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THE ONE VARIABLE IMPLICATIONAL CALCULUS

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Johnstone and Price [1] have given the following "axioms" for the fragment of the implicational calculus with only one variable:

Ax. 1. $*p$
Ax. 2. *If* $\vdash\alpha$ *and* $*\beta$, *then* $*(\alpha \supset \beta)$.
Ax. 3. *If* $*(\alpha \supset \beta)$, *then* $\vdash\alpha$ *and* $*\beta$,

where "$\vdash\alpha$" ("$*\alpha$") symbolizes "α is (not) derivable."

Ax. 2 and *Ax. 3* are not really axioms, but rather are the deductive *rules* of the system. When one examines their proof of

"*Th. 1.* $\vdash(p \supset p)$.
Proof: If $*(p \supset p)$, then, by *Ax. 3*, $\vdash p$.
But this contradicts *Ax. 1*."

one notes that it uses a rule which is not included in the above list, *viz.*:

Ax. 4. If $*\alpha$ *yields both* $\vdash\beta$ *and* $*\beta$, *then* $\vdash\alpha$.

That some additional rule is necessary is clear since it is obvious from the *structure* of the axioms and rules that *no* theorems can be proved from axiom *Ax. 1* using rules *Ax. 2* and *Ax. 3*. With this additional rule all of the proofs of [1] go through as stated; in particular, every one variable tautology is derivable and every one variable non-tautology is rejected. The augmented system is, however, rather inelegant: Two rules are combined in *Ax. 3* (so there are four rules altogether) and *Ax. 4* is a compound (nested) rule.

The following presentation of the system is simpler and the development is slightly easier. Again take $*p$ as the sole axiom. As rules accept the following four:

$$\frac{\vdash\alpha,\ \vdash\beta}{\vdash(\alpha \supset \beta)} \qquad \frac{\vdash\alpha,\ *\beta}{*(\alpha \supset \beta)}$$

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The proof of the main theorem (if α is a one-variable tautology, then $\vdash\alpha$; otherwise $*\alpha$) is a straightforward induction as before. All of these rules yield conclusions longer than the premises and so the calculus is decidable. Moreover, there is a proof procedure: Draw the structural tree of the formula you wish to prove, and observe that at each node only one of the four rules is applicable. Indeed a formula with n "$\supset$"s has a proof which uses exactly n applications of the rules. This proof is unique up to the order in which the rules are applied.

When one observes the similarity between these four rules and the ordinary truth table for implication, it would seem that this system could be extended to the full implicational calculus (in many variables). That this cannot be done by adding additional axioms and rules was, thankfully, brought to my attention by Professor Johnstone. In fact, the last of the above rules is invalid for the full implicational calculus. The problem of finding a presentation of the one variable calculus which can be extended to the full implicational calculus remains open.

It is worthwhile pointing out that what we have done is to use the main idea behind Leśniewski's systems of computable Protothetics [2]. Our system is not a system of Protothetic, however, for there are no functional variables.

In our presentation of the one variable implicational calculus we used four rules, all of which had two premises and none of which is compound. This can be improved on a bit by taking, besides the single axiom $*p$, the three rules

$$\frac{\vdash\beta}{\vdash(\alpha\supset\beta)}\ ,\qquad \frac{*\alpha}{\vdash(\alpha\supset\beta)}\ ,\qquad \frac{\vdash\alpha,\ *\beta}{*(\alpha\supset\beta)}\ .$$

This lacks some of the nice features of the four rule presentation, but does seem to be the simplest system of structural rules where the single axiom is $*p$.

The search for a more conventional presentation will fail for we have the

Theorem. *The tautologies of the one variable implicational calculus are not finitely axiomatizable using the rules of substitution and detachment.*

Proof: Suppose they were. Then let α be a one variable tautology which is longer than any of the axioms. We shall show that $p\supset\alpha$ is not provable from the axioms, so the supposed axiomatization is inadequate.

a) If $p\supset\alpha$ results from substitution then it must come from substitution in $p\supset p$. Then α has length one, which is absurd.

b) If $p\supset\alpha$ comes from detachment it is proved by detaching β from $\beta\supset(p\supset\alpha)$. As we can arrange to make all substitutions before any detachments, this last formula must be a substitution instance of an axiom, namely of $p\supset(p\supset p)$. So again α has length one.

Q.E.D.

Perhaps the most interesting feature of this calculus is that it provides a simple and natural example of a propositional calculus which cannot be finitely axiomatized using the rules of substitution and detachment.

REFERENCES

[1] Johnstone, Henry W., and Robert Price, "Axioms for the implicational calculus with one variable," *Theoria*, vol. 30 (1964), pp. 1-4.

[2] Leśniewski, Stanisław, "Introductory remarks to the continuation of my article: 'Grundzüge eines Neuen Systems der Grundlagen der Mathematik'," English translation of the 1938 original in *Polish Logic*, Storrs McCall, ed., Oxford (1967), pp. 116-169, esp. pp. 149-155.

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