&\, \mathscr{A}_2$, $\mathscr{A}_1 \vee \mathscr{A}_2$, $\neg\mathscr{A}_1$, $(x_i)\,\mathscr{A}_1$, $\mathscr{A}_1 \supset (\mathscr{A}_1 \vee \mathscr{B}_1)$, and $\mathscr{B}_1 \supset \mathscr{A}_1$ are all $\mathscr{A}$-formulas.
2b. If $\neg\mathscr{B}_1$ is an $\mathscr{A}$-formula, then $\mathscr{B}_1$ is an $\mathscr{A}$-formula.
3a. Axioms K3a and K3b are $\mathscr{A}$-formulas (see axioms below).
3b. Theorems deduced from $\mathscr{A}$-formulas by *modus ponens* or generalization are $\mathscr{A}$-formulas.
4. If the closure of $\mathscr{B}_1$ is not antinomic, then it is an $\mathscr{A}$-formula.

Axioms K1 to K12 of K are the same axioms L1 to L12 of L in section 2 (interpreting the script letters as corresponding well-formed predicate formulas); to these we add the following axioms.

K13. $B_j^*(t_1, \ldots, t_n) \,\&\, \neg B_j^*(t_1, \ldots, t_n)$, (for $j = 1, 2, \ldots$).
K14. $(x_i)\,\mathscr{B}_1(x_i) \supset \mathscr{B}_1(t)$, (where t is a term free for x_i in $\mathscr{B}_1(x_i)$).
K15. $(x_i)(\mathscr{B}_1 \supset \mathscr{B}_2) \supset (\mathscr{B}_1 \supset (x_i)\,\mathscr{B}_2)$, (where $\mathscr{B}_1$ does not contain x_i free).
K16. $\neg\mathscr{B}_1(t_1, \ldots, t_n) \supset \neg(x_1) \ldots (x_n)\,\mathscr{B}_1(x_1, \ldots, x_n)$.

In addition to *modus ponens*, we will also use generalization as a rule of inference; i.e., $(x_i)\,\mathscr{B}_1$ *follows from* $\mathscr{B}_1$.

Proposition 4.1. *Every wf* $\mathscr{B}_1$ *that is an instance of a tautology is a theorem of* K.

(Proofs will not be given where there is no essential difference from those of the corresponding propositions of the classical predicate calculus. For example, see [3] for such proofs: we have patterned sections 1 to 4 as much as possible after this reference to simplify our presentation.)

Proposition 4.2. *The system* K *is absolutely consistent, that is, not all wfs in the language of* K *are provable.*

Proof: For each wf $\mathscr{B}_i$ of K, let $h(\mathscr{B}_i)$ be the expression obtained by erasing all the quantifiers and terms in $\mathscr{B}_i$ (together with the associated commas and parentheses). Then $h(\mathscr{B}_i)$ is a statement form with the A's and the B's playing the role of statement letters. Clearly $h(\neg\mathscr{B}_1) = \neg h(\mathscr{B}_1)$, $h(\mathscr{B}_1 \supset \mathscr{B}_2) = h(\mathscr{B}_1) \supset h(\mathscr{B}_2)$, $h(\mathscr{B}_1 \,\&\, \mathscr{B}_2) = h(\mathscr{B}_1) \,\&\, h(\mathscr{B}_2)$, and $h(\mathscr{B}_1 \vee \mathscr{B}_2) = h(\mathscr{B}_1) \vee h(\mathscr{B}_2)$. It can be verified under this transformation that all the axioms of K become tautologies. In addition, if $h(\mathscr{B}_1)$ and $h(\mathscr{B}_1 \supset \mathscr{B}_2)$ are tautologies, then $h(\mathscr{B}_2)$ is a tautology; and if $h(\mathscr{B}_1)$ is a tautology, then so is $h((x_i)\,\mathscr{B}_1)$, which is the same as $h(\mathscr{B}_1)$. Hence, if $\mathscr{B}_1$ is a theorem of K, then $h(\mathscr{B}_1)$ is a tautology. If every wf were provable in K, then for every wf $\mathscr{B}_1$, $h(\mathscr{B}_1)$ would be a tautology under the mapping h. In particular, if $\neg(\mathscr{A}_1 \supset (\neg\mathscr{A}_1 \supset \mathscr{B}_1))$ were a theorem of K, then $h(\neg(\mathscr{A}_1 \supset (\neg\mathscr{A}_1 \supset \mathscr{B}_1))) = \neg(h(\mathscr{A}_1) \supset (\neg h(\mathscr{A}_1) \supset h(\mathscr{B}_1)))$ would be a tautology, which is not the case. Therefore not all wfs are theorems of K, and K is absolutely consistent.

Let $\mathscr{B}_p$ be a wf in a set Γ of wfs; assume that a deduction $\mathscr{B}_1, \ldots, \mathscr{B}_n$ from Γ is given, together with a justification for each step of the deduction. We shall then say that $\mathscr{B}_i$ (for $i = 1, 2, \ldots, n$) depends upon $\mathscr{B}_p$ in this deduction if and only if:

(i) $\mathscr{B}_i$ is $\mathscr{B}_p$ and the justification is that it belongs to Γ; or
(ii) $\mathscr{B}_i$ is a direct consequence by *modus ponens* or generalization of some preceding wfs of the sequence where at least one of these preceding wfs depends upon $\mathscr{B}_p$.

Proposition 4.3. *If* $\mathscr{B}_2$ *does not depend upon* $\mathscr{B}_1$ *in a deduction* $\Gamma, \mathscr{B}_1 \vdash \mathscr{B}_2$, *then* $\Gamma \vdash \mathscr{B}_2$.

Proposition 4.4. (Deduction Theorem) *Assume that* $\Gamma, \mathscr{B}_1 \vdash \mathscr{B}_2$ *where in the deduction no application of the generalization rule to a wf which depends upon* $\mathscr{B}_1$ *has as its quantified variable a free variable of* $\mathscr{B}_1$. *Then* $\Gamma \vdash \mathscr{B}_1 \supset \mathscr{B}_2$.

Corollary 4.5. *If a deduction* $\Gamma, \mathscr{B}_1 \vdash \mathscr{B}_2$ *involves no application of the generalization rule of which the quantified variable is free in* $\mathscr{B}_1$, *then* $\Gamma \vdash \mathscr{B}_1 \supset \mathscr{B}_2$.

Corollary 4.6. *If* $\mathscr{B}_1$ *is a closed wf and* $\Gamma, \mathscr{B}_1 \vdash \mathscr{B}_2$, *then* $\Gamma \vdash \mathscr{B}_1 \supset \mathscr{B}_2$.

Proposition 4.7. *Every theorem of* K *is logically valid.*

Proof: Since all the axioms of K are logically valid, and by properties P11 and P17 *modus ponens* and generalization preserve logical validity, then every theorem of K is logically valid.

Definition of similar wfs. If x_i and x_j are distinct, then $\mathscr{B}_1(x_i)$ and $\mathscr{B}_1(x_j)$ are said to be similar if and only if x_j is free for x_i in $\mathscr{B}_1(x_i)$ and $\mathscr{B}_1(x_i)$ has no free occurrences of x_j. (It is assumed that $\mathscr{B}_1(x_j)$ is obtained from $\mathscr{B}_1(x_i)$ by substituting x_j for all free occurrences of x_i.)

Lemma 4.8. *If* $\mathscr{B}_1(x_i)$ *and* $\mathscr{B}_1(x_j)$ *are similar, then* $\vdash (x_i)\,\mathscr{B}_1(x_i) \equiv (x_j)\,\mathscr{B}_1(x_j)$.

Lemma 4.9. *If a closed wf* $\neg\mathscr{B}_1$ *of a first-order theory* T *based on* K *is not provable in* T, *then the theory* T′ *obtained from* T *by adding* $\mathscr{B}_1$ *as an axiom is absolutely consistent.*

Proof: Assume T′ is not absolutely consistent. Then for any wf $\mathscr{B}_i$ we have $\vdash_{T'} \mathscr{B}_i$ and $\vdash_{T'} \neg\mathscr{B}_i$. By the Deduction Theorem, it follows that $\vdash_T \mathscr{B}_1 \supset \neg\mathscr{B}_1$, and by the tautology $\neg\mathscr{B}_1 \supset \neg\mathscr{B}_1$ we get $\vdash_T \neg\mathscr{B}_1 \supset \neg\mathscr{B}_1$. Thus, by the tautology $(\neg\mathscr{B}_1 \supset \neg\mathscr{B}_1) \supset ((\mathscr{B}_1 \supset \neg\mathscr{B}_1) \supset \neg\mathscr{B}_1)$ and *modus ponens*, $\vdash_T \neg\mathscr{B}_1$ which contradicts our hypothesis that $\neg\mathscr{B}_1$ was not provable in T. Thus the theory T′ we obtain by adding $\mathscr{B}_1$ to the theory T is absolutely consistent.

Lemma 4.10. *The set of expressions of a first-order theory* T *is denumerable.* (Hence the same is true of the set of terms, wfs, and closed wfs of T.)

Definition of Completeness. A first-order theory T is complete if and only if for any closed wf $\mathscr{B}_i$ of T either $\vdash_{\mathrm{T}} \neg \mathscr{B}_i$ or $\vdash_{\mathrm{T}} \mathscr{B}_i$.

Definition of Extension. A first-order theory T′ having the same symbols as the first-order theory T is said to be an extension of T if and only if every theorem of T is a theorem of T′.

Lemma 4.11. (Lindenbaum's Lemma) *If* T *is an absolutely consistent first-order theory, then there is an absolutely consistent and complete extension of* T. *If* $\mathscr{B}_j$ *is a closed formula of* T, *then* $\mathscr{B}_j$ *is an* $\mathscr{A}$*-formula in* T *if and only if* $\mathscr{B}_j$ *is an* $\mathscr{A}$*-formula in that complete extension of* T. *Hence a closed formula* $\mathscr{B}_j$ *is antinomic in* T *if and only if it is antinomic in the complete extension of* T.

