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A THEOREM CONCERNING A RESTRICTED RULE OF SUBSTITUTION IN THE FIELD OF PROPOSITIONAL CALCULI. I

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In this paper the rule of simultaneous substitution ordinarily used in the field of propositional calculi will be called restricted if, in the formalization of the given system, its applications are limited to the axioms of that system. So far as I know, A. Lindenbaum was the first who investigated an instance of this rule. Namely, around 1934 he informed me casually that there are some systems whose axiomatizations have a special structure of the bi-valued propositional calculus in which a replacement of the rule of simultaneous substitution by the restricted one does not affect the strength of these systems. Since Lindenbaum never published his research concerning this and related results, I have no idea exactly how his theorem was formulated and how it was proved. Much later, in [1], pp. 148-151, section **27** (see especially p. 150), A. Church sketches a proof of a theorem which states that any system of the classical propositional calculus or any partial system of that calculus whose only rules of procedure are: detachment for implication and substitution (not necessarily simultaneous) may be reformulated into a system which has the same theorem as the original one and whose single rule of procedure is detachment. An inspection of Church's proof of this theorem shows that it holds simply through replacing each axiom of a system under consideration by the corresponding axiom schema. Since, certainly, Lindenbaum did not intend to reject the rule of substitution totally in formulating his theorem and, probably, he did not use the axiom schemata in the deductions which were needed for a proof of the theorem, the theorems discussed above are rather distinctly different.

In this note we will prove the following theorem concerning the restricted rule of simultaneous substitution:

Theorem A *If* (i) *T is an arbitrary, consistent propositional system whose formalization satisfies the conditions:*

(a) *The set of primitive notions of T contains at least the proposition*

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forming functor for two propositional arguments C which does not necessarily coincide with the classical implication;
(b) *The set of the rules of procedure of T contains at least the following two rules:*

R1 *The rule of simultaneous substitution which is ordinarily used in the field of propositional calculi, but here is adjusted to the primitive notions of system T,*

and

R2 *The rule of detachment in regard to functor C: If the formulas α and $C\alpha\beta$ are the theses of T, then formula β can be added to T, as its new thesis;*

(c) *System T is axiomatizable and its complete axiom set $\mathfrak{A}$, which can be finite or infinite, does not contain the axiom schemata,*

(ii) *Sequence $\mathbf{A} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_m\}$, $1 \leq m < \infty$, is an arbitrary, unempty, finite subset which can be proper or unproper of the axiom set $\mathfrak{A}$,*

(iii) *Formula* **b** *which is a well-formed formula in the field of system T and is not equiform with any term of sequence* **A**,

and

(iv) *Rule* R1* *is the restricted rule* R1,

then in T: Formula **b** *is provable in the field of* **A** *by the applications of the rules* R1 *and* R2, *if and only if, formula* **b** *is provable in the field of* **A** *by the applications of the rules* R1* *and* R2.

Obviously, Theorem A is a very strong generalization of Lindenbaum's theorem mentioned above. It should be noted that the axioms of T are entirely undefined (clearly, we know only that at least one of the axioms belonging to sequence **A** must have a form $C\alpha\beta$), and it is self-evident that Theorem A holds only for the rule of simultaneous substitution.

Proof:

1 In order to prove Theorem A in the most compact way I shall use, in the reasonings presented below, the following abbreviations:

(a) The abbreviations "$\alpha \approx \beta$," "$\mathrm{A}\vdash_{\overline{\mathrm{R1}}} \alpha$," and "$\{\alpha\}\vdash_{\overline{\mathrm{R1}}} \beta$" will mean "formula α is equiform with formula β," "in the field of sequence **A** formula α is provable by R1" and "β is a consequence of α by R1" respectively. The analogous and obvious meanings will have the following abbreviations: "$\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}} \alpha$," "$\mathbf{A}\vdash_{\overline{\mathrm{R1,R2}}} \alpha$," "$\mathbf{A}\vdash_{\overline{\mathrm{R1^*,R2}}} \alpha$," "$\{\alpha\}\vdash_{\overline{\mathrm{R1^*}}} \beta$," "$\{\alpha, \beta\}\vdash_{\overline{\mathrm{R2}}} \gamma$," and "$\mathbf{A}\vdash_{\overline{\mathrm{R1^*}}} V$," and so forth. The last abbreviation given above means "in the field of **A** the set (the sequence) of the formulas V is provable by R1*,"

and the following tacit assumptions:

(b) In any sequence which will be considered below each of its terms occurs without repetition.

(c) If the sequence **Z** considered below contains three terms τ_i, τ_j, and τ_f, $i < j < f$, such that in **Z** $\{\tau_i\}\vdash_{\overline{\text{R1}}}\tau_j$ and $\{\tau_j\}\vdash_{\overline{\text{R1}}}\tau_f$, then tacitly **Z** is substituted by a sequence which possesses exactly the same terms and ordering, but in which $\{\tau_i\}\vdash_{\overline{\text{R1}}}\tau_f$. It is similar if instead of R1 we have R1*. Clearly, we can do it, since our rules of substitution are simultaneous.

(d) If a subsequence **V** of the given sequence **Z** is such that it contains all such and only such terms of **Z** that satisfy all conditions of the given property ϕ and a subsequence **W** = **Z** - **V**, then we assume tacitly that these two subsequences are ordered according to the order in which their terms occur in the sequence **Z**. Moreover, we assume that in **Z** its terms are automatically rearranged in such a way that each term of **V** precedes every term of **W**. Thus, in such a case we have: **Z** = {**V**; **W**}. Obviously, we can do this only under conditions where the subsequences **V** and **W** are disjoint and ϕ is suitably defined.

(e) If a subsequence **V** = $\{\sigma_1, \ldots, \sigma_m\}$, $0 \leqslant m < \infty$, of the given sequence **Z** is such that it contains all such and only such terms of **Z** which satisfy all conditions of the given property ϕ, and formula τ also satisfies these conditions, but it is not a term of **Z**, then the sequence **V*** = $\{\sigma_1, \ldots, \sigma_m, \sigma_{m+1}\}$, $0 \leqslant m + 1 < \infty$, in which $\sigma_{m+1} \approx \tau$, is called an augmentation of **V**. And, if in **Z** we replace **V** by **V***, then this new sequence will be indicated by **Z***. Moreover, we tacitly assume that the augmentation of **Z** is always done in such a way that, if ρ is the last term of **Z**, then it is also the last term of **Z***. Thus, e.g., if **Z** = {**V**; **W**}, then **Z*** = {**V***; **W**}. It is self-evident that if, in the proof given below, it will be established that the given sequence **P** = {**Q**; **R**} possesses a certain required property ψ and its subsequence **Q** is augmented to **Q*** by a formula which does not contradict ψ, then **P*** = {**Q***; **R**} also satisfies ψ.

