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TRANSLATION OF THE SIMPLE THEORY OF TYPES INTO A FIRST ORDER LANGUAGE

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1 *Introduction* In this paper we formulate (1) a simple theory of types, $\mathfrak{T}\omega$, (2) a first order language with postulates, $\mathfrak{T}(t)$, and (3) a set of rules for translating $\mathfrak{T}\omega$ into $\mathfrak{T}(t)$. We prove that a wff of $\mathfrak{T}\omega$ is a theorem of $\mathfrak{T}\omega$ if and only if its translation is a theorem of $\mathfrak{T}(t)$.

The set of postulates of $\mathfrak{T}(t)$ is a modification of the set of formulas D_{11}-D_9 in Hintikka's [4],[1] the main difference being that Hintikka's D_8, which contains a quantified predicate, is replaced by a series of postulates without quantified predicates. Thus we reduce type theory to a first order language where Hintikka's reduction was to a second order language. There is also a difference in approach, this paper being concerned exclusively with syntax, while Hintikka gives considerable attention to model theory.

$\mathfrak{T}(t)$ can be used to talk about the individuals and predicates of type theory in much the same way as we talk about sets in axiomatic set theory. The postulates of $\mathfrak{T}(t)$, which in their intended interpretation assert the existence of the individuals and predicates of $\mathfrak{T}\omega$ and describe their relations to each other, are roughly analogous to the axioms of set theory. We do not suggest that $\mathfrak{T}(t)$ be used to prove results that can be proved in type theory. In doing so one would lose the simplicity and directness of type theory and its capacity to reproduce the structure of intuitive mathematical thinking—a virtue not possessed by any of the popular brands of axiomatic set theory. More promising is the use of $\mathfrak{T}(t)$ to talk about the symbols and syntax of $\mathfrak{T}\omega$. For this purpose one could extend $\mathfrak{T}(t)$ by introducing predicate variables (including those of higher types) and applying quantifiers to such variables. Such an extension of $\mathfrak{T}(t)$ would be a formalized metalanguage of type theory. But this is beyond the scope of the present paper, which is to exhibit a reduction of type theory to a first order language.

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1. [4], p. 84.

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2 *A simple theory of types* The system $\mathfrak{F}\omega$ discussed here is essentially the same as the $\mathfrak{F}\omega$ of [5]. Considerations of convenience, however, have led to some changes in the notation and in the way the primitive basis is presented. The logical constants of the system are Sheffer's stroke, $|$, and the universal quantifier, $\forall$. Formulas containing other logical constants are to be regarded as abbreviations. For certain metatheoretical deductions it is convenient to assume that all wffs are written in unabbreviated form. Scopes of logical constants are shown by dots.

Variables of $\mathfrak{F}\omega$ are of the form $\overset{t}{a}$, where the superscript t is a syntactical expression that stands for a type symbol. The type symbols are composed of iotas and parentheses, where the iota ι denotes the type of individuals and all other type symbols are constructed from the iota by one or more applications of the rule: if $t_1, \ldots, t_n$ are type symbols, then $(t_1 \ldots t_n)$ is a type symbol. The atomic wffs of $\mathfrak{F}\omega$ have the form $\overset{(t_1 \cdots t_n)}{a}(\overset{t_1}{a_1}, \ldots, \overset{t_n}{a_n})$, $n \geq 1$. The letters $a, b, c, \ldots$ will be used as syntactical expressions to represent variables of arbitrary type or of a type specified in the context. In the wff $b(a_1, \ldots, a_n)$ b is said to occur in the *predicate position* and $a_1, \ldots, a_n$ in the *argument positions*. The letters $A, B, C, \ldots$ will be used as syntactical expressions for wffs. $A(b/c)$ stands for the expression which results when b is substituted for all free occurrences of c in A.

There are four axiom schemas:

(1) all truth-functional tautologies,
(2) wffs of the form

$$\forall a B \rightarrow B(c/a)$$

in which c occurs free wherever it is substituted for a,

(3) an extensionality schema of the usual kind,[2]
(4) a predicate-formation schema[3] consisting of wffs having the form

$$\exists b\ \forall a_1 \ldots \forall a_n .\ b(a_1, \ldots, a_n) \leftrightarrow C$$

where $a_1, \ldots, a_n$ are variables occurring free in C.

The rules of inference are

(1) *modus ponens*,
(2) from $A \rightarrow B$, if a is a variable not occurring free in A, to infer $A \rightarrow \forall c B(c/a)$.