Proof: Let $\mathscr{B}_1, \mathscr{B}_2, \ldots$ be an enumeration of all closed wfs of T, by Lemma 4.10. Define a sequence $\mathrm{J}_0, \mathrm{J}_1, \ldots$ of theories in the following way. J_0 is T. Assume J_n is defined, with $n \geqslant 0$. If it is not the case that $\vdash_{\mathrm{J}_n} \neg \mathscr{B}_{n+1}$, then let J_{n+1} be obtained by adding $\mathscr{B}_{n+1}$ as an additional axiom. On the other hand, if $\vdash_{\mathrm{J}_n} \neg \mathscr{B}_{n+1}$, then let $\mathrm{J}_n = \mathrm{J}_{n+1}$. Let J be the first-order theory obtained by taking as axioms all the axioms of all the J_i's, including $\mathrm{J}_0 = \mathrm{T}$. To show that J is absolutely consistent it suffices to prove that the formula $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is not provable in J. If $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ were provable in J, the proof would involve only a finite number of axioms; hence, for some $n \mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ would be provable in J_n. Therefore in order to prove that J is consistent, we must prove that in all the J_i's $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is not provable. If $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is provable in J_0, then by Axiom K8a, Axiom K8b, and Axiom K3a, any formula $\mathscr{B}_k$ would be provable in J_0. But $\mathrm{J}_0 = \mathrm{T}$ is absolutely consistent, therefore $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is not provable in J_0. Now assume that $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is not provable in J_i. If $\mathrm{J}_i = \mathrm{J}_{i+1}$, then $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is not provable in J_{i+1}. If $\mathrm{J}_i \neq \mathrm{J}_{i+1}$, then $\mathscr{B}_{i+1}$ is added to J_i to form J_{i+1}. Suppose that $\vdash_{\mathrm{J}_{i+1}} \mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$, or expressing this another way, $\mathscr{B}_{i+1} \vdash_{\mathrm{J}_i} \mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$. If $\mathscr{B}_{i+1} \vdash_{\mathrm{J}_i} \mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$, then by Axiom K8a and Axiom K8b, $\mathscr{B}_{i+1} \vdash_{\mathrm{J}_i} \mathscr{A}_1$ and $\mathscr{B}_{i+1} \vdash_{\mathrm{J}_i} \neg \mathscr{A}_1$. Hence by the Deduction Theorem, $\vdash_{\mathrm{J}_i} \mathscr{B}_{i+1} \supset \mathscr{A}_1$ and $\vdash_{\mathrm{J}_i} \mathscr{B}_{i+1} \supset \neg \mathscr{A}_1$. By the tautology $(\mathscr{B}_j \supset \mathscr{A}_k) \supset ((\mathscr{B}_j \supset \neg \mathscr{A}_k) \supset \neg \mathscr{B}_j)$, then $\vdash_{\mathrm{J}_i} \neg \mathscr{B}_{i+1}$. But $\neg \mathscr{B}_{i+1}$ is not provable in J_i, so $\mathscr{A}_1 \,\&\, \neg \mathscr{A}_1$ is not provable in J_{i+1}. Therefore J_{i+1} is absolutely consistent and so is J. To prove the completeness of J, let $\mathscr{B}_j$ be any closed wf. Then $\mathscr{B}_j$ is $\mathscr{B}_{i+1}$ for some $i \geqslant 0$. Now either $\vdash_{\mathrm{J}_i} \neg \mathscr{B}_{i+1}$ or $\vdash_{\mathrm{J}_{i+1}} \mathscr{B}_{i+1}$, since if not-$\vdash_{\mathrm{J}_i} \neg \mathscr{B}_{i+1}$, then $\mathscr{B}_{i+1}$ is added as an axiom in J_{i+1}. Thus either $\vdash_{\mathrm{J}} \neg \mathscr{B}_{i+1}$ or $\vdash_{\mathrm{J}} \mathscr{B}_{i+1}$. Hence J is complete.

Assume $\mathscr{B}_i$ is an $\mathscr{A}$-formula in J. If $\mathscr{B}_i$ is not an $\mathscr{A}$-formula in T, then—by condition 4 for $\mathscr{A}$-formulas—since $\mathscr{B}_i$ is closed it is antinomic in T. Because all theorems of T are theorems of J, by Axiom K3a any formula $\mathscr{B}_j$ is provable in J (since $\mathscr{B}_i$ is an $\mathscr{A}$-formula in J). This contradicts the absolute consistency of J. Thus $\mathscr{B}_i$ is an $\mathscr{A}$-formula in T. Suppose now that $\mathscr{B}_j$ is an $\mathscr{A}$-formula in T and $\mathscr{B}_j$ is not an $\mathscr{A}$-formula in J. By condition 4, $\mathscr{B}_j$ is antinomic in J. Since $\mathscr{B}_j$ is not antinomic in T, take the least i for which $\vdash_{J_{i+1}} \mathscr{B}_j$ and $\vdash_{J_{i+1}} \neg\mathscr{B}_j$. By Axiom K6, $\vdash_{J_{i+1}} \mathscr{B}_j \;\&\; \neg\mathscr{B}_j$; using reasoning similar to the above, we can then show that any formula $\mathscr{B}_k$ is provable in J, which contradicts the absolute consistency of J. Therefore for all i not-$\vdash_{J_i} \mathscr{B}_j \;\&\; \neg\mathscr{B}_j$ and $\mathscr{B}_j$ is an $\mathscr{A}$-formula in J. If $\mathscr{B}_j$ is antinomic in J, $\mathscr{B}_j$ is antinomic in T, for if $\mathscr{B}_j$ is not antinomic in T, then (since $\mathscr{B}_j$ is closed) it is an $\mathscr{A}$-formula in T; hence $\mathscr{B}_j$ would be an $\mathscr{A}$-formula in J by the preceding proof. If $\mathscr{B}_j$ is antinomic and is an $\mathscr{A}$-formula in J, then by Axiom K3a any formula $\mathscr{B}_k$ is provable in J. But J is absolutely consistent. Thus $\mathscr{B}_j$ must be antinomic in T. Conversely, since all theorems of T are theorems of J, if $\mathscr{B}_j$ is antinomic in T, then $\mathscr{B}_j$ is antinomic in J.

Proposition 4.12. *Every absolutely consistent first-order theory* T *has a denumerable model.*

Proof: Add to the symbols of T a denumerable set $\{b_1, b_2, \ldots, b_n, \ldots\}$ of new individual constants. Call this new first-order theory T_0. Its axioms are those of T plus those logical axioms which involve the new constants. T_0 is absolutely consistent. For if it were not, then $\vdash_{T_0} \mathscr{B}_j$ for any wf $\mathscr{B}_j$. Replace each b_i appearing in this proof with a variable which does not appear in the proof. This transforms axioms into axioms and preserves the correctness of the applications of the rules of inference. The final proof is then a proof in T. Thus we would have for any wf $\mathscr{B}_k$ in the language of T that $\vdash_T \mathscr{B}_k$, which contradicts the absolute consistency of T.

By Lemma 4.10 let $\mathscr{B}_1(x_{i_1}), \ldots, \mathscr{B}_k(x_{i_k}), \ldots$ be an enumeration of all wfs having at most one free variable. (Let x_{i_k} be the free variable of $\mathscr{B}_k$ if the latter has a free variable, otherwise let x_{i_k} be x_1.) Choose a sequence $b_{j_1}, b_{j_2}, \ldots$ of some individual constants such that b_{j_k} is not contained in $\mathscr{B}_1(x_{i_1}), \ldots, \mathscr{B}_k(x_{i_k})$ and such that b_{j_k} is different from each of $b_{j_1}, b_{j_2}, \ldots, b_{j_{k-1}}$.

If $(x_{i_k})\mathscr{B}_k(x_{i_k})$ is not antinomic in the theory T_0, then we shall let the wf

$$(S_k) \text{ be } \mathscr{B}_k(b_{j_k}) \supset (x_{i_k})\mathscr{B}_k(x_{i_k})$$

where $(x_{i_k})\mathscr{B}_k(x_{i_k})$ is an $\mathscr{A}$-formula by condition 4. If $(x_{i_k})\mathscr{B}_k(x_{i_k})$ is antinomic in the theory T_0, then we shall let the wf

$$(S_k) \text{ be } \neg(x_{i_k})\mathscr{B}_k(x_{i_k}) \supset \neg\mathscr{B}_k(b_{j_k}) \;\&\; \mathscr{B}_k(b_{j_k}).$$

Let T_n be the first-order theory obtained by adding $(S_1), \ldots, (S_n)$ to the axioms of T_0, and let T_∞ be the theory obtained by adding all the (S_i)'s to T_0. If $\neg\mathscr{A}_1 \;\&\; \mathscr{A}_1$ were provable in T_∞, then by Axiom K8a, Axiom K8b, and Axiom K3a, any wf $\mathscr{B}_j$ would be provable in T_∞, which then would not be absolutely consistent. If $\neg\mathscr{A}_1 \;\&\; \mathscr{A}_1$ were provable in T_∞, then its proof would contain a finite number of the (S_i)'s and therefore would be a proof in some T_n. Hence, if in all the T_i's $\neg\mathscr{A}_1 \;\&\; \mathscr{A}_1$ is not provable, T_∞ is absolutely consistent. The proof that in all T_i's $\neg\mathscr{A}_1 \;\&\; \mathscr{A}_1$ is not provable is by induction. In T_0, $\neg\mathscr{A}_1 \;\&\; \mathscr{A}_1$ is not provable because if it were, then by the same reasoning as above, T_0 would not be absolutely consistent. However, T_0 is absolutely consistent, so $\neg\mathscr{A}_1 \;\&\; \mathscr{A}_1$ is not provable in T_0.