2 Now, let us assume the antecedent of Theorem A. Since R1* is a restriction of R1, we know at once that it is sufficient to prove: *In T, if* $\mathbf{A}\vdash_{\overline{\text{R1,R2}}}\mathbf{b}$, *then* $\mathbf{A}\vdash_{\overline{\text{R1*,R2}}}\mathbf{b}$. Hence, assume that in T $\mathbf{A}\vdash_{\overline{\text{R1,R2}}}\mathbf{b}$. Then it follows from this assumption and the standard definition of a proof that there exists an unempty, finite sequence of the formulas:

$$\mathfrak{D} = \{\mathfrak{a}_1, \ldots \mathfrak{a}_m, \mathbf{b}_{m+1}, \ldots, \mathbf{b}_z\},\ 1 < z < \infty,$$

constructed in accordance with the points (b) and (c) of 1 and such that:

(1) The terms $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ of $\mathfrak{D}$ are the axioms of T which belong to **A**;

(2) The terms of subsequence $\mathbf{B} = \{\mathbf{b}_{m+1}, \ldots, \mathbf{b}_z\}$ of $\mathfrak{D}$ are such that if $\mathbf{b}_t$, $m + 1 \leqslant t \leqslant z$, is a term of **B**, then either

(α) there is a term σ of $\mathfrak{D}$ which precedes $\mathbf{b}_t$ and $\{\sigma\}\vdash_{\overline{\text{R1}}}\mathbf{b}_t$,

or

(β) there are the terms σ and τ of $\mathfrak{D}$ which precede $\mathbf{b}_t$ and $\{\sigma, \tau\}\vdash_{\overline{\text{R2}}}\mathbf{b}_t$;

(3) The last term of $\mathfrak{D}$, viz. $\mathbf{b}_z$, is such that $\mathbf{b}_z \approx \mathbf{b}$.

2.1 Since it will be more convenient for our further reasoning, I renumerate the terms of B, as follows: $\mathsf{B} = \{\mathsf{B}_{m+1}, \ldots, \mathsf{b}_z\} = \{\mathsf{b}_1, \ldots, \mathsf{b}_w\}$, $1 \leqslant w < z$, $\mathsf{b}_w \approx \mathsf{b}$. Moreover, since it follows from points (ii) and (iii) of the antecedent of Theorem A that both subsequences A and B of $\mathfrak{D}$ are unempty and since, according to the definition of $\mathfrak{D}$, they are disjoint, we have: $\mathfrak{D} = \{\mathsf{A}; \mathsf{B}\}$, *cf.*, **1** (d).

3 The subsequences of B which will be defined below and which, eventually, can be empty, will be analyzed and used in our proof.

3.1 Let $\mathsf{V}_1 = \{\alpha_1, \ldots, \alpha_w\}$, $0 \leqslant n \leqslant w$, be a subsequence of B containing all such and only such terms of B that for every α_j, $1 \leqslant j \leqslant w$, α_j is a term of V_1 if and only if α_j is a term of $\mathfrak{D}$ and in A there is a term $\mathfrak{a}_i$, $1 \leqslant i \leqslant m$, such that $\{\mathfrak{a}_i\} \vdash_{\overline{\mathrm{R1}}} \alpha_j$.

3.1.1 If V_1 is unempty, then since all terms of V_1 are generated by the application of R1 to the axioms of T belonging to A, it is self-evident that $\mathsf{A} \vdash_{\overline{\mathrm{R1^*}}} \mathsf{V}_1$. Therefore, if $\mathsf{B} = \mathsf{V}_1$, then, since B is not empty, *cf.*, **2.1**, we have, obviously, $\mathsf{V}_1 = \{\mathsf{b}\}$, i.e., that $\mathsf{A} \vdash_{\overline{\mathrm{R1^*}}} \mathsf{b}$. And, in such a case Theorem A is proved. On the other hand, if $\mathsf{B} \neq \mathsf{V}_1$, then for $\mathsf{C} = \mathsf{B} - \mathsf{V}_1$ we have, *cf.*, **1** (c), $\mathsf{B} = \{\mathsf{V}_1; \mathsf{C}\}$, i.e., $\mathfrak{D} = \{\mathsf{A}; \mathsf{V}_1; \mathsf{C}\}$, where V_1 can be empty.

3.2 If C is unempty, let $\mathsf{V}_2 = \{\beta_1, \ldots, \beta_p\}$, $1 \leqslant p \leqslant w$, be a subsequence of C containing all such and only such elements of C that for every β_k, $1 \leqslant k \leqslant w$, β_k is a term of V_2 if and only if β_k is a term of $\mathfrak{D}$ and in $\mathfrak{D}$ there are two terms σ and τ which precede the first term of C and such that $\tau \approx C\sigma\beta_k$ and $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \beta_k$.

3.2.1 If $\mathsf{V}_2 = \mathsf{C}$, then $\mathfrak{D} = \{\mathsf{A}; \mathsf{V}_1; \mathsf{V}_2\}$. Therefore, in such a case if V_1 is empty, then $\mathsf{A} \vdash_{\overline{\mathrm{R2}}} \mathsf{V}_2$, i.e., obviously, *cf.*, **3.2**, $\mathsf{A} \vdash_{\overline{\mathrm{R2}}} \mathsf{b}$, and if V_1 is unempty, then, *cf.*, **3.1.1**, $\mathsf{A} \vdash_{\overline{\mathrm{R1^*,R2}}} \mathsf{V}_2$, i.e., clearly, $\mathsf{A} \vdash_{\overline{\mathrm{R1^*,R2}}} \mathsf{b}$. Thus, if $\mathfrak{D} = \{\mathsf{A}; \mathsf{V}_1; \mathsf{V}_2\}$, then Theorem A is proved. On the other hand, if $\mathsf{V}_2 \neq \mathsf{C}$, we have for $\mathsf{D} = \mathsf{C} - \mathsf{V}_2$, *cf.*, **1** (d), $\mathsf{C} = \{\mathsf{V}_2; \mathsf{D}\}$, i.e., $\mathfrak{D} = \{\mathsf{A}; \mathsf{V}_1; \mathsf{V}_2; \mathsf{D}\}$. In such a case, since D is unempty, its first term, d_1, neither

(a) can be such that $\mathsf{A} \vdash_{\overline{\mathrm{R1}}} \mathsf{d}_1$, since otherwise it would be a term of V_1, *cf.*, **3.1**, or for the same reason, *cf.*, **1** (c), such that $\mathsf{V}_1 \vdash_{\overline{\mathrm{R1}}} \mathsf{d}_1$,

nor

(b) can be such that it would be a consequence by R2 of two terms belonging to $\mathfrak{D}$ which precede the first term of V_2, since otherwise it would be a term of V_2, *cf.*, **3.2**.

Hence, if D is unempty, and d_1 is its first term, then either in V_2 there is a term σ such that $\{\sigma\} \vdash_{\overline{\mathrm{R1}}} \mathsf{d}_1$ or in $\mathfrak{D}$ there are two terms σ and τ such that they precede d_1, at least one of them is a term of V_2, $\tau \approx C\sigma\mathsf{d}_1$ and $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \mathsf{d}_1$. Therefore, if in $\mathfrak{D}$ its subsequence D is not empty, then also V_2 is unempty.

