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ORDERED PAIRS AND CARDINALITY IN NEW FOUNDATIONS

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In any set theory, two sets are said to have the same cardinality if there is a bijection between them. Thus the notion of having the same cardinality (which we shall call 'being equipollent') is dependent on that of function and hence on that of ordered pair. We shall show that in Quine's set theory **NF** (as formulated in [3], for instance) the definition of ordered pair which is used affects whether, or not, two sets are equipollent, and we make some further considerations based on this fact.* The following definitions are made to aid our discussion, and we hope that it is obvious how they could be made precise.

Definition 1 A formula $\psi(x, y, z)$ with exactly three free variables is said to represent an ordered pair relation in a set theory T if

(i) $T \vdash \forall x, y \exists ! z \psi(x, y, z)$,

and

(ii) $T \vdash \forall x, x', y, y', z[(\psi(x, y, z) \wedge \psi(x', y', z)) \rightarrow (x = x' \wedge y = y')]$.

Definition 2 If ψ represents an ordered pair in a set theory T, then $x \approx_{\psi} y$ is a formula which, in a natural way, says that there is a function, represented as a set of ordered pairs which are defined using ψ, which is a bijection from x to y.

We will always assume that $z = \langle x, y\rangle$ is a formula which says that z is the Kuratowski ordered pair (i.e., $\{\{x\}, \{x, y\}\}$) and then in both **ZF** and **NF** this represents an ordered pair relation. Also, $x \approx y$ will always be $x \approx_{\phi} y$, where ϕ is $z = \langle x, y\rangle$.

The next theorem shows that, in a certain sense, the notion of being equipollent is independent of the representation of ordered pairs in **ZF** set theory.

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Theorem 1 *If ψ represents an ordered pair relation in* **ZF**, *then*

$$\mathbf{ZF} \vdash \forall u, v(u \approx v \leftrightarrow u \approx_\psi v).$$

The proof of this result is completely straightforward. For instance, to prove it one way round, suppose that $u \approx v$. Then let f be a bijection from u to v, put $f' = \{z \mid \exists x, y\, \psi(x, y, z) \wedge \langle x, y\rangle \in f\}$ and then verify that $u \approx_\psi v$, using f'.

From a mathematical point of view Theorem 1 is highly desirable as the actual structure of the ordered pair does not seem to be important for two sets being equipollent. However, provided that **NF** is consistent, the analogous form of Theorem 1 is not true for **NF**, even if we restrict ψ to being a stratified formula (see [3] for a definition of stratified). This can be seen as follows.

If $\psi(x, y, z)$ is the formula $z = \{\{x\}, \{x, \{y\}\}\}$, then ψ represents an ordered pair relation in **NF**. By considering Cantor's theorem for **NF** in [3], Quine shows that if $V = \{x \mid x = x\}$ and $S = \{x \mid \exists y\ x = \{y\}\}$, then $\neg(V \approx S)$, but it is straightforward to show that $S \approx_\psi V$ in **NF** and hence the analogous form of Theorem 1 cannot hold.

The key point here, of course, is that we have represented an ordered pair relation using a formula $\psi(x, y, z)$ in which the stratification can only be achieved by attaching different numerals to x and y. It might be argued that this is not desirable in **NF**, but then one must explain the process of stratification in such a way that this becomes highly unreasonable as from a mathematical viewpoint there is no significance in how ordered pairs are represented. The following weak form of Theorem 1 can be proved for **NF**.

Theorem 2 *If $\psi(x, y, z)$ and $\psi'(x', y', z')$ are formulae which represent ordered pair relations in* **NF** *and can be shown to be stratified in such a way that one numeral can be attached to both x and x' and another to both y and y', then* $\mathbf{NF} \vdash \forall u, v(u \approx_\psi v \leftrightarrow u \approx_{\psi'} v)$.

Theorem 2 shows that for considering sets being equipollent in **NF**, it is only the way in which the ordered pair relation can be shown to be stratified (we restrict our attention to stratified definitions from now on) that is important. Thus the following definition of $\approx_i$ is independent of which ψ we choose.

Definition 3 If $\psi(x, y, z)$ represents an ordered pair relation in **NF** and can be shown to be stratified by attaching a numeral n to x and a numeral m to y, and $i = m - n$ then we write $u \approx_i v$ for $u \approx_\psi v$. For definiteness we could take $z = \langle x, \underbrace{\{.\ .\ .\ .\ \{}_{i \text{ brackets}} y\} \ .\ .\ .\ .\}\rangle$ for ψ when $i \geq 0$ and $z = \langle \underbrace{\{.\ .\ .\ .\ \{}_{i \text{ brackets}} x\} \ .\ .\ .\ .\}, y\rangle$ for ψ when $i < 0$.