Assume $\neg\mathscr{A}_1 \& \mathscr{A}_1$ is not provable in all T_i's for $i < n$. The case in which $(x_{i_k})\mathscr{B}_k(x_{i_k})$ is an $\mathscr{A}$-formula in (S_k) is the same as the classical one. The case in which $(x_{i_k})\mathscr{B}_k(x_{i_k})$ in (S_k) is antinomic in T_0 is proved as follows. Assume $\vdash_{\mathsf{T}_n}\neg\mathscr{A}_1 \& \mathscr{A}_1$, or in other words $(S_n)\vdash_{\mathsf{T}_{n-1}}\mathscr{A}_1 \& \neg\mathscr{A}_1$. By the Deduction Theorem, $\vdash_{\mathsf{T}_{n-1}}(S_n) \supset \mathscr{A}_1 \& \neg\mathscr{A}_1$; but since $(S_n) \equiv^{\circ} \neg(S_n)$ is a tautology we also have $\vdash_{\mathsf{T}_{n-1}}\neg(S_n) \supset \mathscr{A}_1 \& \neg\mathscr{A}_1$, so by Axiom K4 $\vdash_{\mathsf{T}_{n-1}}\mathscr{A}_1 \& \neg\mathscr{A}_1$. This contradicts the inductive hypothesis that in the theory $\mathsf{T}_{n-1}\mathscr{A}_1 \& \neg\mathscr{A}_1$ is not provable. Thus $\mathscr{A}_1 \& \neg\mathscr{A}_1$ is not provable in T_n for all n, so T_∞ is absolutely consistent.

For any closed wf $\mathscr{B}_k$, $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_0 if and only if it is an $\mathscr{A}$-formula in T_∞. Suppose $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_0 and not an $\mathscr{A}$-formula in T_∞. By condition 4, $\mathscr{B}_k$ is antinomic in T_∞. If $\mathscr{B}_k$ is antinomic in T_∞, then take the least n for which $(S_n)\vdash_{\mathsf{T}_{n-1}}\mathscr{B}_k$ and $(S_n)\vdash_{\mathsf{T}_{n-1}}\neg\mathscr{B}_k$ where $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_{n-1} and is antinomic in T_n. By the Deduction Theorem, $\vdash_{\mathsf{T}_{n-1}}(S_n) \supset \mathscr{B}_k$ and $\vdash_{\mathsf{T}_{n-1}}(S_n) \supset \neg\mathscr{B}_k$. If (S_n) is antinomic in T_{n-1}, then $\vdash_{\mathsf{T}_{n-1}}\mathscr{B}_k$ and $\vdash_{\mathsf{T}_{n-1}}\neg\mathscr{B}_k$. By Axiom K3a, any formula $\mathscr{B}_i$ is provable in T_{n-1}, which contradicts the absolute consistency of T_{n-1}. If (S_n) is not antinomic in T_{n-1}, then since (S_n) is closed by condition 4, (S_n) is an $\mathscr{A}$-formula in T_{n-1}. By the tautology $((S_n) \supset \mathscr{B}_k) \supset (((S_n) \supset \neg\mathscr{B}_k) \supset \neg(S_n))$, it is the case that $\vdash_{\mathsf{T}_{n-1}}\neg(S_n)$. But (S_n) is added to T_{n-1} to form the theory T_n, and (S_n) is an $\mathscr{A}$-formula in T_n; so by Axiom K3a, any formula $\mathscr{B}_i$ is provable in T_n. But T_n is absolutely consistent. Thus $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_∞. Conversely, assume $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_∞ and not an $\mathscr{A}$-formula in T_0, then $\mathscr{B}_k$ is antinomic in T_0 by condition 4. Since all theorems of T_0 are theorems of T_∞, $\mathscr{B}_k$ is antinomic in T_∞. Since $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_∞, then by Axiom K3a any formula $\mathscr{B}_i$ is provable in T_∞. But T_∞ is absolutely consistent. Thus $\mathscr{B}_k$ is an $\mathscr{A}$-formula in T_0. Hence by reasoning similar to that of Lemma 4.11, it follows that $\mathscr{B}_k$ is antinomic in T_0 if and only if $\mathscr{B}_k$ is antinomic in T_∞. T_∞ is an extension of T_0, and by Lemma 4.11 we shall let J be an absolutely consistent and complete extension of T_∞. The denumerable interpretation $\mathfrak{M}$ shall have as its domain the set of closed terms of T_0 which by Lemma 4.10 is a denumerable set. If c is an individual constant of T_0, then its interpretation shall be c itself. For a predicate letter A_i of K, the associated relation $A_i^!$ in $\mathfrak{M}$ shall hold for arguments $t_1, \ldots, t_n$ if and only if $\vdash_{\mathsf{J}} A_i(t_1, \ldots, t_n)$. For a predicate letter B_j of K, the associated relation B_j' shall hold for arguments $t_1, \ldots, t_n$ if and only if $\vdash_{\mathsf{J}} B_j(t_1, \ldots, t_n)$, and the associated relation B_j'' shall hold for arguments $t_1, \ldots, t_n$ if and only if $\vdash_{\mathsf{J}}\neg B_j(t_1, \ldots, t_n)$. To show that $\mathfrak{M}$ is a model for T_0 (and therefore also for T, since every theorem of T is a theorem of T_0), it suffices to show that any closed wf $\mathscr{B}_j$ of T_0 is true and not antinomic in $\mathfrak{M}$ if and only if $\vdash_{\mathsf{J}}\mathscr{B}_j$ and not-$\vdash_{\mathsf{J}}\neg\mathscr{B}_j$, and that a closed wf $\mathscr{B}_j$ of T_0 is antinomic in $\mathfrak{M}$ if and only if $\vdash_{\mathsf{J}}\mathscr{B}_j$ and $\vdash_{\mathsf{J}}\neg\mathscr{B}_j$, since all theorems of T_0 are theorems of J. The proof is given by induction on the number of connectives and quantifiers in the wf $\mathscr{B}_j$. If $\mathscr{B}_j$ is a closed atomic wf, then by definition $\mathscr{B}_j$ is true and not antinomic in $\mathfrak{M}$ if and only if $\vdash_{\mathsf{J}}\mathscr{B}_j$ and not-$\vdash_{\mathsf{J}}\neg\mathscr{B}_j$, and $\mathscr{B}_j$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_{\mathsf{J}}\mathscr{B}_j$ and $\vdash_{\mathsf{J}}\neg\mathscr{B}_j$. We shall assume for the inductive step that if $\mathscr{B}_h$ is any closed wf with fewer connectives and quantifiers than $\mathscr{B}_j$, then $\mathscr{B}_h$

is true and not antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_h$ and not-$\vdash_J \neg\mathscr{B}_h$, and $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_h$ and $\vdash_J \neg\mathscr{B}_h$.

Case 1. $\mathscr{B}_j$ is $\neg\mathscr{B}_h$.

Subcase 1a. $\mathscr{B}_j$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_j$ and $\vdash_J \neg\mathscr{B}_j$.

Proof: If $\neg\mathscr{B}_h$ is antinomic in $\mathfrak{M}$, then $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$. By the inductive hypothesis, $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_h$ and $\vdash_J \neg\mathscr{B}_h$. By Axiom K7a, we have $\vdash_J \neg\neg\mathscr{B}_h$. Thus if $\neg\mathscr{B}_h$ is antinomic in $\mathfrak{M}$, then $\vdash_J \neg\mathscr{B}_h$ and $\vdash_J \neg\neg\mathscr{B}_h$. Conversely, assume $\vdash_J \neg\mathscr{B}_h$ and $\vdash_J \neg\neg\mathscr{B}_h$. By Axiom K7b, we have $\vdash_J \mathscr{B}_h$ and $\vdash_J \neg\mathscr{B}_h$. Thus by the inductive hypothesis, $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$. If $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$, then $\neg\mathscr{B}_h$ is antinomic in $\mathfrak{M}$.

Subcase 1b. $\neg\mathscr{B}_h$ is true but not antinomic in $\mathfrak{M}$ if and only if $\vdash_J \neg\mathscr{B}_h$ and not-$\vdash_J \neg\neg\mathscr{B}_h$.

Proof: Assume $\neg\mathscr{B}_h$ is true and not antinomic in $\mathfrak{M}$. Therefore $\mathscr{B}_h$ is false and not antinomic in $\mathfrak{M}$. Since J is complete, it follows that $\vdash_J \neg\mathscr{B}_h$ or $\vdash_J \neg\neg\mathscr{B}_h$. Assume first that $\vdash_J \neg\mathscr{B}_h$ and $\vdash_J \neg\neg\mathscr{B}_h$. By Axiom K7b we obtain $\vdash_J \mathscr{B}_h$ and $\vdash_J \neg\mathscr{B}_h$. By the inductive hypothesis, $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$. It was assumed, however, that $\mathscr{B}_h$ is not antinomic in $\mathfrak{M}$, so $\neg\mathscr{B}_h$ is not antinomic in J. Assume that $\vdash_J \neg\neg\mathscr{B}_h$ and not-$\vdash_J \neg\mathscr{B}_h$. It follows by Axiom K7b that $\vdash_J \mathscr{B}_h$ and not-$\vdash_J \neg\mathscr{B}_h$. By the inductive hypothesis it follows that $\mathscr{B}_h$ is true and not antinomic in $\mathfrak{M}$. But $\mathscr{B}_h$ is false and not antinomic in $\mathfrak{M}$. Thus $\vdash_J \neg\mathscr{B}_h$ and not-$\vdash_J \neg\neg\mathscr{B}_h$. On the other hand, assume $\vdash_J \neg\mathscr{B}_h$ and not-$\vdash_J \neg\neg\mathscr{B}_h$. If $\neg\mathscr{B}_h$ is antinomic in $\mathfrak{M}$, then $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$. If $\mathscr{B}_h$ is antinomic in $\mathfrak{M}$, then by the inductive hypothesis, $\vdash_J \mathscr{B}_h$ and $\vdash_J \neg\mathscr{B}_h$. By Axiom K7a we obtain $\vdash_J \neg\neg\mathscr{B}_h$, which is contrary to our assumption, so $\neg\mathscr{B}_h$ is not antinomic in $\mathfrak{M}$. Assume now that $\neg\mathscr{B}_h$ is false and not antinomic in $\mathfrak{M}$, then $\mathscr{B}_h$ is true and not antinomic in $\mathfrak{M}$. By the inductive hypothesis it follows that $\vdash_J \mathscr{B}_h$ and not-$\vdash_J \neg\mathscr{B}_h$. By Axiom K7a, $\vdash_J \neg\neg\mathscr{B}_h$, which is contrary to our assumption. Thus $\mathscr{B}_h$ is true and not antinomic in $\mathfrak{M}$.