3.3 If D is unempty, let $\mathsf{E} = \{\gamma_1, \ldots, \gamma_q\}$, $1 \leqslant q < w$, be a subsequence of D containing all such and only such terms of D that for every γ_k, $1 \leqslant k < w$, γ_k is a term of E if and only if γ_k is a term of $\mathfrak{D}$ and in V_2 there is a term σ such that $\{\sigma\} \vdash_{\overline{\mathrm{R1}}} \gamma_k$.

3.4 If $\mathbf{D} \neq \mathbf{E}$, let $\mathbf{F} = \mathbf{D} - \mathbf{E}$. Hence, if $\mathbf{E}$ is unempty, $\mathbf{D} = \{\mathbf{E};\ \mathbf{F}\}$, i.e., $\mathfrak{D} = \{\mathbf{A};\ \mathbf{V}_1;\ \mathbf{V}_2;\ \mathbf{E};\ \mathbf{F}\}$. It follows at once from the definitions of $\mathbf{D}$ and $\mathbf{E}$ *cf.*, **3.2.1**, and **3.3**, that if $\mathbf{F}$ is not empty, then its first term, say $\mathbf{f}_1$, is such that in $\mathfrak{D}$ there are two terms σ and τ such that they precede $\mathbf{f}_1$, $\tau \approx C\sigma\mathbf{f}_1$ and $\{\sigma, \tau\}\vdash_{\overline{R2}}\mathbf{f}_1$. Moreover, clearly, *cf.*, **3.2.1**., at least one of these terms must be either a term of $\mathbf{V}_2$ or a term of $\mathbf{E}$, since otherwise $\mathbf{f}_1$ would be a term of $\mathbf{V}_2$.

4 In this section it will be shown that if subsequence $\mathbf{E}$ of $\mathbf{D}$ is unempty, then we are always able to replace $\mathfrak{D}$ by its augmentation constructed effectively:

$$\mathfrak{D}^* = \{\mathbf{A};\ \mathbf{V}_1^*;\ \mathbf{V}_2^*;\ \mathbf{F}\}$$

in which its subsequences $\mathbf{A}$ and $\mathbf{F}$ are exactly the same as in $\mathfrak{D}$ and the subsequences $\mathbf{V}_1^*$ and $\mathbf{V}_2^*$ are such augmentations of $\mathbf{V}_1$ and $\mathbf{V}_2$ that each formula which in $\mathfrak{D}$ is a term of $\mathbf{E}$ occurs in $\mathfrak{D}^*$, as a term of $\mathbf{V}_2^*$.

4.1 Let us assume that in $\mathfrak{D}$ its subsequence $\mathbf{E}$ is not empty and, moreover, that γ_k, $1 \leqslant k \leqslant q$, is an arbitrary term of $\mathbf{E}$. Then, according to the definition of $\mathbf{E}$, *cf.*, **3.3**, in $\mathbf{V}_2$ which is not empty, *cf.*, **3.2.1**, there is a term β_h, $1 \leqslant h \leqslant p$, such that $\{\beta_h\}\vdash_{\overline{R1}}\gamma_k$. Since β_h is a term of $\mathbf{V}_2$, in $\mathfrak{D}$, *cf.*, **3.2**, there are two such terms σ and τ that precede the first term of $\mathbf{V}_2$, such that $\tau \approx C\sigma\beta_h$, and $\{\sigma, \tau\}\vdash_{\overline{R2}}\beta_h$. Hence, in accordance with the definition of $\mathbf{V}_2$, there are four possibilities: either both σ and τ are the terms of $\mathbf{A}$, or σ is a term of $\mathbf{A}$ and τ is a term of $\mathbf{V}_1$, or σ is a term of $\mathbf{V}_1$ and τ is a term of $\mathbf{A}$, or both σ and τ are the terms of $\mathbf{V}_1$. Consequently, we have to analyze the four possible cases:

Case 1. Both σ and τ are the terms of $\mathbf{A}$, $\tau \approx C\sigma\beta_h$, $\{\sigma, \tau\}\vdash_{\overline{R2}}\beta_h$ and $\{\beta_h\}\vdash_{\overline{R1}}\gamma_k$. Since both rules of substitution mentioned in the formulation of Theorem A are simultaneous, it follows at once from our present assumptions and the definition of $\mathbf{V}_1$, *cf.*, **3.1**, and **3.1.1**, that

(1) there must exist a formula μ such that $\{\tau\}\vdash_{\overline{R1^*}}\mu$ and $\mu \approx C\rho\gamma_k$,

and that

(2) either $\sigma \approx \rho$ or $\sigma\vdash_{\overline{R1^*}}\rho$.

Hence, we have to investigate the four obvious subcases:

Subcase 1a. $\sigma \approx \rho$ and μ is a term of $\mathfrak{D}$. Whence, if follows from point (1) and the definition of $\mathbf{V}_1$ that μ is a term of $\mathbf{V}_1$. However, such a case is impossible, since otherwise, *cf.*, point (1) and the definition of $\mathbf{V}_2$, γ_k would be a term of $\mathbf{V}_2$.

Subcase 1b. $\sigma \approx \rho$ and μ is not a term of $\mathfrak{D}$. Hence, also μ is not a term of $\mathbf{V}_1$. Now, define:

(a) $\mathbf{V}_1^{*1} = \{\mathbf{V}_1;\ \mu\} = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$, $1 \leqslant n + 1$, $\alpha_{n+1} \approx \mu$.
(b) $\mathbf{V}_2^{*1} = \{\mathbf{V}_2;\ \gamma_k\} = \{\beta_1, \ldots, \beta_p, \beta_{p+1}\}$, $1 \leqslant p + 1$, $\beta_{p+1} \approx \gamma_k$.
(c) $\mathbf{E}^0 = \{\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_q\}$, $1 \leqslant k \leqslant q$.

Now, we replace in $\mathfrak{D}$ its subsequences V_1, V_2, and E by V_1^{*1}, V_2^{*1}, and E^0 respectively obtaining in such a way a new sequence $\mathfrak{D}^{*1} = \{\mathsf{A}; \mathsf{V}_1^{*1}; \mathsf{V}_2^{*1}; \mathsf{E}^0; \mathsf{F}\}$. Since, by assumptions, both σ and τ are the terms of A, $\{\tau\}\vdash_{\overline{\mathrm{R1^*}}}\mu$, $\mu \approx C\rho\gamma_k$ and $\sigma \approx \rho$, we have $\mu \approx C\sigma\gamma_k$ and, therefore, $\{\sigma, \mu\}\vdash_{\overline{\mathrm{R2}}}\gamma_k$. Hence, since in V_1^{*1} its last term $\alpha_{n+1} \approx \mu \approx C\rho\gamma_k \approx C\sigma\gamma_k$ and in V_2^{*1} its last term $\beta_{p+1} \approx \gamma_k$, in $\mathfrak{D}^{*1}$ $\mathsf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\gamma_k$ while in $\mathfrak{D} \approx \mathsf{A}\vdash_{\overline{\mathrm{R1,R2}}}\gamma_k$. Thus, it is self-evident that if this subcase of Case 1 holds for γ_k, then we can always solve it accepting instead of $\mathfrak{D}$ its augmentation $\mathfrak{D}^{*1}$ as a proof sequence of $\mathbf{b}$.