3 *The systems* $\mathfrak{F}(t)$ *and* $\mathfrak{F}$ The logical constants and punctuation marks of $\mathfrak{F}(t)$ are the same as those of $\mathfrak{F}\omega$. The other symbols of $\mathfrak{F}(t)$ are

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2. *Cf.* [5], p. 261.

3. In the $\mathfrak{F}\omega$ of [5], the schema (4) of the present paper is not an axiom schema but a theorem schema. It is equivalent to axiom schema (2b) of [5].

(i) individual variables x, y, z, u, v, w with or without subscripts, (ii) the predicate symbols $=, \alpha, \varepsilon, \pi$, (iii) some expressions to be introduced by definition. Each variable of $\mathfrak{F}(t)$ may be regarded as the name of a finite sequence of variables of $\mathfrak{F}\omega$. The atomic wffs of $\mathfrak{F}(t)$ have the forms

(a) $x \,\alpha\, y$,
(b) $x \,\varepsilon\, y$,
(c) $(x, y)\,\pi\, z$,
(d) $x = y$,

where (a) is to mean that x is the name of a string $a_1, \ldots, a_n$ of variables and y of a variable b such that $b(a_1, \ldots, a_n)$ is wf in $\mathfrak{F}\omega$, (b) that x, y have the foregoing denotations and the expression $b(a_1, \ldots, a_n)$ is not only wf but also true, (c) that z names a string $a_1, \ldots, a_m, b_1, \ldots, b_n$, x names $a_1, \ldots, a_m$ and y names $b_1, \ldots, b_n$. The letters $X, Y, \ldots$ will be used as syntactical expressions for wffs. $X(y/z)$ stands for the result of substituting y for all free occurrences of z in X.

The axiom schemas of $\mathfrak{F}(t)$ are the exact analogues of schemas (1) and (2) of $\mathfrak{F}\omega$ and the rules of inference are the exact analogues of the two rules of $\mathfrak{F}\omega$. The postulates will be shown next. We shall use a system of numbering of postulates, theorems and definitions in which a number followed by **P** refers to a postulate, by **D** to a definition, by **T** to a theorem.

1P $x = x.$
2P $x = y \rightarrow .\, X \rightarrow X(x/y).$
3P $x \varepsilon y \rightarrow x \alpha y.$
4P $\forall x\, \exists y\, \forall z.\, x \alpha y \wedge \neg z \varepsilon y.$

The predicate symbol η is introduced by the definition

5D $x \eta y \leftrightarrow_{def} \exists z\colon x \alpha z \wedge y \alpha z.$

The expression $x \eta y$ is to mean that x and y are names of variables, or of strings of variables, of the same type(s).

6P $x \alpha y \rightarrow .\, x \alpha z \leftrightarrow y \eta z.$
7T *η is an equivalence relation.*

Proof: η is reflexive by **4P**, symmetric by **5D**, transitive by **6P**.

8T $u \eta v \wedge u \alpha x \wedge v \alpha y. \rightarrow x \eta y.$

Proof: The premiss means there is z such that $u \alpha z \wedge v \alpha z$ from which by **6P** $x \eta z \wedge y \eta z$, whence the result by **7T**.

9P $\forall x\, \forall y\, \exists z\, (x, y)\, \pi\, z.$
10P $(x, y)\, \pi\, z \wedge (u, v)\, \pi\, w. \rightarrow\colon x = u \wedge y = v. \leftrightarrow z = w.$
11P $(x, y)\, \pi\, z \wedge (u, v)\, \pi\, w. \rightarrow\colon x \eta u \wedge y \eta v. \leftrightarrow z \eta w.$
12P $(x, y)\, \pi\, z \mid u \alpha z.$
13P $(x, y)\, \pi\, z \wedge z \eta w. \rightarrow \exists u\, \exists v\, (u, v)\, \pi\, w.$

The following definition makes it possible to eliminate the predicate π in some contexts, and so to simplify notation.

14D $\langle x, y\rangle = z \leftrightarrow_{def} (x, y)\,\pi\, z$.

Where the expression $\langle x, y\rangle$ occurs in a formula X, and not in the scope of a quantifier $\forall x$ or $\forall y$, we shall regard X as an abbreviation for

$$\exists z\,.\,(x, y)\,\pi\, z \wedge X(z/\langle x, y\rangle).$$

The associativity rule for concatenation is expressed by

15P $(\langle x, y\rangle, u)\,\pi\, w \leftrightarrow (x, \langle y, u\rangle)\,\pi\, w$

which permits us to use the definition

16D $\langle x, y, z\rangle =_{def} \langle\langle x, y\rangle z\rangle$.