Case 2. $\mathscr{B}_j$ is $\mathscr{B}_1 \supset \mathscr{B}_2$.

Subcase 2a. $\mathscr{B}_1 \supset \mathscr{B}_2$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$ and $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$.

Proof: If $\mathscr{B}_1 \supset \mathscr{B}_2$ is a closed wf, then so are $\mathscr{B}_1$ and $\mathscr{B}_2$. $\mathscr{B}_1 \supset \mathscr{B}_2$ is antinomic only when $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$ and $\mathscr{B}_1$ is true or antinomic in $\mathfrak{M}$. By the inductive hypothesis we have $\vdash_J \mathscr{B}_1$ (or $\vdash_J \mathscr{B}_1$ and $\vdash_J \neg\mathscr{B}_1$), $\vdash_J \neg\mathscr{B}_2$, and $\vdash_J \mathscr{B}_2$. By Axiom K1 we obtain $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$. By Axiom K5 we obtain $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$. Conversely, we assume $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$ and $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$. By Axiom K12 we obtain $\vdash_J \mathscr{B}_1$ & $\neg\mathscr{B}_2$. By Axiom K8a and K8b we obtain $\vdash_J \mathscr{B}_1$ and $\vdash_J \neg\mathscr{B}_2$. So by $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$ and $\vdash_J \mathscr{B}_1 \vdash_J \mathscr{B}_2$. By the inductive hypothesis, $\mathscr{B}_1$ is true or antinomic in $\mathfrak{M}$ and $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$. Therefore $\mathscr{B}_1 \supset \mathscr{B}_2$ is antinomic in $\mathfrak{M}$.

Subcase 2b. $\mathscr{B}_1 \supset \mathscr{B}_2$ is true but not antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$ and not-$\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$.

Proof: Assume that $\mathscr{B}_1 \supset \mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$. There are two subcases to consider. First, $\mathscr{B}_1$ is false and not antinomic in $\mathfrak{M}$ and $\mathscr{B}_2$ is true, false, or antinomic in $\mathfrak{M}$, and second, $\mathscr{B}_2$ is true and not antinomic and $\mathscr{B}_1$ is true, false, or antinomic in $\mathfrak{M}$. Let us first assume that $\mathscr{B}_1$ is false and not antinomic in $\mathfrak{M}$. By Case 1b and the fact that $\neg\mathscr{B}_1$ is true and not antinomic in $\mathfrak{M}$, it follows that $\vdash_J \neg\mathscr{B}_1$ and not-$\vdash_J \neg\neg\mathscr{B}_1$. $\neg\mathscr{B}_1$ is not antinomic in J, therefore—since $\neg\mathscr{B}_1$ is closed—by condition 4, $\neg\mathscr{B}_1$ is an $\mathscr{A}$-formula and $\mathscr{B}_1$ is also an $\mathscr{A}$-formula by condition 2b. By Axiom K1, $\vdash_J \neg\mathscr{B}_2 \supset \neg\mathscr{B}_1$, and by the tautology $(\neg\mathscr{B}_i \supset \neg\mathscr{A}_j) \supset (\mathscr{A}_j \supset \mathscr{B}_i)$, $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$. If $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$, then by Axiom K12, Axiom K8a, and Axiom K7a we have $\vdash_J \neg\neg\mathscr{B}_1$. This is impossible since it is not the case that $\vdash_J \neg\neg\mathscr{B}_1$. Thus not-$\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$. For the second case, let us assume that $\mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$. By the inductive hypothesis, $\vdash_J \mathscr{B}_2$ and not-$\vdash_J \neg\mathscr{B}_2$. By $\vdash_J \mathscr{B}_2$ and Axiom K1, $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$. If $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$, then by Axiom K12 and Axiom K8b, $\vdash_J \neg\mathscr{B}_2$. This is impossible, so not-$\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$. In both cases $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$ and not-$\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$. On the other hand, assume that $\vdash_J \mathscr{B}_1 \supset \mathscr{B}_2$ and not-$\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$. If $\mathscr{B}_1 \supset \mathscr{B}_2$ is antinomic in $\mathfrak{M}$, then $\mathscr{B}_2$ is antinomic and $\mathscr{B}_1$ is true (or antinomic). By the inductive hypothesis, $\vdash_J \mathscr{B}_2$, $\vdash_J \neg\mathscr{B}_2$, and $\vdash_J \mathscr{B}_1$ (or $\vdash_J \mathscr{B}_1$ and $\vdash_J \neg\mathscr{B}_1$). By Axiom K5, $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$, which is contrary to our assumption; so $\mathscr{B}_1 \supset \mathscr{B}_2$ is not antinomic in $\mathfrak{M}$. Now suppose that $\mathscr{B}_1 \supset \mathscr{B}_2$ is false but not antinomic in $\mathfrak{M}$. It follows that $\mathscr{B}_1$ is true and not antinomic in $\mathfrak{M}$ (or $\mathscr{B}_1$ is antinomic in $\mathfrak{M}$) and $\mathscr{B}_2$ is false and not antinomic in $\mathfrak{M}$. So by the inductive hypothesis and Case 1b, $\vdash_J \mathscr{B}_1$, not-$\vdash_J \neg\mathscr{B}_1$ (or $\vdash_J \mathscr{B}_1$ and $\vdash_J \neg\mathscr{B}_1$), and $\vdash_J \neg\mathscr{B}_2$ and not-$\vdash_J \neg\neg\mathscr{B}_2$. By Axiom K5, $\vdash_J \neg(\mathscr{B}_1 \supset \mathscr{B}_2)$, which is contrary to our assumption; so $\mathscr{B}_1 \supset \mathscr{B}_2$ cannot be false and not antinomic in $\mathfrak{M}$. Thus $\mathscr{B}_1 \supset \mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$.

Case 3. $\mathscr{B}_j$ is $\mathscr{B}_1$ & $\mathscr{B}_2$.

Subcase 3a. $\mathscr{B}_1$ & $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_1$ & $\mathscr{B}_2$ and $\vdash_J \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$.

Proof: If $\mathscr{B}_1$ & $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$, then one of the $\mathscr{B}$'s is true or antinomic in $\mathfrak{M}$ and the other is antinomic in $\mathfrak{M}$. Assume that $\mathscr{B}_1$ is true or antinomic in $\mathfrak{M}$ and $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$. By the inductive hypothesis, $\vdash_J \mathscr{B}_1$, $\vdash_J \neg\mathscr{B}_2$, and $\vdash_J \mathscr{B}_2$. By Axiom K6 we obtain $\vdash_J \mathscr{B}_1$ & $\mathscr{B}_2$. By Axiom K9b we obtain $\vdash_J \neg\mathscr{B}_1 \vee \neg\mathscr{B}_2$ and by Axiom K10 we obtain $\vdash_J \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$. Conversely, we assume $\vdash_J \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ and $\vdash_J \mathscr{B}_1$ & $\mathscr{B}_2$. By conditions 2a and 2b, if $\mathscr{B}_1$ and $\mathscr{B}_2$ are $\mathscr{A}$-formulas, then $\neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ is an $\mathscr{A}$-formula. Since $\neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ is a closed wf (because $\mathscr{B}_1$ and $\mathscr{B}_2$ are closed), then by condition 4 $\neg(\mathscr{B}_1$ & $\mathscr{B}_2)$ is not antinomic in J. So $\mathscr{B}_1$ (or $\mathscr{B}_2$) is not an $\mathscr{A}$-formula. If $\mathscr{B}_1$ (or $\mathscr{B}_2$) is not an $\mathscr{A}$-formula, then by condition 4 $\mathscr{B}_1$ (or $\mathscr{B}_2$) is antinomic in J. By the inductive hypothesis, $\mathscr{B}_1$ (or $\mathscr{B}_2$) is antinomic in $\mathfrak{M}$. Since $\vdash_J \mathscr{B}_1$ & $\mathscr{B}_2$, then by Axiom K8a and Axiom K8b $\vdash_J \mathscr{B}_1$ and $\vdash_J \mathscr{B}_2$. If $\mathscr{B}_2$ is not antinomic in J, then by the inductive hypothesis, $\mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$. Thus $\mathscr{B}_1$ & $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$.

Subcase 3b. $\mathscr{B}_1$ & $\mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$ if and only if $\vdash_J \mathscr{B}_1$ & $\mathscr{B}_2$ and not-$\vdash_J \neg(\mathscr{B}_1$ & $\mathscr{B}_2)$.

Proof: Assume that $\mathscr{B}_1 \,\&\, \mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$. Therefore $\mathscr{B}_1$ and $\mathscr{B}_2$ are true and not antinomic in $\mathfrak{M}$. By the inductive hypothesis, $\vdash_J \mathscr{B}_1$, not-$\vdash_J \neg\mathscr{B}_1$, $\vdash_J \mathscr{B}_2$, and not-$\vdash_J \neg\mathscr{B}_2$. So by condition 4, $\mathscr{B}_1$ and $\mathscr{B}_2$ are $\mathscr{A}$-formulas, and by condition 2a, $\mathscr{B}_1 \,\&\, \mathscr{B}_2$ is an $\mathscr{A}$-formula. By Axiom K6, $\vdash_J \mathscr{B}_1 \,\&\, \mathscr{B}_2$. If $\vdash_J \neg(\mathscr{B}_1 \,\&\, \mathscr{B}_2)$, then by Axiom K3a any formula $\mathscr{B}_k$ is provable, which contradicts the absolute consistency of J. Thus $\vdash_J \mathscr{B}_1 \,\&\, \mathscr{B}_2$ and not-$\vdash_J \neg(\mathscr{B}_1 \,\&\, \mathscr{B}_2)$. On the other hand, assume that $\vdash_J \mathscr{B}_1 \,\&\, \mathscr{B}_2$ and not-$\vdash_J \neg(\mathscr{B}_1 \,\&\, \mathscr{B}_2)$. First, if $\mathscr{B}_1 \,\&\, \mathscr{B}_2$ is antinomic in $\mathfrak{M}$, then $\mathscr{B}_1$ is antinomic in $\mathfrak{M}$ and $\mathscr{B}_2$ is true (or antinomic) in $\mathfrak{M}$. The argument is similar if $\mathscr{B}_1$ is true (or antinomic) in $\mathfrak{M}$ and $\mathscr{B}_2$ is antinomic in $\mathfrak{M}$. Therefore, by the inductive hypothesis, $\vdash_J \mathscr{B}_1$ and $\vdash_J \neg\mathscr{B}_1$, and by Axiom K9a $\vdash_J \neg\mathscr{B}_1 \vee \neg\mathscr{B}_2$. By Axiom K10, $\vdash_J \neg(\mathscr{B}_1 \,\&\, \mathscr{B}_2)$. But this is contrary to our assumption, so $\mathscr{B}_1 \,\&\, \mathscr{B}_2$ is not antinomic in $\mathfrak{M}$. Suppose now that $\mathscr{B}_1 \,\&\, \mathscr{B}_2$ is false but not antinomic in $\mathfrak{M}$, hence $\mathscr{B}_1$ (or $\mathscr{B}_2$) is false and not antinomic in $\mathfrak{M}$. By Case 1b, it follows that $\vdash_J \neg\mathscr{B}_1$ and not-$\vdash_J \neg\neg\mathscr{B}_1$. By Axiom K9a, $\vdash_J \neg\mathscr{B}_1 \vee \neg\mathscr{B}_2$, and by Axiom K10, $\vdash_J \neg(\mathscr{B}_1 \,\&\, \mathscr{B}_2)$, which is contrary to our assumption. Thus $\mathscr{B}_1 \,\&\, \mathscr{B}_2$ is true and not antinomic in $\mathfrak{M}$.

