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A RESULT OF EXTENDING BOCHVAR'S 3-VALUED LOGIC

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In this note I shall adopt the notation that Nicholas Rescher uses in [1]. Thus lower case Roman letters are meta-variables, and lower case Greek letters are object variables. We begin with Bochvar's basic system B_3:

p	$\neg p$
T	F
I	I
F	T

	$p \wedge q$: T	I	F	$p \vee q$: T	I	F	$p \rightarrow q$: T	I	F	$p \leftrightarrow q$: T	I	F
T	T	I	F	T	I	T	T	I	F	T	I	F
I	I	I	I	I	I	I	I	I	I	I	I	I
F	F	I	F	T	I	F	T	I	T	F	I	T

First we extend this in the usual way by adopting an assertion operator defined truth-functionally:

p	Ap
T	T
I	F
F	F

and using it to define new connectives:

'$\bar{\neg} p$' for '$\neg Ap$'
'$p \mathbin{\bar{\wedge}} q$' for '$Ap \wedge Aq$'
'$p \mathbin{\bar{\vee}} q$' for '$Ap \vee Aq$'
'$p \Rightarrow q$' for '$Ap \rightarrow Aq$'
'$p \Leftrightarrow q$' for '$Ap \leftrightarrow Aq$'.

This generates the following matrices:

p	$\bar{\neg} p$
T	F
I	T
F	T

	$p \mathbin{\bar{\wedge}} q$: T	I	F	$p \mathbin{\bar{\vee}} q$: T	I	F	$p \Rightarrow q$: T	I	F	$p \Leftrightarrow q$: T	I	F
T	T	F	F	T	T	T	T	F	F	T	F	F
I	F	F	F	T	F	F	T	F	T	F	T	T
F	F	F	F	T	F	F	T	T	T	F	T	T

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But all of this is well known (*cf.* for example [1]). Rescher calls this new system B_3^E, and I shall adopt his usage. We now construct a new extension, starting out in exactly the same way, by adopting the same assertion operator. But where in B_3^E the assertion operator applies to the variables in the definitions, in our new extension, call it $B_3^{E'}$, it will apply to the whole formulae:

$$
\begin{aligned}
&`\bar{p}\text{' for `}A\neg p\text{'}\\
&`p \cdot q\text{' for `}A(p \wedge q)\text{'}\\
&`p \circledvee q\text{' for `}A(p \vee q)\text{'}\\
&`p > q\text{' for `}A(p \rightarrow q)\text{'}\\
&`p <> q\text{' for `}A(p \leftrightarrow q)\text{'}.
\end{aligned}
$$

These definitions generate the following matrices:

p	$\bar{p}$
T	F
I	F
F	T

	$p \cdot q$			$p \circledvee q$			$p > q$			$p <> q$		
	T	I	F	T	I	F	T	I	F	T	I	F
T	T	F	F	T	F	T	T	F	F	T	F	F
I	F	F	F	F	F	F	F	F	F	F	F	F
F	F	F	F	T	F	F	T	F	T	F	F	T

In $B_3^{E'}$ the following hold: one of α or $\bar{\alpha}$ will, for any α, take the value F; $\alpha \cdot \bar{\alpha}$ will, for any α, take the value F; but $\alpha \circledvee \bar{\alpha}$ will *not*, for arbitrary α, take the value T even though $\overline{(\alpha \cdot \bar{\alpha})}$ does, for any α, take the value T. An inspection of the matrices will reveal the following to be tautologies:

(1) $\bar{p} \rightarrow \neg p$
(2) $(p \cdot q) \leftrightarrow (p \mathbin{\overset{\wedge}{\wedge}} q)$
(3) $(p \circledvee q) \rightarrow (p \mathbin{\overset{\vee}{\vee}} q)$
(4) $(p > q) \rightarrow (p \Rightarrow q)$
(5) $(p <> q) \rightarrow (p \Leftrightarrow q)$.

Hence if we take tautologies to be theorems, B_3^E contains $B_3^{E'}$. Furthermore none of the following take the value F, as a look at the matrices will reveal:

(6) $\bar{p} \leftrightarrow \neg p$
(7) $(p \cdot q) \leftrightarrow (p \wedge q)$
(8) $(p \circledvee q) \leftrightarrow (p \vee q)$
(9) $(p > q) \leftrightarrow (p \rightarrow q)$
(10) $(p <> q) \leftrightarrow (p \leftrightarrow q)$.

Interestingly enough, though, the matrices also reveal that

(11) $(p \leftrightarrow q) \leftrightarrow (p \Leftrightarrow q)$

does not take the value F. Yet

(12) $(p <> q) \leftrightarrow (p \Leftrightarrow q)$

is rejected because it *does* take the value F. Thus in these extensions '$\leftrightarrow$' is not transitive, or more paradoxically, equivalence is not an equivalence relation!

From this I conclude that one of the following must be the case. Either we must abandon the idea that logical equivalence (and hence also implication) is transitive. Or we must reject the extensions of B_3 (and it should be noted that since B_3^E contains $B_3^{E'}$, if the latter is rejected, then so must the former) on the grounds that they make B_3 incoherent. Or we must reject B_3 itself on the grounds that these extensions show it to be incoherent. Of these the first is clearly unacceptable. The third is suspect especially in view of the fact that if we take the theses of B_3 to be those formulae that never take the value **F**, we get a system that is isomorphic to the classical two-valued calculus. Thus I suggest that we opt for the second, and abandon the attempt to get tautologies into B_3.*

REFERENCES

[1] Rescher, N., *Many-valued Logics*, McGraw-Hill Book Company (1969).

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