We now introduce as predicates in $\mathfrak{F}(t)$ the type symbols which appear as superscripts to the variables of $\mathfrak{F}\omega$.

17D $x \epsilon\, \iota \leftrightarrow_{def}.\ \neg\exists u\ u\alpha x \wedge \neg\exists y\ \exists z\ (y, z)\,\pi\, x$.

The expression $x \epsilon\, \iota$ is to mean that x denotes an individual variable of $\mathfrak{F}\omega$. The letter ϵ is a syntactical expression denoting set membership (perhaps more appropriately termed the relation of argument to predicate) in $\mathfrak{F}(t)$, not to be confused with ε which is a predicate of $\mathfrak{F}(t)$ denoting set membership in $\mathfrak{F}\omega$.

Other types are defined inductively by the schema

18D $y \epsilon\, (t_1 \ldots t_n) \leftrightarrow_{def} \exists x_1 \ldots \exists x_n\,.\,x_1 \epsilon\, t_1 \wedge \ldots \wedge x_n \epsilon\, t_n \wedge \langle x_1, \ldots, x_n\rangle\, \alpha\, y$,
19P $\exists x\ x \epsilon\, \iota$.

For each type t the theorem

20T $\exists x\ x \epsilon\, t$

follows from **19P, 4P, 9P.**

21P $x \epsilon\, \iota \wedge y \epsilon\, \iota\,. \rightarrow x \eta y$.

For each type t we have the theorem

22T $x \epsilon\, t \rightarrow \forall y\,.\,y \epsilon\, t \leftrightarrow y \eta x$.

First, to prove $x \epsilon\, t \rightarrow \forall y.\ y \epsilon\, t \rightarrow y \eta x$: if t is ι the result is **21P.** For all other t the result may be proved by induction, using **8T, 11P.**

To prove the other half: suppose $x \epsilon\, \iota$ and $x \eta y$. Then $u \alpha y$ is impossible since by **6P** this would imply $u \alpha x$. Similarly $(u, v)\,\pi\, y$ is impossible by **13P.** This proves the result for the case where t is ι. For the higher types the result may be proved by induction, using **6P, 11P.**

This shows that the types are equivalence classes of η. We need also to show that distinct type symbols denote distinct η-classes, or equivalently, that

23T *For every pair of distinct types,* t_1, t_2,

$$x \epsilon\, t_1 | x \epsilon\, t_2.$$

Proof: First suppose that one of the types, say t_1, is ι. Then t_2 is one of the higher types, so that $x \epsilon t_2 \rightarrow \exists u\, u \alpha x$ which is incompatible with $x \epsilon \iota$ by **17D**. That proves the theorem for this special case. The general case can be proved by induction. Let us say that t_i is *subordinate* to t_j if the type symbol represented by t_i is contained in that represented by t_j. Thus (ι) is subordinate to $(\iota(\iota))$ since the latter symbol contains (ι). Clearly ι is subordinate to every type. We shall say that t_i is *properly* subordinate to t_j if t_i and t_j represent distinct type symbols and t_i is subordinate to t_j.

Let us assume, as an induction hypothesis, that the theorem has been proved for every type which is properly subordinate to either or both of t_1, t_2. It is clearly sufficient to show that this hypothesis implies t_1, t_2 are disjoint. Since we have already proved the theorem for the special case where one of the two types is ι, we may assume that both t_1 and t_2 are higher types, that is to say, $t_1 = (t_{11} \ldots t_{1m})$ and $t_2 = (t_{21} \ldots t_{2n})$. In consequence of **17D** and **22T**,

$$\vdash \forall x :.\ x \epsilon t_1 \rightarrow \forall y_1 \ldots \forall y_m \colon y_1 \epsilon t_{11} \wedge \ldots \wedge y_m \epsilon t_{1m} . \leftrightarrow \langle y_1 \ldots y_m \rangle \alpha x$$

and a similar formula can be proved for t_2. In order that a variable x may belong to both of t_1, t_2 we must have $n = m$ in consequence of **12P**. By the induction hypothesis it is also necessary that, for $1 \leq i \leq m = n$, the type symbols represented by t_{1i}, t_{2i} be identical. But these conditions are satisfied only if t_1 and t_2 represent the same symbol. This completes the proof.