Case 4. $\mathscr{B}_j$ is $\mathscr{B}_1 \vee \mathscr{B}_2$.

Proof: Both subcases are similar to the subcases in Case 3.

Case 5. $\mathscr{B}_j$ is $(x_n)\mathscr{B}_h$.

Since we have an enumeration of all formulas with at most one free variable, we may assume that $\mathscr{B}_h$ is $\mathscr{B}_k(x_{ik})$.

Subcase 5a. $(x_n)\mathscr{B}_k(x_{ik})$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_J (x_n)\mathscr{B}_k(x_{ik})$ and $\vdash_J \neg(x_n)\mathscr{B}_k(x_{ik})$.

Assume first that x_n is not x_{ik}, then $\mathscr{B}_k(x_{ik})$ is closed and does not contain x_n free. If $\mathscr{B}_k(x_{ik})$ is closed, then it is clear from the definition of satisfiability that $(x_n)\mathscr{B}_k(x_{ik})$ is antinomic in $\mathfrak{M}$ if and only if $\mathscr{B}_k(x_{ik})$ is antinomic in $\mathfrak{M}$. We shall show that $(x_n)\mathscr{B}_k(x_{ik})$ is antinomic in J if and only if $\mathscr{B}_k(x_{ik})$ is antinomic in J. Assume $\mathscr{B}_j$ is antinomic in J. If $\mathscr{B}_k(x_{ik})$ is not antinomic in J, being a closed wf it is an $\mathscr{A}$-formula. Therefore, by condition 2a, $(x_n)\mathscr{B}_k(x_{ik})$ is also an $\mathscr{A}$-formula. Since $\vdash_J \mathscr{B}_j$ and $\vdash_J \neg\mathscr{B}_j$, then by Axiom K3a any wf $\mathscr{B}_l$ is provable in J, which contradicts the absolute consistency of J. Therefore $\mathscr{B}_k(x_{ik})$ is antinomic in J. On the other hand, assume that $\mathscr{B}_k(x_{ik})$ is antinomic in J. By generalization, $\vdash_J (x_n)\mathscr{B}_k(x_{ik})$, and by Axiom K16 and $\vdash_J \neg\mathscr{B}_k(x_{ik})$, we also have that $\vdash_J \neg(x_n)\mathscr{B}_k(x_{ik})$. Thus $\mathscr{B}_j$ is antinomic in J. By the inductive hypothesis, subcase 5a is proved for x_n different from x_{ik}.

Assume now that x_n is x_{ik}. If $(x_{ik})\mathscr{B}_k(x_{ik})$ is antinomic in $\mathfrak{M}$, then for some closed term t we have that $\mathscr{B}_k(t)$ is antinomic in $\mathfrak{M}$ by property P22. By the inductive hypothesis, $\vdash_J \mathscr{B}_k(t)$ and $\vdash_J \neg\mathscr{B}_k(t)$. We shall prove that $(x_{ik})\mathscr{B}_k(x_{ik})$ is not an $\mathscr{A}$-formula. Assume that it is, then by the completeness of J either $\vdash_J (x_{ik})\mathscr{B}_k(x_{ik})$ or $\vdash_J \neg(x_{ik})\mathscr{B}_k(x_{ik})$. If $\mathscr{B}_j$ is antinomic in J, then by Axiom K3a any wf $\mathscr{B}_l$ is provable in J contradicting the absolute consistency of J. Therefore $\mathscr{B}_j$ is not antinomic in J. Assume that $\vdash_J \mathscr{B}_j$ and not-$\vdash_J \neg \mathscr{B}_j$. Since $\vdash_J \neg \mathscr{B}_k(t)$, then by Axiom K16 $\vdash_J \neg (x_{ik}) \mathscr{B}_k(x_{ik})$, which contradicts the assumption. Assume now that $\vdash_J \neg \mathscr{B}_j$ and not-$\vdash_J \mathscr{B}_j$. Since $\mathscr{B}_j$ is not antinomic in J, (S_k) is $\mathscr{B}_k(b_{ik}) \supset (x_{ik}) \mathscr{B}_k(x_{ik})$. $\mathscr{B}_j$ is antinomic in $\mathfrak{M}$, then by property P22 and definition 6a of satisfiability, $\mathscr{B}_k(x_{ik})$ is true or antinomic in $\mathfrak{M}$. By the inductive hypothesis, it is the case that $\vdash_J \mathscr{B}_k(b_{ik})$ (or $\vdash_J \mathscr{B}_k(b_{ik})$ and $\vdash_J \neg \mathscr{B}_k(b_{ik})$), which together with (S_k) leads to $\vdash_J \mathscr{B}_j$, contrary to the assumption. Hence $\mathscr{B}_j$ is not an $\mathscr{A}$-formula, and it follows from condition 4 for $\mathscr{A}$-formulas that $\mathscr{B}_j$ is antinomic in J. On the other hand, assume that $(x_{ik}) \mathscr{B}_k(x_{ik})$ is antinomic in J. By Proposition 4.11 and the reasoning offered earlier, $(x_{ik}) \mathscr{B}_k(x_{ik})$ is antinomic in T_0. Hence (S_k) is $\neg (x_{ik}) \mathscr{B}_k(x_{ik}) \supset \neg \mathscr{B}_k(b_{ik})$ & $\mathscr{B}_k(b_{ik})$. Let t be any closed term of T_0. Since $\vdash_J (x_{ik}) \mathscr{B}_k(x_{ik})$, by Axiom K14 it is the case that $\vdash_J \mathscr{B}_k(t)$. By the inductive hypothesis, $\mathscr{B}_k(t)$ is true in $\mathfrak{M}$ for every t in the domain of $\mathfrak{M}$, therefore $(x_{ik}) \mathscr{B}_k(x_{ik})$ is true in $\mathfrak{M}$. By the inductive hypothesis, $\mathscr{B}_k(b_{ik})$ is antinomic in $\mathfrak{M}$ since $\mathscr{B}_k(b_{ik})$ is antinomic in J. Thus $(x_{ik}) \mathscr{B}_k(x_{ik})$ is antinomic in $\mathfrak{M}$ by property P22.

Subcase 5b. $(x_n) \mathscr{B}_k(x_{ik})$ is true and not antinomic in $\mathfrak{M}$ if and only if $\vdash_J (x_n) \mathscr{B}_k(x_{ik})$ and not-$\vdash_J \neg (x_n) \mathscr{B}_k(x_{ik})$.

Assume first that x_n is not x_{ik}, then $\mathscr{B}_k(x_{ik})$ is closed and $(x_n) \mathscr{B}_k(x_{ik})$ is true and not antinomic in $\mathfrak{M}$ if and only if $\mathscr{B}_k(x_{ik})$ is true and not antinomic in $\mathfrak{M}$ by definition of satisfiability. We shall prove that $\vdash_J (x_n) \mathscr{B}_k(x_{ik})$ if and only if $\vdash_J \mathscr{B}_k(x_{ik})$. Assume that $\vdash_J (x_n) \mathscr{B}_k(x_{ik})$; by Axiom K14, $\vdash_J \mathscr{B}_k(x_{ik})$. Conversely, assume that $\vdash_J \mathscr{B}_k(x_{ik})$, then by generalization, $\vdash_J (x_n) \mathscr{B}_k(x_{ik})$. By the inductive hypothesis, subcase 5b is proved for x_n different from x_{ik}.