Subcase 1c. $\sigma\vdash_{\overline{\mathrm{R1^*}}}\rho$ and in $\mathfrak{D}$ μ is a term of V_1. The case that ρ is a term of $\mathfrak{D}$, i.e., ρ is a term of V_1 is impossible, since otherwise γ_k would be a term of V_2. Therefore, ρ is not a term of $\mathfrak{D}$. Hence, define:

(d) $\mathsf{V}_1^{*2} = \{\mathsf{V}_1; \rho\} = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$, $1 \leqslant n+1$, $\alpha_{n+1} \approx \rho$.

Then, using V^* and E^0 as defined in points (b) and (c) above, we replace $\mathfrak{D}$ by its augmentation $\mathfrak{D}^{*2} = \{\mathsf{A}; \mathsf{V}_1^{*2}; \mathsf{V}_2^{*1}; \mathsf{E}^0; \mathsf{F}\}$. Since τ is a term of A and $\{\tau\}\vdash_{\overline{\mathrm{R1^*}}}\mu$, in $\mathfrak{D}^{*2}$ $\mathsf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\gamma_k$. Hence, if this subcase of Case 1 holds for γ_k, then we can always solve it accepting instead of $\mathfrak{D}$ its augmentation $\mathfrak{D}^{*2}$ as a proof sequence of $\mathbf{b}$.

Subcase 1d. $\{\sigma\}\vdash_{\overline{\mathrm{R1^*}}}\rho$ and μ is not a term of $\mathfrak{D}$. Whence, μ is also not a term of V_1. On the other hand, since either ρ is a term of $\mathfrak{D}$ or ρ is not a term of $\mathfrak{D}$, each of these two possibilities must be assumed and investigated separately. Wherefore:

Subcase 1d$_1$. $\{\sigma\}\vdash_{\overline{\mathrm{R1^*}}}\rho$; ρ is a term of $\mathfrak{D}$ and μ is not a term of V_1. Since σ is a term of A, $\{\sigma\}\vdash_{\overline{\mathrm{R1^*}}}\rho$ and ρ is a term of $\mathfrak{D}$, it follows from the definition of V_1, *cf.*, **3.1** and **3.1.1**, that ρ is a term of V_1. Hence, define:

(e) $\mathsf{V}_1^{*3} = \{\mathsf{V}_1, \mu\} = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}\}$, $1 \leqslant n+1$, $\alpha_{n+1} \approx \mu$.

Then, we replace $\mathfrak{D}$ by its augmentation $\mathfrak{D}^{*3} = \{\mathsf{A}; \mathsf{V}_1^{*3}; \mathsf{V}_2^{*1}; \mathsf{E}^0; \mathsf{F}\}$. Since both terms σ and τ are the terms of A, $\{\sigma\}\vdash_{\overline{\mathrm{R1^*}}}\rho$, $\{\tau\}\vdash_{\overline{\mathrm{R1^*}}}\mu$, ρ is a term of V_1, i.e., clearly, ρ is a term of V_1^{*3}, and, moreover, μ is a term of V_1^{*3}, in $\mathfrak{D}^{*3}$ $\mathsf{A}\vdash_{\overline{\mathrm{R1^*,R2}}}\gamma_k$. Therefore, if subcase 1d$_1$ holds for γ_k, then we can always solve it accepting instead of $\mathfrak{D}$ its augmentation $\mathfrak{D}^{*3}$ as a proof sequence of $\mathbf{b}$.

Subcase 1d$_2$. $\{\sigma\}\vdash_{\overline{\mathrm{R1^*}}}\rho$, ρ is not a term of $\mathfrak{D}$ and μ is not a term of V_1. Hence, ρ is not a term of V_1. Now, if in A σ precedes τ, we define

(f) $\mathsf{V}_1^{*4} = \{\mathsf{V}_1, \rho, \mu\} = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}\}$, $1 \leqslant n+1$, $\alpha_{n+1} \approx \rho$ and $\alpha_{n+2} \approx \mu$.

On the other hand, if in A τ precedes σ, then we define V_1^{*4} as follows:

(g) $\mathsf{V}_1^{*4} = \{\mathsf{V}_1, \mu, \rho\} = \{\alpha_1, \ldots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}\}$, $1 \leqslant n+1$, $\alpha_{n+1} \approx \mu$ and $\alpha_{n+2} \approx \rho$.

Remark I: Clearly, each of the sequences which are defined in points (f) and (g) above can be used in order to obtain a solution of subcase 1d$_2$. Since the obtained solution must be unique, it must be uniquely determined which of the sequences presented above should be accepted in regard to the term γ_k under consideration. Since the assumptions concerning γ_k inform only that both σ and τ are the terms of $\mathbf{A}$, $\{\sigma\}\vdash_{\overline{R1^*}}\rho$, $\{\tau\}\vdash_{\overline{R1^*}}\mu$, both ρ and μ are not the terms of $\mathbf{V}_1$ and $\{\rho, \mu\}\vdash_{\overline{R2}}\gamma_k$, the order of σ and τ in $\mathbf{A}$ is only one possible determinent which uniquely selects a proper sequence in regard to γ_k.

Then, we replace $\mathfrak{D}$ by its augmentation $\mathfrak{D}^{*4} = \{\mathbf{A}; \mathbf{V}_1^{*4}; \mathbf{V}_2^{*1}; \mathbf{E}^0; \mathbf{F}\}$ in which $\mathbf{V}_1^{*4}$ is defined either as in (f) or as in (g) according to Remark I. Since both σ and τ are terms of $\mathbf{A}$, $\{\sigma\}\vdash_{\overline{R1^*}}\rho$ and $\{\tau\}\vdash_{\overline{R1^*}}\mu$, in $\mathfrak{D}^{*4}$ both ρ and μ are the terms of $\mathbf{V}_1^{*4}$. Whence, in $\mathfrak{D}^{*4}$ $\mathbf{A}\vdash_{\overline{R1^*,R2}}\gamma_k$. Therefore, if subcase $1d_2$ holds for γ_k, then we can always solve it accepting instead of $\mathfrak{D}$ its augmentation $\mathfrak{D}^{*4}$ as a proof sequence of $\mathbf{b}$.

Thus, subcase 1d is solved because it is established above that for each its possible instances, i.e., subcases $1d_1$ and $1d_2$, *cf.*, also Remark I, we are able to construct in an effective way such unique augmentation of $\mathfrak{D}$, viz. $\mathfrak{D}^*_{S1d}$, that $\mathfrak{D}^*_{S1d}$ is a proof sequence of $\mathbf{b}$ and that in $\mathfrak{D}^*_{S1d}$ $\mathbf{A}\vdash_{\overline{R1^*,R2}}\gamma_k$. Since subcases $1d_1$ and $1d_2$ are disjoint, for the term γ_k under consideration there is only one solution. Namely, if the given subcase ($1d_1$ or $1d_2$) holds for γ_k, then instead of $\mathfrak{D}$ such form of $\mathfrak{D}^*_{S1d}$ should be accepted as a proof sequence of $\mathbf{b}$ which corresponds to that subcase.