Next we present some postulates which have the consequence that the translations in $\mathfrak{F}(t)$ of axioms of $\mathfrak{F}\omega$ belonging to schemas (3) and (4) are theorems of $\mathfrak{F}(t)$. For the extensionality schema

24P $\forall x\, \forall y :.\ \exists u.\ u \alpha x \wedge u \alpha y . \rightarrow : \forall u.\ u \varepsilon x \leftrightarrow u \varepsilon y . \rightarrow x = y,$

and for the predicate formation schema the seven postulates

25P $\forall x\, \forall y\, \exists z :: z \eta x \wedge \forall u \therefore u \alpha x \wedge u \alpha y . \rightarrow : u \varepsilon z \leftrightarrow . u \varepsilon x | u \varepsilon y,$

26P $\forall x\, \forall y\, \exists z\, \forall u\, \forall v :.\ u \alpha x \wedge v \alpha y . \rightarrow \langle u, v \rangle \alpha z : \wedge : \langle u, v \rangle \varepsilon z \leftrightarrow . u \varepsilon x \,|\, v \varepsilon y,$

27P $\forall x\, \forall y\, \exists z\, \forall u.\ \langle u, y \rangle \alpha x \rightarrow u \alpha z . \wedge .\ u \varepsilon z \leftrightarrow \langle u, y \rangle \varepsilon x,$

28P $\forall x\, \exists z\, \forall u\, \forall v.\ \langle u, v \rangle \alpha x \rightarrow \langle v, u \rangle \alpha z . \wedge . \langle v, u \rangle \varepsilon z \leftrightarrow \langle u, v \rangle \varepsilon x.$

From **27P**, **28P** one may deduce

29T $\forall x\, \exists z\, \forall u\, \forall v\, \forall w \colon \langle u, v, w \rangle \alpha x \rightarrow \langle v, u, w \rangle \alpha z . \wedge . \langle v, u, w \rangle \varepsilon z$
$\leftrightarrow \langle u, v, w \rangle \varepsilon x,$

30P $\forall x\, \exists z\, \forall u \colon \langle u, u \rangle \alpha x \rightarrow u \alpha z . \wedge . u \varepsilon z \leftrightarrow \langle u, u \rangle \varepsilon x.$

From **27P**, **30P** one may deduce

31T $\forall x\, \exists z\, \forall u\, \forall v \colon \langle u, u, v \rangle \alpha x \rightarrow \langle u, v \rangle \alpha z . \wedge . \langle u, v \rangle \varepsilon z \leftrightarrow \langle u, u, v \rangle \varepsilon x,$

32P $\forall x\, \exists z\, \forall y\, \forall u \colon x \alpha y \rightarrow \langle x, y \rangle \alpha z . \wedge . \langle u, y \rangle \varepsilon z \leftrightarrow u \varepsilon y,$

33P $\forall x\, \exists z \colon \forall u\, \forall v.\ \langle u, v \rangle \alpha x \rightarrow u \alpha z . \wedge . u \varepsilon z \leftrightarrow \forall v \langle u, v \rangle \varepsilon x.$

Each of the last seven postulates has three parts: first, a prefix which begins with $\forall x\, \forall y\, \exists z$ ($\forall x\, \exists z$); second, a part from which, if we know the type(s) of x and y (of x) we can determine the type of z; third, a part that specifies how the extension of z is determined by the extension(s) of x and y (of x). The third parts are adequate for description in $\mathfrak{F}(t)$ of all those procedures, such as set intersection, Cartesian product, relative product, by which predicates can be constructed in type theory, as will be shown in section **6**.

This completes the description of the basis of $\mathfrak{F}(t)$.

The system $\mathfrak{F}$ has the same wffs, the same axioms and rules of inference as $\mathfrak{F}(t)$, but no postulates.

4 *Two lemmas* By a *valuation* of a set of wffs of $\mathfrak{F}$ or of $\mathfrak{F}\omega$ we mean a mapping of that set into the set of truth values, $\{\mathbf{t}, \mathbf{f}\}$. Given any wff of $\mathfrak{F}$ having the form $\forall xY$, by an *instance* of that wff we mean a wff of the form $Y(y/x)$ where y is free wherever it is substituted for x.[4] We shall say that a valuation $\mathbf{v}$ of a set of wffs is *consistent* if it satisfies these two conditions:

(1) given any wff of the form $X|Y$, $\mathbf{v}(X|Y) = \mathbf{t}$ if and only if $\mathbf{v}$ assigns value $\mathbf{f}$ to at least one of the components X, Y,
(2) given any wff of the form $\forall xY$, $\mathbf{v}(\forall xY) = \mathbf{t}$ iff, for every instance $Y(y/x)$[5] of the given wff, $\mathbf{v}(Y(y/x)) = \mathbf{t}$.

