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MANY-ONE DEGREES ASSOCIATED WITH PARTIAL PROPOSITIONAL CALCULI

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Introduction Throughout this paper we shall use **PPC** as an abbreviation for partial propositional calculus and **PIPC** as an abbreviation for partial implicational propositional calculus. At the Princeton Bicentennial in 1946, Tarski raised the question as to whether certain problems associated with **PPC**'s were recursively unsolvable. This ultimately triggered a series of papers concerned with these problems, central among which are Linial and Post [4], Yntema [11], Gladstone [2], Ihrig [3], and Singletary [7], [8], [9], and [10].

Here we shall be concerned with the nature of the sets represented by decision problems for **PPC**'s and **PIPC**'s. In [3] Ihrig showed that every recursively enumerable (r.e.) degree of unsolvability could be represented by a **PPC**. In Gladstone [2] and Singletary [8] it is shown that every r.e. degree of unsolvability can be represented by a **PIPC** (and hence also by a **PPC**). In particular we now show that every many-one r.e. degree of unsolvability may be represented by the decision problem for a **PIPC** (**PPC**), and, furthermore, that this result is "best possible" in the sense that not every one-one degree may be so represented.

This result seems somewhat surprising to us in view of the well-known result that not every many-one degree may be represented by the decision problem for a first order theory; see, e.g., Rogers [6]. The obvious conclusion, of course, is that the class of sets represented by decision problems for **PIPC**'s (**PPC**'s) is richer than the class of sets represented by decision problems for first order theories.

Preliminary Definitions In order to expedite the exposition to follow, we shall use the following somewhat non-standard formulation of a semi-Thue system which is easily shown to be equivalent to the standard formulation.

A *semi-Thue system* shall consist of a finite alphabet A and a finite set of defining relations U where the members of U are pairs of words over A.

$$A : a_1, a_2, \ldots, a_n$$
$$U : A_1 \rightarrow B_1, A_2 \rightarrow B_2, \ldots, A_m \rightarrow B_m.$$

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A *word* is a finite (possibly empty) string of symbols over A, with possible repetitions. We shall define $C \vdash D$, where C and D are words over A to be the assertion that there exists a finite sequence of statements, $C_1 \vdash D_1$, $C_2 \vdash D_2, \ldots, C_e \vdash D_e$ such that C_1 is C and D_e is D, D_i is C_{i+1} for $1 \leqslant i \leqslant e - 1$, such that each statement $C_i \vdash D_i$ is justified by one of the following rules:

1. *C_i is WC_j, D_i is WD_j, for some j, $1 \leqslant j < i$, and for some word W.*
2. *C_i is C_jW, D_i is D_jW, for some j, $1 \leqslant j < i$, and for some word W.*
3. *C_i is D_i.*
4. *C_i is A_j and D_i is B_j for some j, $1 \leqslant j \leqslant m$.*
5. *C_i is C_j, D_i is D_k, and D_j is C_k for some j, k, $1 \leqslant j < i$; $1 \leqslant k < i$.*

A possibly clearer, if less explicit, summary of these rules may be given as follows:

1. *If $C \vdash D$, then $WC \vdash WD$.*
2. *If $C \vdash D$, then $CW \vdash DW$.*
3. *$C \vdash C$.*
4. *If $C \rightarrow D$, then $C \vdash D$.*
5. *If $C \vdash E$ and $E \vdash D$, then $C \vdash D$.*

A **PIPC** is a system having $\supset$, [,] and an infinite list of propositional variables $p_1, q_1, r_1, s_1, p_2, q_2, r_2, s_2, \ldots$ as primitive symbols. Its well-formed formulas (wffs) are (1) a propositional variable standing alone, and (2) $[A \supset B]$, where A and B are wffs. Its axioms are a finite set of tautologies and its rules of inference are *modus ponens* and substitution.

A **PPC** is a system having as primitive symbols all of the primitive symbols of a **PIPC** and, in addition, the primitive symbol $\sim$. Its wffs are (1) a propositional variable standing alone, (2) $\sim A$, where A is a wff, and (3) $[A \supset B]$, where A and B are wffs. Its axioms are a finite set of tautologies and its two rules of inference are *modus ponens* and substitution.