4.1.1 Consequently, since it is established in **4.1** that for each possible subcase of Case 1, we are able to construct in an effective way such unique augmentation of $\mathfrak{D}$, viz. $\mathfrak{D}^*_{C1}$, such that $\mathfrak{D}^*_{C1}$ is a proof sequence of $\mathbf{b}$ and that in $\mathfrak{D}^*_{C1}$ $\mathbf{A}\vdash_{\overline{R1^*,R2}}\gamma_k$, then Case 1 is solved. Moreover, since subcases a-d of Case 1 are obviously mutually disjoint, we can conclude that for the term γ_k under consideration there is only one solution of Case 1.

4.2 There are three remaining cases, *cf.*, **4.1**, which we have to investigate. Namely:

Case 2. σ is a term of $\mathbf{A}$, τ is a term of $\mathbf{V}_1$, $\tau \approx C\sigma\beta_h$, $\{\sigma, \tau\}\vdash_{\overline{R2}}\beta_h$ and $\{\beta_h\}\vdash_{\overline{R1}}\gamma_k$.
Case 3. σ is a term of $\mathbf{V}_1$, τ is a term of $\mathbf{A}$, $\tau \approx C\sigma\beta_h$, $\{\sigma, \tau\}\vdash_{\overline{R2}}\beta_h$ and $\{\beta_h\}\vdash_{\overline{R1}}\gamma_k$.
Case 4. Both σ and τ are terms of $\mathbf{V}_1$, $\tau \approx C\sigma\beta_h$, $\{\sigma, \tau\}\vdash_{\overline{R2}}\beta_h$ and $\{\beta_h\}\vdash_{\overline{R1}}\gamma_k$.

Remark II: It is obvious, that if one of the cases 1 or 3 holds for γ_k, then in $\mathfrak{D}$ there must be two distinct terms such that each of them is a term of $\mathbf{A}$ and γ_k is a consequence of them by R1 and R2. On the other hand, if one of the cases 2 or 4 holds for γ_k, then it follows from the definitions of these cases that in $\mathfrak{D}$ there can be only one term such that it is a term of $\mathbf{A}$ and γ_k is a consequence of it by R1 and R2. Hence, in our proof it is not excluded as a possibility that $\mathbf{A} = \{\mathfrak{a}_1\}$.

Using reasonings entirely analogous to these which are presented above we can prove without any difficulty that the cases 2, 3, and 4 can be solved always in a similar way as Case 1. Namely, if one of these cases holds for γ_k, then we are able to construct in an effective way the unique augmentation of $\mathfrak{D}$ such that this augmentation is a proof sequence of $\mathbf{b}$ in which $\mathbf{A}\vdash_{\overline{R1^*,R2}}\gamma_k$. Therefore, since the cases 1-4 are mutually disjoint and only one of them holds for γ_k under consideration, we can conclude that if in $\mathfrak{D}$ its subsequence E is not empty and γ_k, $1 \leqslant k \leqslant q$, is an arbitrary term of E, then there is the unique augmentation of $\mathfrak{D}$ such that this augmentation which can be constructed in an effective way is a proof sequence of $\mathbf{b}$ in which $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \gamma_k$.

4.3 Since in sections **4.1** and **4.2** it is assumed that γ_k, $1 \leqslant k \leqslant q$, is an arbitrary term of E, it is self-evident that if we shall apply the methods of a proof which was presented in those sections consecutively to each term of E, then finally we shall obtain in an effective way the unique augmentation of $\mathfrak{D}$ such that this augmentation will be a proof sequence of $\mathbf{b}$ in which $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \mathsf{E}$. More precisely:

4.3.1 Let us assume that E is not empty. Since, *cf.*, **3.3**, $\mathsf{E} = \{\gamma_1, \ldots, \gamma_q\}$, $1 \leqslant q < w < \infty$, E is a finite sequence. Then, define

Df.1 For any $n = 1, 2, 3, \ldots, q$:

$$\mathsf{E}_{\gamma_n} = \begin{cases} \mathsf{E}_{\gamma_1} = \{\gamma_2, \ldots, \gamma_q\}, \text{ i.e., } \mathsf{E}_{\gamma_1} \text{ is } \mathsf{E} \text{ from which } \gamma_1 \text{ is removed,} \\ \mathsf{E}_{\gamma_m} = \{\gamma_{m+1}, \ldots, \gamma_q\}, \text{ i.e., } \mathsf{E}_{\gamma_m} \text{ is } \mathsf{E}_{\gamma_{m-1}} \text{ from which } \gamma_m \text{ is removed.} \end{cases}$$

Since q is finite, it follows at once from Df.1 that E_{γ_q} is the empty sequence.

Now, in the same manner as in **4.1** and **4.2** we construct in an effective way the unique augmentation of $\mathfrak{D}$ in regard to the first term of E, viz. γ_1. Let us indicate this augmentation by: $\mathfrak{D}^*_{\gamma_1} = \{\mathsf{A};\ \mathsf{V}^*_{1\gamma_1};\ \mathsf{V}^*_{2\gamma_1};\ \mathsf{E}_{\gamma_1};\ \mathsf{F}\}$ where $\mathsf{V}^*_{1\gamma_1}$ and $\mathsf{V}^*_{2\gamma_1}$ are respectively V_1 and V_2 augmented in regard to γ_1, $\mathsf{E}_{\gamma_1} = \{\gamma_2, \ldots, \gamma_q\}$. Thus, $\mathfrak{D}^*_{\gamma_1}$ is a proof sequence of $\mathbf{b}$ in which $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \gamma_1$. Since $\mathfrak{D}^*_{\gamma_1}$ is a proof sequence of $\mathbf{b}$, we can obtain its augmentation in regard to γ_2, viz. $\mathfrak{D}^*_{\gamma_2} = \{\mathsf{A};\ \mathsf{V}^*_{1\gamma_2};\ \mathsf{V}^*_{2\gamma_2};\ \mathsf{C}_{\gamma_2};\ \mathsf{F}\}$. Clearly, $\mathfrak{D}^*_{\gamma_2}$ is a proof sequence of $\mathbf{b}$ in which $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \{\gamma_1, \gamma_2\}$. Applying consecutively the preceding method to all the terms of E according to their order we obtain a finite sequence $\mathfrak{G} = \{\mathfrak{D};\ \mathfrak{D}^*_{\gamma_1};\ \ldots;\ \mathfrak{D}^*_{\gamma_q}\}$ containing $q + 1$ terms and such that its first term is $\mathfrak{D}$ and if σ_n, $2 \leqslant n \leqslant q + 1$, is a term of $\mathfrak{G}$, then σ_n is an augmentation of the term σ_{n-1} such that σ_n is a proof sequence of $\mathbf{b}$ in which $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \{\gamma_1, \ldots, \gamma_{n-1}\}$. Obviously, the last term of $\mathfrak{G}$, i.e., $\mathfrak{D}^*_{\gamma_q} = \{\mathsf{A};\ \mathsf{V}^*_{1\gamma_q};\ \mathsf{V}^*_{2\gamma_q};\ \mathsf{E}_{\gamma_q};\ \mathsf{F}\}$, is a proof sequence of $\mathbf{b}$ in which E_{γ_q} is empty, and in which each term of E is a term of $\mathsf{V}^*_{2\gamma_q}$ and $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \mathsf{E}$.