For $\mathfrak{F}\omega$ the definition of instance is the exact analogue of that given above. A valuation of a set of wffs of $\mathfrak{F}\omega$ is consistent if it satisfies the analogues of (1), (2) and also

(3) if A is an axiom, then $\mathbf{v}(A) = \mathbf{t}$.

For the first and second axiom schemas the third condition of consistency is entailed by the first two, but for the third and fourth it is independent.

Lemma 1. *Let* $\mathbf{v}$ *be an arbitrary valuation of the atomic wffs of a pure first order predicate calculus, in particular, of* $\mathfrak{F}$. *Then* $\mathbf{v}$ *can be extended to a consistent valuation of all the wffs of that system.*

Proof: An extension $\mathbf{w}$ of $\mathbf{v}$ is defined inductively by the following sequence of rules.

Rule 1. If X is quantifier-free, $\mathbf{w}(X)$ is the value determined, through the truth-table of X, by the values assigned by $\mathbf{v}$ to the atomic components of X.
Rule $2n$ ($n \geq 1$). If values have been assigned by the preceding rules to the instances $Y(y/x)$ of a wff $\forall xY$, then $\mathbf{v}(\forall xY) = \mathbf{t}$ if the value $\mathbf{t}$ has been assigned to all the instances; otherwise $\mathbf{v}(\forall xY) = \mathbf{f}$.
Rule $2n + 1$. If X is a truth-functional combination of wffs $Y_1, \ldots, Y_n$ to

4. If any occurrence of x in Y is in the scope of a quantifier $\forall y$, let z be the first variable (in alphabetical order) which does not occur in Y, and let Y' be the expression which results when z is substituted for y in Y. Then $Y'(y/x)$ is to be considered an instance of $\forall x Y$.

5. Or $Y'(y/x)$ defined in the preceding note.

which values have been assigned by the preceding rules, then $\mathsf{w}(X)$ is the value which results by the truth-table of that combination from the values assigned to $Y_1, \ldots, Y_n$.

It is easy to see that for every wff X the value $\mathsf{w}(X)$ is uniquely determined in such a way that the conditions of consistency are satisfied.

Lemma 2. *Let A be a wff of $\mathfrak{F}\omega$. Then A is a theorem if and only if, for every consistent valuation* v *of all the wffs,* $\mathsf{v}(A) = \mathbf{t}$. *An exactly analogous result holds for the system* $\mathfrak{F}$.

Proof: To prove the "only if" one may verify for every axiom A belonging to schema (1) or schema (2) that $\mathsf{v}(A) = \mathbf{t}$ if v is consistent, and for the two rules of inference that every consistent valuation which assigns $\mathbf{t}$ to the premiss or premisses must assign this value to the conclusion.

To prove the "if" one can use the fact, proved by Henkin in [2] for the first order calculus and in [3] for type theory, that if $\neg A$ is not a theorem, then there is a maximal consistent set Γ of wffs of which A is a member. Clearly a valuation v such that $\mathsf{v}(A) = \mathbf{t}$ if $A \in \Gamma$ and $\mathsf{v}(A) = \mathbf{f}$ otherwise is a consistent valuation in the sense of the foregoing definition.[6]

5 *Translations* Let n be a one-one mapping of the set composed of all finite sequences $a_1, \ldots, a_n$ $(n \geq 1)$ of variables of $\mathfrak{F}\omega$ onto the set of variables of $\mathfrak{F}(t)$. The syntactical expression $x = n(a_1, \ldots, a_n)$ may be read "x is the name of $a_1, \ldots, a$." The variables of $\mathfrak{F}(t)$ which are names of sequences composed of two or more variables of $\mathfrak{F}\omega$ are not used in writing translations of the wffs of $\mathfrak{F}\omega$, but we shall have a use for them in section **6**.

For each wff A of $\mathfrak{F}\omega$ we define the following wffs of $\mathfrak{F}(t)$: $\mathrm{tr}(A)$ which we call the translation of A, $\mathrm{r}(A)$, and if there are free variables in A, $\mathrm{t}(A)$. In the definitions which follow it is to be understood that $\mathbf{y}$, $\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_n$ are the names, respectively, of b, a, a_1, a_n.