Clearly, the set of theorems of any **PIPC** is also the set of theorems of some **PPC** and hence our results for **PIPC**s hold equally as well for **PPC**s.

Results and Proofs We shall establish the following result.

Theorem 1 *For each* r.e. *many-one degree of unsolvability d there exists a* **PIPC** *with decision problem of degree d.*

This result is to be proved by exhibiting a uniformly effective procedure P which, when applied to any semi-Thue system T, no word in a defining relation of which is the empty word, will produce a **PIPC**, P_T, such that the word problem for T and the decision problem for P_T are of the same many-one degree. We then appeal to a result of Overbeek [5] that there exists such a semi-Thue (actually Thue) system of each r.e. many-one degree.

Let T be a semi-Thue system defined by:

$$A_T : 1, b$$
$$U_T : G_i \rightarrow \overline{G}_i, i = 1, 2, \ldots, m.$$

If W is a non-empty word over A_{T}, define W^* to be the wff of a **PIPC** given by the following recursive definition.

$$1^* \text{ is } p_2 \supset [p_2 \supset p_2]$$
$$b^* \text{ is } p_2 \supset 1^*$$
$$(W1)^* \text{ is } [W^* \vee 1^*]$$

and

$$(Wb)^* \text{ is } [W^* \vee b^*]$$

where W is any non-empty word over A_{T} and $[A \vee B]$ is an abbreviation for $[A \supset B] \supset B$. If W is a non-empty word over A_{T}, define W' to be $W^* \vee h$, where h is an abbreviation for the fixed wff $p_2 \supset b^*$. Note that here as well as in the remainder of this paper abbreviations of wffs are made in accordance with the conventions of Church [1].

If we let ϕ be a variable which may be replaced by 1^* or b^* we may now define $\mathsf{P_T}$ to be the **PIPC** specified by the following set of axiom schemes.

1. $[\phi \vee h] \supset [\phi \vee h]$
2. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[p_1 \vee \phi] \vee h] \supset [[q_1 \vee \phi] \vee h]$
3. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[\phi \vee p_1] \vee h] \supset [[\phi \vee q_1] \vee h]$
4. $G_i' \supset \overline{G}_i'$, *for* $i = 1, 2, \ldots, m$
5. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[r_1 \vee h] \supset [s_1 \vee h]] \supset [[[p_1 \vee r_1] \vee h] \supset [[q_1 \vee s_1] \vee h]]$
6. $[[[p_1 \vee q_1] \vee r_1] \vee h] \supset [[[p_1 \vee q_1] \vee r_1] \vee h] \supset_{\blacksquare} [[[p_1 \vee q_1] \vee r_1] \vee h]$ $\supset [[p_1 \vee [q_1 \vee r_1]] \vee h]$
7. $[[[p_1 \vee q_1] \vee r_1] \vee h] \supset [[[p_1 \vee q_1] \vee r_1] \vee h] \supset_{\blacksquare} [[p \vee [q_1 \vee r_1]] \vee h]$ $\supset [[[p_1 \vee q_1] \vee r_1] \vee h]$
8. $[p_1 \vee h] \supset [q_1 \vee h] \supset_{\blacksquare} [[q_1 \vee h] \supset [r_1 \vee h]] \supset [[p_1 \vee h] \supset [r_1 \vee h]]$

We now prove a sequence of eight lemmas. Of these Lemmas 7 and 8 are sufficient to establish Theorem 1. Of the preliminary Lemmas 1 through 6 perhaps Lemma 2 and Lemma 6 are the most crucial as together they completely characterize the theorems of $\mathsf{P_T}$. As we shall see, it is almost an immediate consequence of these two lemmas that the decision problem for $\mathsf{P_T}$ many-one reduces to the word problem for T. In the proofs that follow the symbol □ shall be used to designate the end of an argument.

Lemma 1 *The following two propositions hold for wffs of* $\mathsf{P_T}$.