Thus, it has been proved in this section that if $\mathfrak{D} = \{\mathsf{A};\ \mathsf{V}_1;\ \mathsf{V}_2;\ \mathsf{E};\ \mathsf{F}\}$ is a proof sequence of $\mathbf{b}$, and in $\mathfrak{D}$ its subsequence E is not empty, then there is the unique augmentation of D, viz. $\mathsf{D}^*_{\gamma_q}$, such that $\mathsf{D}^*_{\gamma_q}$ is a proof sequence of $\mathbf{b}$ in which $\mathsf{A} \vdash_{\overline{\mathsf{R1^*,R2}}} \mathsf{E}$.

4.4 Since in $\mathfrak{D}^*_{\gamma_q}$ E_{γ_q} is empty, $\mathfrak{D}^*_{\gamma_q} = \{\mathsf{A};\ \mathsf{V}^*_{1\gamma_q};\ \mathsf{V}^*_{2\gamma_q};\ \mathsf{F}\}$. And, in order to simplify this rather cumbersome notation, instead of $\mathfrak{D}^*_{\gamma_q}$, $\mathsf{V}^*_{1\gamma_q}$, and $\mathsf{V}^*_{2\gamma_q}$ we shall use $\mathfrak{D}_0$, $\mathsf{V}_{1\mathsf{E}}$, and $\mathsf{V}_{2\mathsf{E}}$, respectively. Thus, $\mathfrak{D}_0 = \{\mathsf{A};\ \mathsf{V}_{1\mathsf{E}};\ \mathsf{V}_{2\mathsf{E}};\ \mathsf{F}\}$ will mean the same as $\mathfrak{D}^*_{\gamma_q} = \{\mathsf{A};\ \mathsf{V}^*_{1\gamma_q};\ \mathsf{V}^*_{2\gamma_q};\ \mathsf{F}\}$.

5 Now, let us assume that in $\mathfrak{D}$ its subsequence D, *cf.*, **3.2.1**, is not empty.

Whence, $\mathsf{D} = \{\mathsf{E};\ \mathsf{F}\}$, *cf.*, **3.3** and **3.4**, and, therefore, $\mathfrak{D} = \{\mathsf{A};\ \mathsf{V}_1;\ \mathsf{V}_2;\ \mathsf{E};\ \mathsf{F}\}$ in which at least one of the subsequences, E or F, must be unempty. Hence, if F is empty, then $\mathfrak{D} = \{\mathsf{A};\ \mathsf{V}_1;\ \mathsf{V}_2;\ \mathsf{E}\}$. In such a case, *cf.*, **4**, we are always able to replace $\mathfrak{D}$ by its augmentation $\mathfrak{D}_0 = \{\mathsf{A};\ \mathsf{V}_{1\mathsf{E}};\ \mathsf{V}_{2\mathsf{E}}\}$ such that $\mathfrak{D}_0$ is a proof sequence of b in which $\mathsf{A} \vdash_{\overline{\mathrm{R1^*,R2}}} \mathsf{b}$. Thus, if in $\mathfrak{D}$ its subsequence F is empty, Theorem A is proved.

5.1 Therefore, let us assume that in $\mathfrak{D}$ its subsequence F is not empty. Then, if in $\mathfrak{D}$, E is empty, $\mathfrak{D} = \{\mathsf{A};\ \mathsf{V}_1;\ \mathsf{V}_2;\ \mathsf{F}\}$. On the other hand, if in $\mathfrak{D}$, E is not empty, then $\mathfrak{D} = \{\mathsf{A};\ \mathsf{V}_1;\ \mathsf{V}_2;\ \mathsf{E};\ \mathsf{F}\}$ and, therefore, *cf.*, **4**, we are always able to replace $\mathfrak{D}$ by its augmentation $\mathfrak{D}_0 = \{\mathsf{A};\ \mathsf{V}_{1\mathsf{E}};\ \mathsf{V}_{2\mathsf{E}};\ \mathsf{F}\}$ such that $\mathfrak{D}_0$ is a proof sequence of b in which $\mathsf{A} \vdash_{\overline{\mathrm{R1^*,R2}}} \mathsf{E}$. Since it is self-evident that $\mathfrak{D}$, in which E is empty but F is not empty, is a particular instance of $\mathfrak{D}_0$, in the future only $\mathfrak{D}_0$ will be investigated.

Remark III: In order to avoid misunderstanding and confusion it should be noted that if $\mathfrak{D}^*$ is an arbitrary augmentation of $\mathfrak{D}$ such that $\mathfrak{D}^*$ is a proof sequence of b, then the subsequences V_1^* and V_2^* of $\mathfrak{D}^*$ are always defined in exactly the same way as V_1 and V_2, *cf.*, **3.1** and **3.2**, but, obviously, their definitions are automatically adjusted to $\mathfrak{D}^*$. Hence, e.g., in $\mathfrak{D}_0$ $\mathsf{A} \vdash_{\overline{\mathrm{R1^*,R2}}} \{\mathsf{V}_{1\mathsf{E}};\ \mathsf{V}_{2\mathsf{E}}\}$.

5.2 Assume that in $\mathfrak{D}_0$

$$\mathsf{F} = \{\mathsf{f}_1, \ldots, \mathsf{f}_t\},\ 1 \leqslant t < w,\ \mathsf{f}_t \approx \mathsf{b}.$$

Since in $\mathfrak{D}_0$, E is empty, it follows from the definition of F, *cf.*, **3.4**, that in $\mathfrak{D}_0$ there are two terms κ and λ such that both κ and λ precede f_1, i.e., the first term of F, and at least one of them must be a term of $\mathsf{V}_{2\mathsf{E}}$ and $\{\kappa,\ \lambda\} \vdash_{\overline{\mathrm{R2}}} \mathsf{f}_1$. Whence, clearly, if $\mathsf{F} = \{\mathsf{f}_1\}$, then $\mathsf{f}_1 \approx \mathsf{b}$ and, therefore, in $\mathfrak{D}_0$, $\mathsf{A} \vdash_{\overline{\mathrm{R1^*,R2}}} \mathsf{b}$. Since in such a case Theorem A is proved, let us assume that $\mathsf{F} \neq \{\mathsf{f}_1\}$. Consequently, *cf.*, **3.4**, if in $\mathfrak{D}_0$ f_k, $2 \leqslant k \leqslant t$, is a term of F, then either

(1) in F there is a term λ such that λ precedes f_k and $\{\lambda\} \vdash_{\overline{\mathrm{R1}}} \mathsf{f}_k$,

or

(2) in $\mathfrak{D}_0$ there are two terms μ and ν such that both μ and ν precede f_k, and at least one of them is either a term of $\mathsf{V}_{2\mathsf{E}}$ or a term of F, and $\{\mu,\ \nu\} \vdash_{\overline{\mathrm{R2}}} \mathsf{f}_k$.