If there are free variables in A let $a_1, \ldots, a_n$ be a complete list of these variables and let $t_1, \ldots, t_n$ be their types. Then

$$\mathrm{t}(A) \text{ is } \mathbf{x}_1 \in t_1 \wedge \ldots \wedge \mathbf{x}_n \in t_n.$$

If A is the atomic wff $b(a)$, then

$$\mathrm{r}(A) \text{ is } \mathbf{x}\, \varepsilon\, \mathbf{y}.$$

If A is the atomic wff $b(a_1, \ldots, a_n)$, then

$$\mathrm{r}(A) \text{ is } \langle \mathbf{x}_1, \ldots, \mathbf{x}_n \rangle\, \varepsilon\, \mathbf{y}.$$

If A is the wff $a = b$, then

$$\mathrm{r}(A) \text{ is } \mathbf{x} = \mathbf{y}.$$

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6. The simplified proof of Henkin's theorem in [1], §54, can easily be adapted for use here.

If A is $B|C$, then

$$\mathsf{r}(A) \text{ is } \mathsf{r}(B)|\mathsf{r}(C).$$

If A is $\forall aB$ and t is the type of a, then

$$\mathsf{r}(A) \text{ is } \forall \mathbf{x}.\ \mathbf{x} \in t \rightarrow \mathsf{r}(B).$$

If A is a closed wff, then

$$\mathsf{tr}(A) \text{ is } \mathsf{r}(A),$$

otherwise,

$$\mathsf{tr}(A) \text{ is } \mathsf{t}(A) \rightarrow \mathsf{r}(A).$$

6 *The main result*

Let A be a wff of $\mathfrak{F}\omega$. Then A is a theorem of $\mathfrak{F}\omega$ if and only if $\mathsf{tr}(A)$ *is a theorem of* $\mathfrak{F}(t)$.

Throughout the proof of this theorem let it be understood that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{u}$ are names of a, b, c, d respectively except where otherwise specified, and that $\mathbf{w}_1$ names the string $a_1, \ldots, a_m$, $\mathbf{w}_2$ the string $b_1, \ldots, b_n$.

We prove the "if" in its contrapositive form. Suppose, then, that A is not a theorem. By Lemma 2 there is a consistent valuation $\mathbf{V}$ of all the wffs of $\mathfrak{F}\omega$ such that $\mathbf{V}(A) = \mathbf{f}$.

We now construct a valuation $\mathbf{v}$ of the wffs of $\mathfrak{F}$, i.e., of $\mathfrak{F}(t)$ by the following rules. Let Y be a wff of $\mathfrak{F}$.

(1) If Y is the wff $\mathbf{w}_1 \varepsilon \mathbf{y}$, let $\mathbf{v}(Y) = \mathbf{t}$ if the expression $b(a_1, \ldots, a_m)$ is wf in $\mathfrak{F}\omega$ and value $\mathbf{t}$ is assigned to it by $\mathbf{V}$; otherwise $\mathbf{v}(Y) = \mathbf{f}$.
(2) If Y is the wff $\mathbf{w}_1 \alpha \mathbf{y}$, let $\mathbf{v}(Y) = \mathbf{t}$ if $b(a_1, \ldots, a_m)$ is wf in $\mathfrak{F}\omega$; otherwise $\mathbf{v}(Y) = \mathbf{f}$.
(3) If Y is the wff $\mathbf{w}_1 = \mathbf{w}_2$, let $\mathbf{v}(Y) = \mathbf{t}$ if $m = n$, $\mathbf{V}(a_1 = b_1) = \ldots = \mathbf{V}(a_m = b_n) = \mathbf{t}$; otherwise $\mathbf{v}(Y) = \mathbf{f}$.
(4) If Y is the wff $(\mathbf{w}_1, \mathbf{w}_2)\pi\mathbf{w}$, let $\mathbf{v}(Y) = \mathbf{t}$ if $\mathbf{w} = \mathsf{n}(c_1, \ldots, c_{m+n})$ and $\mathbf{V}(a_1 = c_1) = \ldots = \mathbf{V}(a_m = c_m) = \mathbf{V}(b_1 = c_{m+1}) = \ldots = \mathbf{V}(b_n = c_{m+n}) = \mathbf{t}$; otherwise $\mathbf{v}(Y) = \mathbf{f}$.

These four rules assign a value to every atomic wff of $\mathfrak{F}$. Let $\mathbf{v}$ be the consistent valuation of all the wffs which result by the proof of Lemma 1 from the values assigned by these rules.

