Notre Dame Journal of Formal Logic
Volume XV, Number 4, October 1974
NDJFAM

DESCRIPTION THEORY: CRITICAL DEFENSE OF A RUSSELLIAN APPROACH

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1 Scope of the Present Study The present paper contains a systematic discussion of a standard version of Russell's theory of descriptions (this is not another historical study on Russell or descriptions). More specifically, we will briefly present a formal version of the theory which will be argued to be both formally and materially adequate. In the latter respect, we will be concerned with showing that the theory is flexible enough in applications to ordinary discourse to accommodate alternative methods of paraphrase.

The adequacy of Russell's theory has repeatedly been challenged at both the formal and the applied levels. In the present account the core of Russell's well-known method of contextual definition for descriptions will be retained. At the formal level, we will show that the method suffices to justify a law of *Substitutivity of Identity* for descriptions; this law in turn allows us to extend standard quantification rules to description containing contexts in general. At the applied level, we will indicate how the adroit use of certain special contexts of descriptions (called *primary* contexts) can secure maximum flexibility in formalization.

2 The Basic Theory Quine has in effect proposed the following law

$$(y)((x)(Fx \equiv x = y) \equiv (\iota x)Fx = y)$$

as a fundamental condition of adequacy for any theory of descriptions.[1] In addition to the above, I would like to mention also the following as equally basic:

$$((y)Fy \cdot (\exists y)((\iota x)Gx = y)) \supset F(\iota x)Gx \qquad \text{RUI}$$
$$(F(\iota x)Gx \cdot (\exists y)((\iota x)Gx = y)) \supset (\exists y)Fy \qquad \text{REG}$$

As their names indicate, RUI and REG are *restricted* analogues of standard universal instantiation and existential generalization, respectively. They differ from the standard laws only in containing the additional clause: '$(\exists y)((\iota x)Gx = y)$'. This well-known method for formalizing claims of the form 'The so-and-so exists' goes back to Russell. To see that it does

Received January 12, 1973

capture *nontrivially* the intended import of such claims, we must bear in mind the status of the description as a *nonbindable* singular term (whether the method can adequately be extended to *names* generally need not concern us here). In short, the overall rationale for RUI and REG is to allow instantiation and generalization with respect to descriptions under the proviso that the descriptions be *referential*, i.e., have referents in the domain of interpretation.

Now, in order to obtain the above laws for descriptions, we need as our underlying logic standard quantification with identity[2] supplemented by the following *definitional schema*:

$$`G(\iota x)(Fx)\text{'} =_{Df} `(\exists y)(Gy \cdot (x)(Fx \equiv x = y))\text{'} \qquad \text{DS}$$

To secure unique eliminability of '$(\iota x)(Fx)$' from all contexts, we can resort to Quine's expedient of limiting 'G' to *atomic* replacements (predicates that are not further analyzable into truth-functional or quantificational components).[3] It will be shown below that this expedient not only is formally adequate but also provides a very flexible basis for applications.

Other provisos and explanations concerning DS are in order here. First, the parentheses enclosing 'Fx' in the *definiendum* are required to avoid systematic ambiguity in cases where 'G' is taken as a *polyadic* predicate (e.g., '$H(\iota x)Fxz$').[4] Such parentheses may of course generally be suppressed in practice. Secondly, it is to be understood that no free 'y' in 'F' gets captured by the outlying quantifier, '$(\exists y)$'. And finally, we make a provision to the effect that wherever two or more descriptions (or occurrences of the same description) are to be eliminated from the same atomic context, the elimination proceeds from left to right; in the reverse process, i.e., that of introduction of the *definiens*, the procedure should correspondingly be from right to left. This third proviso, however, is for the purist (the curious reader can verify for himself that introduction or elimination in whatever order is in this case equivalence preserving).

3 Proof of Logical Adequacy That DS satisfies the fundamental law for descriptions may easily be verified by taking 'G' as '$= z$'.[5] We get by identity:

Th1 $(y)((\iota x)Fx = y) \equiv (x)(Fx \equiv x = y)$

Two consequences of Th1 worth noting here are:

Th2 $(\exists y)((\iota x)Fx = y) \equiv (\exists y)(x)(Fx \equiv x = y)$

i.e., "The so-and-so exists just in case there is a unique so-and-so"; and:

Th3 $\sim(\exists y)((\iota x)Fx = y) \equiv (y)(\sim Fy \vee (\exists x)(Fx \cdot x \neq y))$

i.e., "The so-and-so does not exist just in case either there is no so-and-so or there is more than one." Evidently, these basic laws are desiderata in any theory of descriptions.

Let us now address ourselves to the problem of proving RUI and REG on the basis of the underlying logic supplemented by DS. But we will need to prove first a more general principle, central to any logic of terms:

Th4 $a = b \supset (Fa \equiv Fb)$

i.e., the *Substitutivity of Identity*. We will prove that Th4 holds for any replacements of 'a' and 'b' by descriptions or variables in any context 'F' however complex (provided it is *extensional*).[6] Furthermore—and this is the philosophically interesting point—it will be shown that Th4 obtains irrespective of the semantical status of descriptions occurring therein.