Assume now that x_n is x_{ik}. Further, assume that $\mathscr{B}_j$ is true and not antinomic in $\mathfrak{M}$. Since J is complete, either $\vdash_J \mathscr{B}_j$ or $\vdash_J \neg \mathscr{B}_j$. We shall show that $\neg \mathscr{B}_j$ is not provable in J and $\mathscr{B}_j$ is not antinomic in J. Suppose that $\neg \mathscr{B}_j$ is provable in J. (S_k) is either $\mathscr{B}_k(b_{ik}) \supset (x_{ik}) \mathscr{B}_k(x_{ik})$ or $\neg (x_{ik}) \mathscr{B}_k(x_{ik}) \supset \neg \mathscr{B}_k(x_{ik})$ & $\mathscr{B}_k(x_{ik})$. By the tautology $(\mathscr{B}_1 \supset \mathscr{A}_1) \supset (\neg \mathscr{A}_1 \supset \neg \mathscr{B}_1)$, we obtain $\neg (x_{ik}) \mathscr{B}_k(x_{ik}) \supset \neg \mathscr{B}_k(b_{ik})$ whenever $\mathscr{B}_j$ is not antinomic (and therefore an $\mathscr{A}$-formula by condition 4). If $\neg \mathscr{B}_j$ is provable in J, whatever the form of (S_k) it follows that $\vdash_J \neg \mathscr{B}_k(b_{ik})$ (using Axiom K8a when necessary). If $\mathscr{B}_k(b_{ik})$ is antinomic in J, then by the inductive hypothesis, $\mathscr{B}_k(b_{ik})$ is antinomic in $\mathfrak{M}$. By property P22, $(x_{ik}) \mathscr{B}_k(x_{ik})$ is then antinomic in $\mathfrak{M}$, which is a contradiction. Therefore $\mathscr{B}_k(b_{ik})$ is not antinomic in J. If $\neg \mathscr{B}_k(b_{ik})$ is provable in J, then by Case 1b $\neg \mathscr{B}_k(b_{ik})$ is true and not antinomic in $\mathfrak{M}$. Since $(x_{ik}) \mathscr{B}_k(x_{ik})$ is true in $\mathfrak{M}$, $\mathscr{B}_k(b_{ik})$ is true in $\mathfrak{M}$. This would mean that $\mathscr{B}_k(b_{ik})$ is antinomic in $\mathfrak{M}$. By property P22, $(x_{ik}) \mathscr{B}_k(x_{ik})$ is again antinomic in $\mathfrak{M}$, which is a contradiction. Therefore $\neg \mathscr{B}_k(b_{ik})$ is not provable in J. Hence $\neg \mathscr{B}_j$ is not provable in J. That $\mathscr{B}_j$ is not antinomic in J is proved as follows. If $\mathscr{B}_j$ is antinomic in J, then (S_n) is $\neg (x_{ik}) \mathscr{B}_k(x_{ik}) \supset \neg \mathscr{B}_k(b_{ik})$ & $\mathscr{B}_k(b_{ik})$; as a consequence, $\mathscr{B}_k(b_{ik})$ is antinomic in J and therefore in $\mathfrak{M}$. Thus $\mathscr{B}_j$ is antinomic in $\mathfrak{M}$—a contradiction. Hence $\vdash_J \mathscr{B}_j$ and not-$\vdash_J \neg \mathscr{B}_j$. On the other hand, assume that $\vdash_J \mathscr{B}_j$ and not-$\vdash_J \neg \mathscr{B}_j$. First suppose that $\mathscr{B}_j$ is false and not antinomic in $\mathfrak{M}$. Then for some closed term t in $\mathfrak{M}$, $\mathscr{B}_k(t)$ is false and not antinomic in $\mathfrak{M}$. By Case 1b, $\vdash_J \neg \mathscr{B}_k(t)$ and not-$\vdash_J \neg\neg \mathscr{B}_k(t)$. Hence not-$\vdash_J \mathscr{B}_k(t)$.

By condition 4, $\mathscr{B}_k(t)$ is an $\mathscr{A}$-formula. By Axiom K14, $(x_{ik})\mathscr{B}_k(x_{ik}) \supset \mathscr{B}_k(t)$, and the tautology $(\mathscr{B}_1 \supset \mathscr{A}_1) \supset (\neg\mathscr{A}_1 \supset \neg\mathscr{B}_1)$, we obtain $\vdash_{\mathrm{J}} \neg(x_{ik})\mathscr{B}_k(x_{ik})$, which is a contradiction. So $\mathscr{B}_j$ cannot be false and not antinomic in $\mathfrak{M}$. Suppose now that $\mathscr{B}_j$ is antinomic in $\mathfrak{M}$, then for some closed term t, $\mathscr{B}_k(t)$ is antinomic in $\mathfrak{M}$. By the inductive hypothesis, $\mathscr{B}_k(t)$ is antinomic in J, but then $\vdash_{\mathrm{J}} \neg(x_{ik})\mathscr{B}_k(x_{ik})$ by Axiom K16, which is contrary to our assumption. Hence $\mathscr{B}_j$ cannot be antinomic in $\mathfrak{M}$ and must be true and not antinomic in $\mathfrak{M}$.

Proposition 4.13. *If a wf $\mathscr{B}_j$ is logically valid, then $\mathscr{B}_j$ is provable in* T.

Proof: We shall consider only closed wfs, since a wf $\mathscr{B}_j$ is logically valid if and only if its closure is logically valid, and $\mathscr{B}_j$ is provable in T if and only if its closure is provable in T. Assume that $\mathscr{B}_j$ is a logically valid closed wf. If $\mathscr{B}_j$ is antinomic in the theory T, then $\mathscr{B}_j$ is of course provable in T. Assume that $\mathscr{B}_j$ is not provable in T, then by Lemma 4.9 we can add $\neg\mathscr{B}_j$ to T to form the theory T$'$, which will be absolutely consistent. Suppose that $\neg\mathscr{B}_j \vdash_{\mathrm{T}} \mathscr{B}_j$. It follows that $\vdash_{\mathrm{T}} \neg\mathscr{B}_j \supset \mathscr{B}_j$ by the Deduction Theorem, and since $\mathscr{B}_j \supset \mathscr{B}_j$ is a tautology, we have $\vdash_{\mathrm{T}} \mathscr{B}_j$ by Axiom K4, which is contrary to our assumption. Therefore $\neg\mathscr{B}_j$ is not antinomic in T$'$ by Axiom K7b. By condition 4 for $\mathscr{A}$-formulas, $\neg\mathscr{B}_j$ is an $\mathscr{A}$-formula in T$'$. By Proposition 4.12, we can construct a model $\mathfrak{M}$ such that $\neg\mathscr{B}_j$ is true and not antinomic in $\mathfrak{M}$ if and only if $\vdash_{\mathrm{J}} \neg\mathscr{B}_j$ and not-$\vdash_{\mathrm{J}} \neg\neg\mathscr{B}_j$, and $\neg\mathscr{B}_j$ is antinomic in $\mathfrak{M}$ if and only if $\vdash_{\mathrm{J}} \neg\mathscr{B}_j$ and $\vdash_{\mathrm{J}} \neg\neg\mathscr{B}_j$, where J is the complete and absolutely consistent extension of T_∞ constructed as in the proof of Proposition 4.12. If $\neg\mathscr{B}_j$ is antinomic in $\mathfrak{M}$, then $\vdash_{\mathrm{J}} \neg\mathscr{B}_j$ and $\vdash_{\mathrm{J}} \neg\neg\mathscr{B}_j$. Since $\neg\mathscr{B}_j$ is an $\mathscr{A}$-formula in T$'$, then it must be an $\mathscr{A}$-formula in T_0' (where T_0' is T$'$ with the addition of the constants $b_1, b_2, \ldots$ as in the proof of Proposition 4.12). If this were not the case, then we would have proofs in T_0' of $\neg\mathscr{B}_j$ and $\mathscr{B}_j$. If we were to replace all occurrences of the b_i's by variables not occurring in the proofs of $\neg\mathscr{B}_j$ and $\mathscr{B}_j$, then the resulting proofs would be proofs in T$'$. But $\mathscr{B}_j$ is an $\mathscr{A}$-formula in T$'$, and by Axiom K3a any wf would be provable in T$'$, which contradicts the absolute consistency of T$'$. Hence $\neg\mathscr{B}_j$ is an $\mathscr{A}$-formula in T_0'. $\neg\mathscr{B}_j$ is also an $\mathscr{A}$-formula in T_∞' (see the proof of Proposition 4.12 and also Lemma 4.11), therefore $\neg\mathscr{B}_j$ is an $\mathscr{A}$-formula in J. But then by Axiom K3a any wf $\mathscr{B}_l$ would be provable in J. Since J is absolutely consistent, $\neg\mathscr{B}_j$ cannot be antinomic in $\mathfrak{M}$. Because $\neg\mathscr{B}_j$ is provable in J, it is true and not antinomic in $\mathfrak{M}$ by the proof of Proposition 4.12. Since $\mathscr{B}_j$ is logically valid, $\mathscr{B}_j$ is also true in $\mathfrak{M}$. $\mathscr{B}_j$ is not antinomic in $\mathfrak{M}$ because $\neg\mathscr{B}_j$ is not antinomic in $\mathfrak{M}$. Hence $\neg\mathscr{B}_j$ is both true and false, and also not antinomic in $\mathfrak{M}$, which is impossible. Thus, $\mathscr{B}_j$ must be a theorem of T.

Proposition 4.14. *A well-formed formula of* T *is logically valid if and only if it is a theorem of* T.

Proof: Propositions 4.7 and 4.13.

III. AN ANTINOMIC SET THEORY

5 Antinomic Sets and Ordinals We shall use two predicate letters, = and ϵ, the first an A-letter, the second a B-letter (see section 3). Definitions D1 to D4 introduce respectively inclusion, union, intersection, and complementation. These definitions are identical to those of classical set theory, except that here $\equiv$ is replaced by $\equiv^\circ$ in each definition. In addition, we need the following.