We assert that (i) for each wff A of $\mathfrak{F}\omega$, $\mathbf{v}(\mathsf{tr}(A)) = \mathbf{V}(A)$, and (ii) to each of the postulates of $\mathfrak{F}(t)$ $\mathbf{v}$ assigns value $\mathbf{t}$. We now prove (i).

(a) Let Y be a wff of the form $\mathbf{x} \in t$. By induction on the types one can deduce from Rules (2) and (4) that $\mathbf{v}(Y) = \mathbf{t}$ if and only if $\mathbf{x}$ is the name of a variable of type t.
(b) It follows that for every wff A of $\mathfrak{F}\omega$, $\mathbf{v}(\mathsf{t}(A)) = \mathbf{t}$. In view of the tautology $p \rightarrow: p \rightarrow q. \leftrightarrow q$ and the consistency of $\mathbf{v}$, this implies $\mathbf{v}(\mathsf{tr}(A)) = \mathbf{v}(\mathsf{r}(A))$, so it is sufficient to prove $\mathbf{v}(\mathsf{r}(A)) = \mathbf{V}(A)$.
(c) If A is the wff $b(a)$, it is immediate from Rule (1) that $\mathbf{v}(\mathsf{r}(A)) = \mathbf{V}(A)$.

(d) Suppose A is $b(a_1, a_2)$ and consider first the case $\mathbf{V}(A) = \mathbf{t}$. By Rule (1) $\mathbf{v}(\mathsf{w}_1 \varepsilon \mathsf{y}) = \mathbf{t}$. By (4) $\mathbf{v}((\mathsf{x}_1, \mathsf{x}_2) \pi \mathsf{w}_1) = \mathbf{t}$. So, since $\mathbf{v}$ is consistent it must assign $\mathbf{t}$ to

$$\exists \mathsf{w}.\ (\mathsf{x}_1, \mathsf{x}_2) \pi \mathsf{w} \wedge \mathsf{w} \varepsilon \mathsf{y}$$

which is the unabbreviated form of $\mathrm{r}(A)$.

Next consider the case $\mathbf{V}(A) = \mathbf{f}$. By Rules (4) and (3) $\mathbf{v}$ can assign $\mathbf{t}$ to $(\mathsf{x}_1, \mathsf{x}_2) \pi \mathsf{w}$ only if there are c_1, c_2 such that $\mathsf{w} = n(c_1, c_2)$ and $\mathbf{V}(c_1 = a_1) = \mathbf{V}(c_2 = a_2) = \mathbf{t}$, but the latter condition in the present case implies $\mathbf{V}(b(c_1, c_2)) = \mathbf{f}$, hence by Rule (1) $\mathbf{v}(\mathsf{w} \varepsilon \mathsf{y}) = \mathbf{f}$. Thus every instance of the negation of $\mathrm{r}(A)$ gets value $\mathbf{t}$, so by consistency of $\mathbf{v}$, $\mathbf{v}(\mathrm{r}(A)) = \mathbf{f}$. So our assertion is proved for $b(a_1, a_2)$. A straightforward extension of the proof applies to $b(a_1, \ldots, a_m)$, $m > 2$.

(e) If A is $a = b$, the statement $\mathbf{v}(\mathrm{r}(A)) = \mathbf{V}(A)$ is immediate by Rule (3).

(f) By (c) and (d), $\mathbf{v}(\mathrm{r}(A)) = \mathbf{V}(A)$ holds for all atomic A. Since both $\mathbf{V}$ and $\mathbf{v}$ satisfy the first condition of consistency (*cf.* section **4**), this identity must hold for all quantifier-free A, so by (b) assertion (i) holds for all quantifier-free A.

(g) Now suppose A is $\forall aB$ where B is quantifier-free, and consider first the case $\mathbf{V}(A) = \mathbf{t}$. By consistency of $\mathbf{V}$, all instances of $\forall aB$ have value $\mathbf{t}$ under $\mathbf{V}$. If $\mathbf{t}$ is the type of a, all instances of $\mathrm{r}(A)$ have the form

$$* \qquad \mathsf{y} \epsilon t \rightarrow \mathrm{r}(B)(\mathsf{y}/\mathsf{x})$$

where y is an arbitrary variable. If y is the name of a variable of type t, then $\mathrm{r}(B)(\mathsf{y}/\mathsf{x})$ is the image under r of an instance of $\forall aB$, hence has value $\mathbf{t}$ under $\mathbf{v}$ by (f). So the formula * has value $\mathbf{t}$ under $\mathbf{v}$ in this case.