As a preliminary to the proof, we will describe a systematic method for assigning a numerical value to the degree of logical complexity of a formula. Suppose, then, that our primitive basis consists of negation, the conditional, universal quantification and identity, together with an unspecified number of atomic predicates. We can recursively define the complexity value of a formula as follows:

(1) If 'P' is atomic, its complexity value is 0.
(2) If 'Fx' has complexity value n, '$(x)Fx$' has complexity value $n + 1$.
(3) If 'P' has complexity value n, '$\sim P$' has complexity value $n + 1$.
(4) If 'P' has a complexity value of m, and 'Q' of n, and moreover $m \leqslant n$, then each of '$P \supset Q$' and '$Q \supset P$' has a complexity value of $n + 1$.

We are now ready for the proof of Th4. The two major steps to be established are that

I. Th4 obtains for all atomic contexts, 'F' (*Base Case*);
II. Th4 obtains for all atomic contexts, 'F', having complexity value $k + 1$ whenever it obtains for all contexts of complexity value, k. (*Inductive Step*).

If we can establish I and II, we will have shown by (weak) induction that Th4 holds for *all* contexts, 'F'.

Base Case: 'F' depicts an atomic context.
Subcase 1: 'a' and 'b' are both variables. We get Th4 by the underlying theory of identity.
Subcase 2: 'a' is a description and 'b' a variable. We leave it to the reader to satisfy himself by elementary logic that the closure of

$$(\iota x)Gx = y \supset (F(\iota x)Gx \equiv Fy)$$

can be derived as a theorem on the basis of DS.
Subcase 3: 'a' is a variable and 'b' a description. This case turns out to be a mere corollary of the preceding one by virtue of the equivalence:

$$(\iota x)Gx = y \equiv y = (\iota x)Gx$$

Subcase 4: 'a' and 'b' are both descriptions. Again the reader can verify for himself that the following is provable:

$$(\iota x)Gx = (\iota x)Hx \supset (F(\iota x)Gx \equiv F(\iota x)Hx)$$

Inductive Step: Let 'F^n' in general represent (stand for) a predicate of complexity value, n. We will consider four basic ways in which

(i) $a = b \supset (F^{k+1}{}_a \equiv F^{k+1}{}_b)$

can be derived from

(ii) $a = b \supset (F^k{}_a \equiv F^k{}_b)$

Cases 1, 2, and 3: Let '$F^{k+1}{}_a$' stand for '$\sim F^k{}_a$' or '$F^k{}_a \supset P$' or '$P \supset F^k{}_a$' (for some formula 'P' of degree j, $j \leqslant k$). And similarly for '$F^{k+1}{}_b$'. (i) then follows from (ii) truth-functionally.
Case 4: Let '$F^{k+1}{}_a$' represent '$(z)F^k{}_a$' (assuming 'z' has a free occurrence in 'F^k'). Similarly for '$F^{k+1}{}_b$'. By elementary logic:

(1)	$a = b \supset (F^k{}_a \equiv F^k{}_b)$	Hypothesis
(2)	$a = b \supset (z)(F^k{}_a \equiv F^k{}_b)$	from (1)
(3)	$a = b \supset ((z)F^k{}_a \equiv (z)F^k{}_b)$	from (2)

This completes our proof of Th4. The derivations of RUI and REG on this basis present no problem. Taking 'a' as '$(\iota x)Gx$' and 'b' as 'y', we obtain the following instance of Th4:

$$(\iota x)Gx = y \supset (F(\iota x)Gx \equiv Fy)$$

Both RUI and REG can be derived from the above by elementary logical transformations.

4 Applications to Ordinary Discourse We are now in a position to assess the practical ramifications of the formal theory. The clue to the successful application of the apparatus of description theory lies, as we hope to show, in the use of the fundamental law, Th1. Th1 indicates how any description occurring in the context '$(\iota x)Fx = y$' can be systematically eliminated *without the detour of* DS. Because of their great usefulness in applications to ordinary discourse, such contexts of descriptions will be given a special name: they will be called *primary* contexts.

The fundamental technique of formalization using primary contexts may roughly be described as follows[7]:

Stage One: Paraphrase into the idiom of quantification with all descriptions confined to primary contexts;
Stage Two: Eliminate descriptions from primary contexts *via* Th1.
The above technique will be referred to briefly as the *method of confinement.* A few typical examples will make abundantly clear the utmost flexibility of the standard theory of descriptions when applied in conformity with the method of confinement.

(1) The square circle is square.
(2) Pegasus does not fly.[8]
(3) Pegasus is Pegasus.

Let me insist right away that I do not intend to defend any *a priori* preconceptions concerning what "must be" (or "cannot be") the truth-value of any of the above examples. My concern rather is to indicate how the formal theory of descriptions is *neutral* with regards to alternative and at the same time mutually compatible construals of each of (1)-(3).