D5. $x \subset y \equiv^\circ (x \subseteq y \ \&\ x \neq y)$ (Proper Inclusion)
D6. $(t)(t \in \emptyset \equiv^\circ t \neq t)$ (Null Set)
D7. $(t)(t \in \mathsf{V} \equiv^\circ t = t)$ (Universal Set)
D8. $x =^* y \equiv^\circ (t)(t \in x \equiv^\circ t \in y)$
D9. $(t)(t \in \{x, y\} \equiv^\circ t =^* x \vee t =^* y)$ (Unordered Pair)
D10. $(t)(t \in \langle x, y\rangle \equiv^\circ t =^* \{x\} \vee t =^* \{x, y\})$ (Ordered Pair)
D11. $(t)(t \in w \times z \equiv^\circ t = ^* \langle u, v\rangle \ \&\ u \in w \ \&\ v \in z)$ (Cartesian Product)
D12. $\mathsf{Rel}(x) \equiv^\circ x \subseteq \mathsf{V} \times \mathsf{V}$ (Binary Relation)
D13. $(t)(t \in \mathsf{E} \equiv^\circ t =^* \langle u, v\rangle \ \&\ u \in v)$ ($\in$-relation)
D14. $x \ \mathsf{Irr}\ y \equiv^\circ (t)(t \in y \supset \langle t, t\rangle \notin x) \ \&\ \mathsf{Rel}(x)$ (x is an irreflexive relation on y)
D15. $x \ \mathsf{We}\ y \equiv^\circ \mathsf{Rel}(x) \ \&\ x \ \mathsf{Irr}\ y \ \&\ (z)(z \subseteq y \ \&\ (\exists t)(t \in z) \supset (\exists w)(w \in z \ \&$
$(v)(v \in z \ \&\ v \neq^* w \supset \langle w, v\rangle \in x \ \&\ (\langle w, v\rangle \in x \ \&\ \langle v, w\rangle \in x \supset$
$w =^* v))))$ (x well orders y)
D16. $\mathsf{Trans}(x) \equiv^\circ (t)(t \in x \supset t \subseteq x)$ (x is transitive)
D17. $\mathsf{O}(x) \equiv^\circ \mathsf{Trans}(x) \ \&\ \mathsf{E}\ \mathsf{We}\ x$
D18. $\mathsf{Ord}(x) \equiv^\circ \mathsf{O}(x) \ \&\ (y)(\mathsf{O}(y) \ \&\ y \in y \supset x \in y) \ \&\ (y)(\mathsf{O}(y) \ \&\ x \in x \supset y \in x) \ \&$
$(y)(t)(y \subseteq x \ \&\ t \in y \ \&\ t \notin y \supset y \subseteq t) \ \&\ (y)(y \in x \supset \mathsf{O}(y))$
(x is an ordinal)
D19. $x \in \mathsf{On} \equiv^\circ \mathsf{Ord}(x)$

Sets y for which there is a t such that $t \in y$ and $t \notin y$ will be called antinomic, and t will be called an antinomic member of y. Sets without antinomic members will be called consistent. Axioms for a Set Theory S follow (not all independent).

S1. $(x)(x = x)$
S2a. $(u)(v)(z)(v = z \supset (\mathscr{B}_1(u, v) \supset \mathscr{B}_1(u, z)))$
S2b. $(u)(v)(z)(u = z \supset (\mathscr{B}_1(u, v) \supset \mathscr{B}_1(z, v)))$

In Axioms S2a and S2b $\mathscr{B}_1$ is any wf containing an arbitrary finite number of free variables.

S3. $(x)(y)(z)(x = y \ \&\ y = z \supset x = z)$
S4. $(x)(y)(x =^* y \equiv x = y)$ (Axiom of Extensionality)

This axiom states that two sets are identical if they contain the same consistent and antinomic members.

S5. $(x)(y)(\exists z)(t)(t \in z \equiv^\circ t =^* x \vee t =^* y)$ (Axiom of Unordered Pairs)
S6. $(\exists z)(x)(x \in z \equiv^\circ x \neq x)$ (Null Set Axiom)
S7. $(x)(y)(\exists z)(t)(t \in z \equiv^\circ t \in x \ \&\ t \in y)$ (Intersection)
S8. $(x)(\exists y)(t)(t \in y \equiv^\circ t \notin x)$ (Complementation)
S9. $(\exists y)(x)(x \in y \equiv^\circ \mathscr{B}_1(x))$, *where x occurs free in $\mathscr{B}_1(x)$ but y does not, and where $\mathscr{B}_1(x)$ is a wf that contains only occurrences of $\in$, or only occurrences of $=$, or if it contains occurrences of both predicate letters then not-$\vdash \mathscr{B}_1(x) \equiv x \neq x$.* (Axiom of Comprehension)

Proposition 5.1. $\mathrm{Ord}(x) \supset x \notin x$ & $(u)(u \in x \supset u \notin u)$.

Proof: Since $\mathrm{Ord}(x)$, then E is Irreflexive on x; so $(u)(u \in x \supset u \notin u)$. If $x \in x$, then $x \notin x$.

Proposition 5.2. E Irr On.

Proof: Let $x \in \mathrm{On}$, then $\mathrm{Ord}(x)$. By Proposition 5.1 it follows that $x \notin x$.

Proposition 5.3. $\mathrm{Ord}(x)$ & $\mathrm{Trans}(y)$ & $y \subset x \supset y \in x$.

Proof: Let us consider the set z defined in the following way: $v \in z \equiv^{\circ} v \in x$ & $v \notin y$ (i.e., z is the intersection of x and the complement of y). It follows that $z \subseteq x$. Since $\mathrm{Ord}(x)$, x is well-ordered and therefore there is a w which is the least member of z (then, for all v, $v \neq^* w$ & $v \in z \supset w \in v$). Since $w \in x$, if $y =^* w$, then $y \in x$ by Axiom S4 and Axiom S2b. In order to prove $y =^* w$, the following cases should be proved (see definition of $y =^* x$): Case 1, $t \in y \supset t \in w$ (i.e., $y \subseteq w$); Case 2, $t \in w \supset t \in y$ (i.e., $w \subseteq y$); Case 3, $t \notin y \supset t \notin w$; Case 4, $t \notin w \supset t \notin y$.

Case 1. Assume $t \in y$ and $t \notin w$. Since x is well-ordered and the set $\{t, w\}$ is a subset of x, then $\{t, w\}$ has a least member. There are two cases to consider: (i) when it is the case that $t \notin w$ and $t \in w$; (ii) when it is not the case that $t \in w$. In either case we shall reach the conclusion that $x \in x$. First let us assume that $t \in w$ and $t \notin w$. Since $\mathrm{Ord}(x)$ and $w \in x$, then $t \in x$ and $\mathrm{O}(t)$. $\mathrm{Ord}(x)$, $w \in x$, $t \in w$, and $t \notin w$ imply $w \subseteq t$. But $t \in w$ and $w \subseteq t$ imply that $t \in t$. $\mathrm{Ord}(x)$, $\mathrm{O}(t)$, $t \in t$ imply $x \in t$. $\mathrm{Trans}(x)$, $x \in t$, and $t \in x$ imply $x \in x$. We shall now assume that it is not the case that $t \in w$, then $w \in t$ (there is a least member of the set $\{t, w\}$). $\mathrm{Trans}(y)$, $w \in t$, and $t \in y$ imply $w \in y$. Because $w \in z$, it follows from the definition of z that $w \notin y$ also. By the definition of $\mathrm{Ord}(x)$ it follows that $\mathrm{Ord}(x), y \subset x$, $w \in y$, and $w \notin y$ imply $y \subseteq w$. Since $w \in y$ and $y \subseteq w$, then $w \in w$. Since $w \in x$ and $\mathrm{Ord}(x)$, then $\mathrm{O}(w)$. By the definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $\mathrm{O}(w)$, and $w \in w$ imply $x \in w$. Since $x \in w$ and $w \in x$, then by transitivity of x, $x \in x$. Since x is well-ordered and y is a subset of x (by assumption), then y is well-ordered (a subset of a well-ordered set is well-ordered). Since by assumption $\mathrm{Trans}(y)$ and E We y, then $\mathrm{O}(y)$. By the definition of $\mathrm{Ord}(x), \mathrm{Ord}(x)$, $\mathrm{O}(y)$, and $x \in x$ imply $y \in x$. Also, $x \in w$ and $w \subseteq y$ imply $x \in y$ ($x \in w$ and $w \subseteq y$ in both cases above). Consider the set $\{x, y\}$, which is a well-ordered subset of x. Thus the set $\{x, y\}$ has a least member. By the definition of well-ordering, $x \neq^* y$, $x \in y$, and $y \in x$ imply $x =^* y$. If $x =^* y$, then by Axiom S4 $x = y$. Therefore by assuming $t \notin w$ we obtain $y = x$ (i.e., $t \notin w \supset x = y$). Since $x = y$ is an $\mathcal{A}$-formula, it follows by the tautology $(\neg \mathcal{B}_1 \supset \mathcal{A}_1) \supset (\neg \mathcal{A}_1 \supset \mathcal{B}_1)$ (in which $\mathcal{B}_1$ is $t \in w$ and $\mathcal{A}_1$ is $x = y$) that $x \neq y \supset t \in w$. By assumption y is a proper subset of x, so $x \neq y$ by the definition of proper subset. Thus $x \neq y$ and $x \neq y \supset t \in w$ imply $t \in w$.

Case 2. Assume $t \in w$ and $t \notin y$. $t \in x$ and $t \notin y$ by assumption, so it follows from the definition of z that $t \in z$. Since $t \in z$, then $w \in t$ (where w is the least member of z). Consider the set $\{t, w\}$, which is a subset of x. x is well-ordered because $\mathrm{Ord}(x)$, therefore $\{t, w\}$ has a least member. As in

Case 1, $t \in w$ and $w \in t$ imply $t = w$. By Axiom S2b, $t \in w$ and $t = w$ imply $w \in w$. By definition of $\mathrm{Ord}(x)$, $w \in x$ implies $\mathrm{O}(w)$. By definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $\mathrm{O}(w)$, and $w \in w$ imply $x \in w$. Consider the set $\{x, w\}$, which is a subset of the well-ordered set w. The set $\{x, w\}$ has a least member. Again, as in Case 1, $x \in w$ and $w \in x$ imply $w = x$; furthermore, also as in Case 1, $\mathrm{O}(y)$. By Axiom S2a, $w = x$ and $x \in w$ imply $x \in x$. $\mathrm{Ord}(x)$, $\mathrm{O}(y)$, and $x \in x$ imply $y \in x$ by the definition of $\mathrm{Ord}(x)$. Let us consider the two cases $y \in y$ and $y \notin y$. If $y \in y$ then, since $\mathrm{Ord}(x)$ and $\mathrm{O}(y)$, it follows from the definition of $\mathrm{Ord}(x)$ that $x \in y$. Consider the set $\{x, y\}$, which is a subset of y. Since y is well-ordered, there is a least member of $\{x, y\}$. By the same argument used in Case 1, $x \in y$ and $y \in x$ imply $y = x$. If $y \notin y$, and since $y \in x$, then by the definition of z, $y \in z$. It follows then that $w \in y$ because w is the least member of z. By Axiom S2b, $w = x$ and $w \in y$ imply $x \in y$. Again by considering the set $\{x, y\}$, it follows through the same reasoning used above that $x \in y$ and $y \in x$ imply $y = x$. By Axiom K4 (where $\mathcal{B}_1$ is $y \in y$ and $\mathcal{B}_2$ is $y = x$), it follows that $y = x$. So by assuming $t \notin y$, we obtain $y = x$ (i.e., $t \notin y \supset y = x$). By the tautology $(\neg \mathcal{B}_1 \supset \mathcal{A}_1) \supset (\neg \mathcal{A}_1 \supset \mathcal{B}_1)$ (where $\mathcal{B}_1$ is $t \in y$ and $\mathcal{A}_1$ is $y = x$), and because $y \neq x$ (since y is a proper subset of x), then $t \in y$. Thus $t \in w \supset t \in y$.