5.3 Now, we introduce the following two definitions:

Df. 2 For any $n = 1, 2, 3, \ldots < \infty$:

$$\mathsf{F}_n = \begin{cases} \mathsf{F}_1 = \mathsf{F} \text{ from which its first term } \mathsf{f}_1 \text{ and every other term, if any,} \\ \qquad \text{which is a consequence of } \mathsf{f}_1 \text{ by R1 are removed.} \\ \mathsf{F}_m = \mathsf{F}_{m-1} \text{ from which its first term and every other term, if any,} \\ \qquad \text{which is a consequence of this first term by R1 are removed.} \end{cases}$$

Df. 3 For any $n = 1, 2, 3, \ldots < \infty$:

$$\mathsf{S}_n = \begin{cases} \mathsf{S}_1 = \mathsf{F} - \mathsf{F}_1. \\ \mathsf{S}_m = \mathsf{F}_{m-1} - \mathsf{F}_m. \end{cases}$$

Thus, for any k, $1 < k < n$, since F_k and S_k are disjoint, $\mathsf{F}_{k-1} = \{\mathsf{S}_k; \mathsf{F}_k\}$ where F_k can be empty. And, if F_k is unempty, then $\mathsf{F}_{k-1} = \{\mathsf{S}_k; \mathsf{S}_{k+1}; \mathsf{F}_{k+1}\}$ and so forth. In order to have a convenient notation in the future we use for an arbitrary k, $1 \leqslant k \leqslant n$, $\mathsf{S}_k = \{\mathsf{s}_1^k, \ldots, \mathsf{s}_x^k\}$, $1 \leqslant x \leqslant t$, assuming that if for j, $1 \leqslant j \leqslant n$, $\mathsf{S}_j \neq \mathsf{S}_k$, then S_j and S_k can have different numbers of terms. This convention will not lead to any misunderstanding in our further deductions.

Remark IV: It follows at once from our assumption concerning the structures of the sequences under investigation, *cf.*, point (c), in section **1**, that if s_j^k, $2 \leqslant j \leqslant x$, is a term of S_k, $1 \leqslant k \leqslant n$, then in $\mathsf{S}_k \{\mathsf{s}_1^k\} \vdash_{\overline{\mathrm{R1}}} \mathsf{s}_j^k$.

5.4 Let us assume that for the given k, $1 < k < n$, it was already proved that in $\mathfrak{D}_0$ its subsequence $\mathsf{F} = \{\mathsf{S}_1; \ldots; \mathsf{S}_{k-1}; \mathsf{S}_k; \mathsf{F}_k\}$ and, moreover, suppose that in F its subsequence F_k is not empty. Then, the first term of F, say s_1^{k+1}, cannot be a consequence by R1 of any term of $\mathfrak{D}_0$ which precedes it, since otherwise s_1^{k+1} would be a term of one of the subsequences $\mathsf{V}_{1\mathsf{E}}$, $\mathsf{V}_{2\mathsf{E}}$, $\mathsf{S}_1, \ldots, \mathsf{S}_{k-1}, \mathsf{S}_k$, contrary to the definition of F_k, *cf.*, Remark IV, **3.1**, **3.2**, **3.3**, **4.4**, Remark III, **5.1** and **5.3**. Hence in $\mathfrak{D}_0$ there must be two terms μ and ν such that both μ and ν precede s_1^{k+1} and $\{\mu, \nu\} \vdash_{\overline{\mathrm{R2}}} \mathsf{s}_1^{k+1}$. Therefore, in accordance with definitions Df. 2 and Df. 3 in F_k, s_1^{k+1} generates a new subsequence, viz., S_{k+1}. And, since by assumption F_k is unempty, $\mathsf{F}_k = \{\mathsf{S}_{k+1}, \mathsf{F}_{k+1}\}$. Therefore, we proved that: *For an arbitrary k, $1 < k < n$, if $\mathfrak{D}_0 = \{\mathsf{A}; \mathsf{V}_{1\mathsf{E}}; \mathsf{V}_{2\mathsf{E}}; \mathsf{S}_1; \ldots; \mathsf{S}_k; \mathsf{F}_k\}$, then $\mathfrak{D}_0 = \{\mathsf{A}; \mathsf{V}_{1\mathsf{E}}; \mathsf{V}_{2\mathsf{E}}; \mathsf{S}_1; \ldots; \mathsf{S}_k; \mathsf{S}_{k+1}; \mathsf{F}_{k+1}\}$.* This statement, together with the facts that $\mathfrak{D}_0$ is finite and that in $\mathfrak{D}_0$ S_1 is not empty, *cf.*, **5.2** and Df. 3, allows us to conclude by an elementary induction that for a certain finite y, $1 \leqslant y \leqslant t$,

$$\mathfrak{D}_0 = \{\mathsf{A}; \mathsf{V}_{1\mathsf{E}}; \mathsf{V}_{2\mathsf{E}}; \mathsf{S}_1, \ldots, \mathsf{S}_y\}, \ \mathsf{s}_z^y \approx \mathbf{b}$$

where for an arbitrary S_k, $1 \leqslant k \leqslant y$, S_k is not empty.

5.5 It follows from the definitions of F and S_n and the fact, *cf.*, **5.4**, that in $\mathfrak{D}_0$ $\mathsf{F} = \{\mathsf{S}_1; \ldots; \mathsf{S}_y\}$, $1 \leqslant y \leqslant t$, that for S_k, $1 \leqslant k \leqslant y$, in $\mathfrak{D}_0$ there must be two terms σ and τ such that both σ and τ precede s_1^k, i.e., the first term of S_k and $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \mathsf{s}_1^k$. Since $\mathfrak{D}_0 = \{\mathsf{A}; \mathsf{V}_{1\mathsf{E}}; \mathsf{V}_{2\mathsf{E}}; \mathsf{S}_1; \ldots; \mathsf{S}_y\}$, $\mathfrak{D}_0$ is a sequence of $y + 3$ mutually disjoint subsequences. Hence, since σ and τ can be the terms of the arbitrary subsequences of $\mathfrak{D}_0$ which precede S_k and they can even belong to two different subsequences, in $\mathfrak{D}_0$ there are many possible combinations such that each of them can be eventually the actual instance which satisfies $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \mathsf{s}_1^k$. In the future we shall call such possibilities in regard to S_k the generic cases of S_k. Since for our further deductions it is important to know the exact number of the generic cases for each S_h, $1 \leqslant h \leqslant y$, this problem will be investigated below.