We can now reformulate the results which we noted earlier as $\neg(V \approx_0 S)$ and $S \approx_1 V$, and another result of [3] shows that $\neg(V \approx_1 V)$ although, of course, $V \approx_0 V$. Our next theorem notes some properties of being i-equipollent (i.e., $\approx_i$) and it is obvious how these are generalizations of properties of being 0-equipollent, which is the usual definition of being equipollent in **NF**.

Definition 4 $x^{(m)} = \{y \mid \exists t \in x\, y = \underbrace{\{.\ .\ .\ .\ \{t\}\ .\ .\ .\ .\}}_{m \text{ brackets}}\}$.

Theorem 3 *The universal closures of the following are provable in* **NF**

(i) $x \approx_0 x$ (i.e., $x \approx x$),
(ii) $x \approx_i y \rightarrow y \approx_{-i} x$,
(iii) $x \approx_i y \wedge y \approx_j z \rightarrow x \approx_{i+j} z$,
(iv) $x^{(m)} \approx_m x$.

The proof of Theorem 3 is straightforward and it might be interesting to investigate further properties of i-equipollence, but we now consider a method of extending **NF**. It is reasonable to suggest that if $u \approx_i v$, for any i, then u and v are equipollent in an intuitive sense and thus we let **ENF** be **NF** extended by adding a new symbol $\approxeq$ together with the axiom

$$u \approxeq v \longleftrightarrow \textit{for some integer } i,\ u \approx_i v. \qquad (*)$$

We shall not consider methods for formalizing this axiom in first order terms but will continue to treat it in the intuitive sense. Theorem 3 shows that $\approxeq$ has the properties

(i) $x \approxeq x$,
(ii) $x \approxeq y \rightarrow y \approxeq x$,
(iii) $x \approxeq y \wedge y \approxeq z \rightarrow x \approxeq z$,
(iv) $x^{(m)} \approxeq x$,

and thus $\approxeq$ seems a more reasonable formulation of being equipollent than $\approx$ in **NF** as $\approxeq$ also possesses the intuitively true property (iv). To actually work with **ENF** we would probably also have to add axioms asserting the existence of cardinals, as equivalence classes under $\approxeq$, and other comprehension principles, but we shall leave these problems and we shall finally consider the interpretations, when $\approx$ is replaced by $\approxeq$, of two of the results which have been proved for **NF**.

In [2] it is shown that if **NF** is consistent then the axiom of counting is not provable in **NF**. This axiom is the intuitively true statement

$$\forall n\ (\mathsf{Nn}(n) \rightarrow \{m \mid \mathsf{Nn}(m) \wedge m < n\} \in n),$$

where $\mathsf{Nn}(n)$ is a formula saying 'n is a natural number,' in which $\approx$ is used for sets being equipollent. Hence the axiom of counting says that if $\mathsf{Nn}(n)$ then for some $t \in n$,

$$\{m \mid \mathsf{Nn}(m) \wedge m < n\} \approx t. \qquad (**)$$

To consider this axiom in **ENF** we should really consider natural numbers defined using $\approxeq$, but for comparison we will use Nn. Intuitively, the reason why (**) is not derivable in **NF** is that the objects on the left and the right are of different "types," but it is straightforward to show that the following form of the axiom of counting is provable in **NF**.

$$\textit{If } \mathsf{Nn}(n) \textit{ then for some } t \in n, \{m \mid \mathsf{Nn}(m) \wedge m < n\} \approx_2 t,$$

so that in **ENF** $\{m \mid \mathsf{Nn}(m) \wedge m < n\} \approxeq t$ and this again suggests that $\approxeq$ is a better notion of being equipollent than $\approx$.

Henson shows in [1] that it is relatively consistent with **NF** that for finite sets x, if $\mathsf{Nc}(x)$ is the cardinal of x and $\mathsf{P}(x)$ is the power set of x, then we can have $\mathsf{Nc}(\mathsf{P}(x)) < \mathsf{Nc}(x)$ or $\mathsf{Nc}(\mathsf{P}(x)) > \mathsf{Nc}(x)$. He also shows that we can have $\mathsf{Nc}(x^{(1)}) <, =$ or $> \mathsf{Nc}(x)$. We have already noted that in **ENF** $x \approxeq x^{(1)}$ and hence the latter pathologies are eliminated. In **ENF** $x \approxeq x^{(1)} \in \mathsf{P}(x)$ so that $\mathsf{P}(x)$ will probably be at least as big as x, but we do not seem to get an immediate answer to this.

There are a number of problems which could be investigated concerning natural extensions of **ENF**, but it could also prove worthwhile to consider other properties which depend on the definition of ordered pairs in **NF**.

REFERENCES

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