In the case where y is not the name of a variable of type t, $\mathbf{v}(\mathsf{y} \epsilon t) = \mathbf{f}$ and again * has value $\mathbf{t}$. Thus $\mathbf{v}$ assigns $\mathbf{t}$ to all instances of $\mathrm{r}(\forall aB)$ and so to $\mathrm{r}(\forall aB)$ itself.

If $\mathbf{V}(\forall aB) = \mathbf{f}$, then there is an instance of $\forall aB$ which also has value $\mathbf{f}$ under $\mathbf{V}$, and it is easy to see that in this case there is also an instance of $\mathrm{r}(\forall aB)$ to which $\mathbf{v}$ assigns value $\mathbf{f}$.

By a sequence of applications of the arguments under (f) and (g) one can prove $\mathbf{v}(\mathrm{r}(A)) = \mathbf{V}(A)$ for any wff of $\mathfrak{F}\omega$ having one or more quantifiers. This completes the proof of (i).

Assertion (ii) can be verified for each postulate, one by one, a tedious routine which we omit. The "if" part of the theorem follows easily from (i) and (ii).

To prove the "only if" part it is sufficient to show that (iii) if A is an axiom of $\mathfrak{F}\omega$ then $\mathrm{tr}(A)$ is a theorem of $\mathfrak{F}(t)$, and (iv) if B follows from A and $A \rightarrow B$ by the first (from A by the second) rule of inference, then $\mathrm{tr}(B)$ is deducible in $\mathfrak{F}(t)$ from $\mathrm{tr}(A)$ and $\mathrm{tr}(A \rightarrow B)$ (from $\mathrm{tr}(A)$).[7] The proof that (iii)

7. See [1], pp. 312-313, for those properties of Γ in consequence of which Γ is the set of wffs to which value t is assigned by a consistent valuation.

holds for axioms of $\mathfrak{F}\omega$ belonging to schemas (1) and (2) is elementary. (iv) follows quite easily from **20T**. It follows easily from **24P** that (iii) holds for (3), the extensionality schema.

For schema (4) the proof of (iii) is not quite so simple. First consider the special case (a) where the wff C in the formula of schema (4) is a quantifier-free wff in which the string of variables in the argument positions (i.e., inside the parentheses) is the same in each of the atomic components of C and in the expression $b(a_1, \ldots, a_n)$ occurring in the left side of the formula. In this case the predicates in atomic components of C all belong to the same type as b. Such an axiom is

$$\exists c\, \forall d\colon c(d) \leftrightarrow .\, a(d) \,|\, b(d),$$

of which the translation is (if t is the type of d),

$$\mathbf{x},\, \mathbf{y} \in (t) \rightarrow\!:\cdot:\, \exists \mathbf{z}::\mathbf{z} \in (t) \wedge \forall \mathbf{u}:.\mathbf{u} \in t \rightarrow: \mathbf{u} \varepsilon \mathbf{z} \leftrightarrow .\, \mathbf{u} \varepsilon \mathbf{x} \,|\, \mathbf{u} \varepsilon \mathbf{y},$$

which is easily deduced from **25P**. In instances of case (a) where C is more complex, the translation of the axiom can be proved by a sequence of applications of **25P**, consisting of one application for each stroke occurring in the unabbreviated form of C.

Next, consider the case of a quantifier-free wff in which (α) no variable occurs more than once in the argument positions of C, (β) the string of variables in the argument positions of $b(a_1, \ldots, a_n)$ lists *all* the variables in the argument positions of C, and (γ) lists them in the *order* in which they occur in those positions. In this case the translation of the axiom can be proved by a sequence of applications of **26P** and **25P**. Thus, the translation of

$$\exists c\colon c(d_1, d_2) \leftrightarrow .\, a(d_1) \wedge b(d_2)$$

can be proved by one application of **26P** followed by one of **25P**.

Next, suppose that conditions (α) and (β) are satisfied but not (γ). Then the translation of the axiom can be proved by first proceeding as in case (b), then using **28P**, **29T**, or both. These two (with **9P**) are adequate for all permutations. If (α) is not satisfied one may use **30P**, **31T**, or both, in conjunction where necessary with **28P**, **29T**. If (β) is not satisfied, one may use **27P** if the string $a_1, \ldots, a_n$ does not include all the variables occurring in the argument positions of C, and **32P** if this string includes some variables occurring in predicate positions and not in argument positions of C. So it is proved that all translations of axioms of schema (4) in which C is quantifier-free are theorems of $\mathfrak{F}(t)$. It is easy to see that **33P** permits us to remove the restriction to quantifier-free C. This completes the proof.

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