5.5.1 Clearly, for an arbitrary S_h, $1 \leqslant h \leqslant y$, if in $\mathfrak{D}_0$ there are two terms σ and τ such that both σ and τ precede s_1^h, i.e., the first term of S_h and $\{\sigma, \tau\} \vdash_{\overline{\mathrm{R2}}} \mathsf{s}_1^h$, then the following generic cases (α) both σ and τ are the terms of A; (β) σ is a term of A and τ is a term of $\mathsf{V}_{1\mathsf{E}}$; (γ) σ is a term of $\mathsf{V}_{1\mathsf{E}}$ and τ is a term of A; and (δ) both σ and τ are the terms of $\mathsf{V}_{1\mathsf{E}}$; are impossible, since otherwise $\mathbf{s}_1^h$ would be a term of $\mathbf{V}_{2\mathbf{E}}$, *cf.*, **3.2**. Therefore, at least one of the terms, σ or τ, must be a term of $\mathbf{V}_{2\mathbf{E}}$ or a term of $\mathbf{S}_f$, $1 \leqslant f < h$. Thus, if $h = 1$, i.e., $\mathbf{S}_h = \mathbf{S}_1$, and $\{\sigma, \tau\} \vdash_{\overline{R2}} \mathbf{s}_1^1$, there are five and only five generic cases of $\mathbf{S}_1$, namely:

(a) σ is a term of $\mathbf{A}$ and τ is a term of $\mathbf{V}_{2\mathbf{E}}$,
(b) σ is a term of $\mathbf{V}_{1\mathbf{E}}$ and τ is a term of $\mathbf{V}_{2\mathbf{E}}$,
(c) σ is a term of $\mathbf{V}_{2\mathbf{E}}$ and τ is a term of $\mathbf{A}$,
(d) σ is a term of $\mathbf{V}_{2\mathbf{E}}$ and τ is a term of $\mathbf{V}_{1\mathbf{E}}$,
(e) Both σ and τ are the terms of $\mathbf{V}_{2\mathbf{E}}$.

5.5.2 It is self-evident that these five generic cases of $\mathbf{S}_1$ are also the generic cases of any $\mathbf{S}_h$, $1 < h \leqslant y$. But, since in $\mathfrak{D}_0$ the number of subsequences which precede such $\mathbf{S}_h$ is bigger than the number of subsequences which precede $\mathbf{S}_1$, there are additional generic cases of $\mathbf{S}_h$. Thus, for example, if $h = 2$, i.e., $\mathbf{S}_h = \mathbf{S}_2$, then since in $\mathfrak{D}_0$, $\mathbf{S}_2$ is preceded by $\mathbf{A}$, $\mathbf{V}_{1\mathbf{E}}$, $\mathbf{V}_{2\mathbf{E}}$, and $\mathbf{S}_1$, there are seven new generic cases of $\mathbf{S}_2$, viz. (α) σ is a term of $\mathbf{A}$, or of $\mathbf{V}_{1\mathbf{E}}$, or of $\mathbf{V}_{2\mathbf{E}}$ and τ is a term of $\mathbf{S}_1$; (β) σ is a term of $\mathbf{S}_1$ and τ is a term of $\mathbf{A}$, or of $\mathbf{V}_{1\mathbf{E}}$, or of $\mathbf{V}_{2\mathbf{E}}$; (γ) both σ and τ are the terms of $\mathbf{S}_1$; such that in $\mathbf{S}_2$ $\{\sigma, \tau\} \vdash_{\overline{R2}} \mathbf{s}_1$. Thus, there are 12 generic cases of $\mathbf{S}_2$. Similarly, there are 21 generic cases of $\mathbf{S}_3$, 32 of $\mathbf{S}_4$, 45 of $\mathbf{S}_5$ and so forth.

5.5.3 The discussion presented above enables us to establish the following formula:

Formula $\mathfrak{S}$ *For any h, $1 \leqslant h \leqslant y$, if $\mathbf{S}_h$ is a subsequence of $\mathfrak{D}_0$, then there are $h^2 + 4h$ generic cases of $\mathbf{S}_h$.*

We prove Formula $\mathfrak{S}$ as follows:

(1) If for the given m, $1 \leqslant m < y$, $\mathbf{S}_{m+1}$ is a subsequence of $\mathfrak{D}_0$, then in $\mathfrak{D}_0$ there are $3 + m$ subsequences which precede $\mathbf{S}_{m+1}$. Hence, it is self-evident that the number of all new generic cases of $\mathbf{S}_{m+1}$ is:

$$((3 + m) - 1) + ((3 + m) - 1) + 1 = 2m + 5.$$

Using the formula obtained above we define the following function:

Df. 4 For any $n = 0, 1, 2, 3, \ldots$

$$\varphi_n = \begin{cases} \varphi_0 = 5. \\ \varphi_m = \varphi_{m-1} + 2. \end{cases}$$

Clearly, the value of φ_0 is the number of generic cases of $\mathbf{S}_1$ and the values of $\varphi_1, \varphi_2, \varphi_3, \ldots$ are respectively the numbers of the new generic cases of $\mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4, \ldots$.

(2) It follows from **5.5.1**, **5.5.2**, and point (1) that for any f and g, $1 \leqslant f < g < n$, all generic cases of $\mathbf{S}_f$ are also the generic cases of $\mathbf{S}_g$, but besides them there are $2(g - 1) + 5$ new generic cases of $\mathbf{S}_g$. Then, the function defined in (1) allows us to calculate, for the given finite m, the number of all generic cases of $\mathbf{S}_m$ as a finite series of φ_n containing m components. Namely:

$$
\begin{aligned}
\sum_{n=0}^{n=m-1} \varphi_n &= \varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \ldots + \varphi_{m-2} + \varphi_{m-1} \\
&= 5 + 7 + 9 + 11 + \ldots + (2(m - 2) + 5) + (2(m - 1) + 5) \\
&= 5m + 2 + 4 + 6 + \ldots + 2(m - 2) + 2(m - 1) \\
&= 5m + 2(1 + 2 + 3 + \ldots + (m - 2) + (m - 1)) \\
&= 5m + 2\left(\frac{(m - 1)m}{2}\right) = 5m + (m^2 - m) = m^2 + 4m.
\end{aligned}
$$

Thus, for the given finite m Formula $\mathfrak{S}$ is established.

(3) It remains to prove by induction that for every $\mathbf{S}_h$, $1 \leqslant h \leqslant y$, Formula $\mathfrak{S}$ holds. Since we have

(a) Formula $\mathfrak{S}$ holds for $\mathbf{S}_1$.
(b) Assume that for the given k, $1 < k < y$, Formula $\mathfrak{S}$ holds for $\mathbf{S}_k$. Hence, *cf.*, point (1), the number of all generic cases of $\mathbf{S}_{k+1}$ is: $(k^2 + 4k) + (2k + 5) = (k^2 + 2k + 1) + (4k + 4) = (k + 1)^2 + 4(k + 1)$. Therefore, Formula $\mathfrak{S}$ holds for $\mathbf{S}_{k+1}$.

The proof of Formula $\mathfrak{S}$ is complete.

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To be continued

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