Case 3. $t \notin y \supset t \notin w$. Assume $t \notin y$ and $t \in w$, then $y = x$ where the proof is like that in Case 2 (i.e., $t \in w \supset y = x$). By the tautology $(\mathcal{B}_1 \supset \mathcal{A}_1) \supset (\neg \mathcal{A}_1 \supset \neg \mathcal{B}_1)$ (where $\mathcal{B}_1$ is $t \in w$ and $\mathcal{A}_1$ is $y = x$), and because $y \neq x$, it follows that $t \notin w$.

Case 4. $t \notin w \supset t \notin y$. Assume $t \notin w$ and $t \in y$. Assuming that $t \in y$, we obtain $y = x$ (i.e., $t \in y \supset y = x$) with the same proof used in Case 1. By the tautology $(\mathcal{B}_1 \supset \mathcal{A}_1) \supset (\neg \mathcal{A}_1 \supset \neg \mathcal{B}_1)$ (where $\mathcal{B}_1$ is $t \in y$ and $\mathcal{A}_1$ is $y = x$), and the fact that $y \neq x$, it follows that $t \notin y$.

Proposition 5.4. $\mathrm{Ord}(x) \;\&\; \mathrm{Ord}(y) \supset (x \in y \vee y \in x \vee x = y)$.

Proof: Assume $\mathrm{Ord}(x)$, $\mathrm{Ord}(y)$, and $x \neq y$. Now $x \cap y \subseteq x$ and $x \cap y \subseteq y$. Since x and y are transitive and well-ordered, then $x \cap y$ is transitive and well-ordered; so $\mathrm{O}(x \cap y)$. If $x \cap y \subset x$ and $x \cap y \subset y$, then $x \cap y \in x$ and $x \cap y \in y$ by Proposition 5.3, hence $x \cap y \in x \cap y$. By definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $\mathrm{O}(x \cap y)$, and $x \cap y \in x \cap y$ imply $x \in x \cap y$, and similarly for $\mathrm{Ord}(y)$, $y \in x \cap y$. The set $\{x, x \cap y\} \subseteq x \cap y$ and contains a least member because x is well-ordered. As in the reasoning used in Case 1 of Proposition 5.3, $x \in x \cap y$ and $x \cap y \in x$ imply $x = x \cap y$; similarly, $y = x \cap y$. By Axiom S3, $x = y$, which is contrary to our assumption. Thus $x \cap y \subseteq x$ or $x \cap y \subseteq y$. Since $x \neq y$, then $x \cap y \subset x$ or $x \cap y \subset y$; so $x \subset y$ or $y \subset x$. Thus by Proposition 5.3, $x \in y$ or $y \in x$.

Proposition 5.5. $\mathrm{Ord}(x) \;\&\; y \in x \supset \mathrm{Ord}(y)$.

Proof: Since by assumption $y \in x$, then $\mathrm{O}(y)$ by the definition of $\mathrm{Ord}(x)$. In order to prove that $\mathrm{Ord}(y)$ we must prove the following:

(1) $(z)(\mathrm{O}(z) \;\&\; z \in z \supset y \in z)$;
(2) $(z)(\mathrm{O}(z) \;\&\; y \in y \supset z \in y)$;

(3) $(z)(t)(z \subseteq y \;\&\; t \in z \;\&\; t \notin z \supset z \subseteq t)$;
(4) $(u)(u \in y \supset \mathrm{O}(u))$.

(1) Assume $\mathrm{O}(z)$ and $z \in z$. By the definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $\mathrm{O}(z)$, and $z \in z$ imply $x \in z$. Since $y \in x$, $x \in z$, and $\mathrm{Trans}(z)$, then $y \in z$. Hence $(z)(\mathrm{O}(z) \;\&\; z \in z \supset y \in z)$.

(2) Assume $\mathrm{O}(z)$ and $y \in y$. By the definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $\mathrm{O}(y)$, and $y \in y$ imply $x \in y$. Since $\mathrm{O}(y)$, then y is well-ordered, so the set $\{x, y\}$, which is a subset of y, has a least member. As in Case 1 of Proposition 5.3, $x \in y$ and $y \in x$ imply $x = y$. By Axiom S2a, $x = y$ and $x \in y$ imply $x \in x$. By the definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $\mathrm{O}(z)$, and $x \in x$ imply $z \in x$. By Axiom S2a, $z \in x$ and $x = y$ imply $z \in y$. Hence $(z)(\mathrm{O}(z) \;\&\; y \in y \supset z \in y)$.

(3) Assume $z \subseteq y$, $t \in z$, and $t \notin z$. Since by assumption $y \in x$, the transitivity of x implies $y \subseteq x$, so because $z \subseteq y$ and $y \subseteq x$ it follows that $z \subseteq x$. By the definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$, $z \subseteq x$, $t \in z$, and $t \notin z$ imply $z \subseteq t$. Hence $(z)(t)(z \subseteq y \;\&\; t \in z \;\&\; t \notin z \supset z \subseteq t)$.

(4) Assume $u \in y$. Since $\mathrm{Trans}(x)$, then $u \in y$ and $y \in x$ imply $u \in x$. By the definition of $\mathrm{Ord}(x)$, $\mathrm{Ord}(x)$ and $u \in x$ imply $\mathrm{O}(u)$. Hence $(u)(u \in y \supset \mathrm{O}(u))$.

Proposition 5.6. $\mathrm{E\ We\ On}$.

Proof: Assume $x \subseteq \mathrm{On}$ and $x \neq^* \emptyset$. If y is the least member of x, then the proof is finished. If y is a member of x but not its least member, then $x \cap y \neq^* \emptyset$. Since $x \cap y \subseteq y$ and y is an ordinal, then $x \cap y$ is well-ordered. Let w be the least member of $x \cap y$; w is an ordinal because $w \in x$. If t is any ordinal of x, then by Proposition 5.4 $t \in w$ or $w \in t$ or $t = w$. If for all t in x, $w \in t$ or $w = t$, then w is the least member of x. On the other hand, if for some t $t \in w$, we shall show that $w = t$. Assume $t \in w$. Since $t \in w$ and $w \in y$, then $t \in y$ by the transitivity of y. Thus $t \in x \cap y$; so it follows that $w \in t$, since w is the least member of $x \cap y$. Consider the set $\{w, t\}$ included in y. The set y is well-ordered because y is an ordinal, so $\{w, t\}$ has a least member. As in Case 1 of Proposition 5.3, $w \in t$ and $t \in w$ imply $t = w$. Thus w is the least member of x.

Proposition 5.7. $\mathrm{Trans}(\mathrm{On})$.

Proof: Assume $u \in \mathrm{On}$. We shall show that if $v \in u$, then $v \in \mathrm{On}$. Since $u \in \mathrm{On}$, then by definition of On it follows that $\mathrm{Ord}(u)$. By Proposition 5.5, $v \in u$ and $\mathrm{Ord}(u)$ imply $\mathrm{Ord}(v)$, therefore $v \in \mathrm{On}$.

Proposition 5.8. $\mathrm{O}(\mathrm{On})$.

Proof: By Propositions 5.6 and 5.7.

Proposition 5.9. $(\mathrm{Ord}(x) \;\&\; x \in x \;\&\; \mathrm{Ord}(y) \;\&\; y \in y) \supset x = y$.

Proof: x and y are any two ordinals that are members of themselves. Consider the set $\{x, y\} \subseteq \mathrm{On}$. Since On is well-ordered, the set $\{x, y\}$ has a least member. By the definition of ordinal, from $\mathrm{Ord}(x), \mathrm{Ord}(y)$, $x \in x$, and $y \in y$ we obtain $x \in y$ and $y \in x$. By the same argument used in Case 1 of Proposition 5.3, $x \in y$ and $y \in x$ imply $x = y$.

Proposition 5.10. $\mathrm{Ord}(w)$ & $w \in w \supset w = \mathrm{On}$.

Proof: Since $\mathrm{Ord}(w)$, $w \in w$, and $\mathrm{O}(\mathrm{On})$, it follows from the definition of $\mathrm{Ord}(w)$ that $\mathrm{On} \in w$. Consider the set $\{\mathrm{On}, w\} \subseteq w$, which has a least member. As in Case 1 of Proposition 5.3, $w \in \mathrm{On}$ and $\mathrm{On} \in w$ imply $w = \mathrm{On}$.

Proposition 5.11. $\mathrm{Ord}(\mathrm{On})$ *and hence* $\mathrm{On} \in \mathrm{On}$.

Proof: By Axiom S9, there is a w such that $w \in w$ and $\mathrm{Ord}(w)$. $\mathrm{On} = w$ follows from Proposition 5.10. Therefore by Axiom S2b, $\mathrm{Ord}(\mathrm{On})$.

Proposition 5.12. $(\exists y)(x)(x \in y \equiv^{\circ} x \notin x)$.

Proof: $x \notin x$ satisfies the conditions of the Comprehension Axiom.

The set y in Proposition 5.12 is Russell's antinomic set, which in this theory does not have the same radical consequences that it had in classical set theory.

Since the set of $\mathcal{A}$-formulas of our antinomic predicate calculus is not recursively decidable, it is not possible to assign to every one of them an identifiable Gödel number. Therefore, it is not possible to arithmetize the metamathematics of systems based on such a calculus and as a consequence the proof of Gödel's incompleteness theorem does not apply to these systems. Furthermore, having admitted inconsistencies, it is now possible for set theory to be complete. Which specific axioms are required to produce such a complete set theory is still an open question.

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