Let us proceed straight to the first example. A standard reading suggests the following formalization (in line with the method of confinement):

(a) $(\exists y)((\iota x)(Cx \cdot Sx) = y \cdot Sy)$

Now, it may be rashly concluded on the basis of (a) that (1) must be false. But this conclusion cannot be justified unless we are also prepared to regard (a) as *the* correct (the only possible) formalization of (1). Such a view of the situation springs from too rigid an attitude concerning the interrelationships between grammatical and logical form.[9] A reader with a preconception to the effect that (to put it lamely) "(1) is true because if there were any square circle it would be square," can also accommodate *his* intuitions (without resorting to a dubious counterfactual in his formalized sentence!). Moreover, he can do so without contradicting (a), through the following version:

(b) $(y)((\iota x)(Cx \cdot Sx) = y \supset Sy)$

(b) happens to be logically true and carries no commitment to the existence of any square circles.[10] What is more, it is logically compatible with (a). Again, however, it is futile to argue which is the correct formalization of (1). The point is that the English sentence lends itself to either interpretation (and perhaps other interpretations as well).

Consider now example (2). Readers familiar with Quine's well-known textbook[11] will recall how the author handles this very same example. Having in effect analyzed

(4) Pegasus flies.

as

(c) $(\exists y)((\iota x)Px = y \cdot Fy)$

he then interprets (2) as the negation of (c) and consequently true (given the nonexistence of Pegasus). Here again we witness too close an adherence to ordinary grammar. Of course nothing prevents us from construing 'Pegasus does not fly' as the (logical) negation of 'Pegasus flies.' But neither are we constrained to interpret the pair as contradictories. We could just as well have formalized (2) by:

(d) $(\exists y)((\iota x)Px = y \cdot \sim Fy)$

which makes 'Pegasus does not fly' come out no less false than 'Pegasus flies!'[12]

But that is not all. It is also instructive to note that we could have so formalized (2) and (4) as to make them come out equally true. Using the same method as in the analysis of (1):

(e) $(y)((\iota x)Px = y \supset \sim Fy)$
(f) $(y)((\iota x)Px = y \supset Fy)$

These vacuous truths are as innocuous as conditionals with false antecedents generally.

Our final example, (3), is probably the one which has raised the most dust.[13] On the face of it, (3) looks as patently true as 'Socrates is Socrates.' But if we are anxious to save the appearances here, we will beware of translating (3) by:

(g) $(\iota x)Px = (\iota x)Px$

which, given DS, is patently false. But there are alternative ways of handling (3) which succeed in analyzing it as a *truism*. Indeed the alert reader will probably have already guessed what alternative I had in mind:

(h) $(y)((\iota x)Px = y \supset (\iota x)Px = y)$

and notice that (h) works for any description—irrespective of semantical status.

5 Conclusion I have tried to show not that Russell's theory of descriptions is "the right one" but that it is logically adequate and furthermore can successfully cope with certain moot problems of application. What more can be demanded of a theory?

NOTES

1. For Quine's perceptive discussion of descriptions, see [3], pp. 181-190; and [4], pp. 227-234.
2. Any system of standard (referential) quantification will serve equally well. Also it is irrelevant to our purposes whether the underlying logic countenances primitive names, not reducible to descriptions.
3. See [4], pp. 231-233. Note that atomicity is relative to a particular formalization.
4. This was brought to my attention by Prof. R. Barrett.
5. In formal languages where identity occurs as a defined predicate (hence, not atomic) the derivation of Th1 will obviously be more devious.
6. The proof may suitably be adjusted to accommodate primitive names.
7. For axiomatic treatments of description theory based directly on primary contexts, see [2] and [5]. But the idea behind the technique of *application* comes essentially from [3], pp. 181-190.
8. Note for the captious reader: in this and subsequent contexts, 'Pegasus' is being used as an abbreviation for a description ('The winged-horse captured by Bellerophon', if you like).
9. For more on this theme, see especially [1], section 6.
10. The reader may have noticed that the method exemplified in (b) is reminiscent of a standard practice in traditional logic of construing singular statements as "*A* propositions". Notice also that (b) is an "*A* proposition" without existential pre-suppositions.

11. [4], *ibid.*

12. Recall how Russell would handle (2) with his distinction between "primary" and "secondary" occurrences of descriptions. (d) in effect analyzes 'Pegasus' as having "primary occurrence" in (2). (This use of 'primary' must of course not be confused with what we are calling 'primary contexts'.)

13. For a critical discussion, see [1], *ibid.*

REFERENCES

[1] Goe, G., "Reconstructing formal logic: Further developments and considerations," *Notre Dame Journal of Formal Logic*, vol. XI (1970), pp. 37-75.

[2] Hintikka, K. J. J., "Towards a theory of definite descriptions," *Analysis*, vol. 19 (1959), pp. 79-85.

[3] Quine, W. V., *Word and Object*, The M.I.T. Press, Cambridge, Massachusetts (1960).

[4] Quine, W. V., *Methods of Logic*, Third Edition, Holt, Rinehart and Winston, Inc., New York (1972).

[5] Van Fraassen, B. C., "The completeness of free logic," *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 12 (1966), pp. 219-234.

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