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A SIMPLE PROOF OF HERBRAND'S THEOREM

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Hilbert and Bernays (*cf.*, *Grundlagen der Mathematik*, vol. II) prove Herbrand's theorem using their first ε-theorem. We can avoid this step by employing, instead of a Hilbert-type formalization of logic, the Beth's tableaux. But as we cannot dispense with function signs, we must supplement their usual rules in three cases. a) Existential quantifier to the right: in the next line of the tableau we must write down all substitutions of the corresponding bound variable by terms built up out of all free variables and function signs which have already occurred in the tableau. We get in the general case an effectively denumerable list of formulae; the strict finitistic character of Beth's tableaux is lost but the procedure is thoroughly constructive. b) Universal quantifier to the left: the corresponding change. c) Cancellation of identical members of a disjunction.

The proof of completeness of the semi-formal Beth's tableaux follows the well known pattern.

Given a formula of quantificational logic in prenex normal form

prefix nucleus

$$\bigwedge_x \bigwedge_y \bigvee_z \bigwedge_h \bigvee_m \bigwedge_l \mathfrak{A}(x, y, z, h, m, l) \tag{1}$$

According to Hilbert and Bernays' proof, we must substitute in the nucleus the variables which are bound by universal quantifiers occurring at the beginning of the prefix by different free variables not occurring in (1). All other variables of the nucleus which are bound by universal quantifiers must be substituted by different function signs which have as many arguments as there are existential quantifiers in the prefix preceding the corresponding universal quantifier. The arguments of these function signs must be filled by the bound variables of the preceding existential quantifiers. In this way we get

$$\bigvee_z \bigvee_m \mathfrak{A}(a_1, a_2, z, \phi(z), m, \psi(z, m)) \tag{2}$$

Let us suppose that (1) has a closed development (*cf.*, P. Lorenzen: "Dialogkalküle," *Archiv für mathematische Logik und Grundlagenforschung*, Bd. 15/3-4, (1972)), then (2) has also a closed development. In its n'th line

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(n is the number of existential quantifiers in (2)) we have a list of formulae obtained from the nucleus of (2) by the substitution of its bound variables by terms out of the list: $a_1, a_2, \phi(a_1), \psi(a_1, a_2) \ldots$. In the closed development of (2) beneath the n'th line, there are only a finite number of bifurcations and each branch has only a finite number of formulae. It follows that only a finite number of formulae in the infinite list of the n'th line have really been used. We build a finite disjunction thereof; this disjunction has also a closed development. Hilbert and Bernays use their first ε-theorem for this last step.

We sketch now the rest of the proof; for details we refer to Hilbert and Bernays. Let us suppose that a disjunction, built up out of the nucleus $\mathfrak{A}(x, y, z, h, m, l)$ by the substitution method described above, has a closed development. If we consider the terms containing function signs as indices of new free variables (for example, we consider $\phi\phi(a_1)$ as $a_{\phi\phi(a_1)}$) then this new formula has also a closed development. Using the three rules

$$\text{a)}\ \frac{\mathfrak{A} \vee \mathfrak{B}(a_1) \vee \mathfrak{C}}{\mathfrak{A} \vee \bigvee x\ \mathfrak{B}(x) \vee \mathfrak{C}} \qquad \text{b)}\ \frac{\mathfrak{A} \vee \mathfrak{B}(a_1) \vee \mathfrak{C}}{\mathfrak{A} \vee \bigwedge x\ \mathfrak{B}(x) \vee \mathfrak{C}}$$

(a_1 does not occur in the conclusio)

$$\text{c)}\ \frac{\mathfrak{A} \vee \mathfrak{B} \vee \mathfrak{B} \vee \mathfrak{C}}{\mathfrak{A} \vee \mathfrak{B} \vee \mathfrak{C}}$$

we can obtain (1). The substitution method was chosen in order to permit always—if necessary—the application of rule b).

This completes the proof of Herbrand's theorem in a way whose simplicity can be hardly improved. The theorem affirms: *to every valid formula in prenex normal form of quantificational logic we can effectively find a valid disjunction without quantifiers from which the original formula is deducible using only rules* a), b), *and* c).

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