(a) *A wff of the form* $[A_1 \vee B] \supset [A_2 \vee B]$ *cannot take the form* $[X \vee Y]$, *where* $A_1, A_2, B, X,$ *and* Y *are wffs.*
(b) *A wff of the form* $[A_1 \vee B] \supset [A_2 \vee B] \supset_{\blacksquare} [X_1 \vee B] \supset [X_2 \vee B]$, *where* A_1, A_2, B, X_1 *and* X_2 *are wffs, cannot take the form* $[Y_1 \vee Y_2]$, *where* Y_1 *and* Y_2 *are wffs.*

Proof: Suppose (a) is false. Then Y must be identified with both B and $[A_2 \vee B]$. This is impossible so (a) holds. Suppose (b) is false. Then Y_2 must be identified with both $[A_2 \vee B]$ and $[X_1 \vee B] \supset [X_2 \vee B]$. By (a) this is impossible, and hence (b) holds. □

If A is a wff of $\mathrm{P_T}$, then A is *regular* if and only if (1) A is 1^*, or A is B^*, or (2) A is of the form $[A_1 \vee A_2]$ where A_1 and A_2 are regular. It should be noted that the only variable occurring in a regular wff is p_2.

If A is a regular wff of $\mathrm{P_T}$, then $\langle A\rangle$ is the word over $\mathrm{P_T}$ obtained by replacing each occurrence of 1^* and b^* in A by 1 or b, respectively, and then removing all occurrences of $[\,,\,]$ and $\vee$. For any regular wff A, $\langle A\rangle$ is unique.

Lemma 2 *Every theorem of* $\mathrm{P_T}$ *may be abbreviated into one of the following forms.*

Form 1. *Substitution instances of* Axioms 2, 3, 5, 6, 7, *and* 8.
Form 2. *Substitution instances of* $[[r_1 \vee h] \supset [s_1 \vee h]] \supset_{\blacksquare} [[[p_1 \vee r_1] \vee h] \supset [[q_1 \vee s_1] \vee h]$, *where* $[p_1 \vee h] \supset [q_1 \vee h]$ *is a theorem of* $\mathrm{P_T}$.
Form 3. *Substitution instances of* $[q_1 \vee h] \supset [r_1 \vee h] \supset_{\blacksquare} [p_1 \vee h] \supset [r_1 \vee h]$, *where* $[p_1 \vee h] \supset [q_1 \vee h]$ *is a theorem of* $\mathrm{P_T}$.
Form 4. *Substitution instances of* $[W_1 \vee h] \supset [W_2 \vee h]$, *where* W_1 *and* W_2 *are regular and* $\langle W_1\rangle \vdash_{\mathrm{T}} \langle W_2\rangle$.

Proof: Lemma 2 is to be established by mathematical induction on n, the number of lines in a given proof in $\mathrm{P_T}$. Let B be a theorem of $\mathrm{P_T}$ and let $B_1, B_2, \ldots, B_n$, where B_n is B, be a proof of B in P_{T}; i.e., each B_i for $i = 1, 2, \ldots, n$ is either a substitution instance of an axiom or is deduced by a use of *modus ponens* with minor premiss B_q and major premiss B_r, where $q, r < n$. We first consider the following special case.

Case 0. B_n is a substitution instance of an axiom. Then if B_n is a substitution instance of Axiom 2, 3, 5, 6, 7 or 8 B is of Form 1 and the lemma holds. If B_n is a substitution instance of Axiom 1, B is of Form 4 as is apparent from rule 3 for semi-Thue systems. Finally, if B_n is a substitution instance of Axiom 4 then B is of Form 4 as is apparent from rule 4 for semi-Thue systems.
Case 1. Suppose $n = 1$. Then the conclusion follows from Case 0.
Case 2. Assume that $n > 1$ and that the conclusion holds for all positive integers less than n.
Case 2a. B_n is a substitution instance of an axiom. Again the conclusion follows from Case 0.
Case 2b. Assume B_q is of Form 4 and B_r is of Form 1. If B_r is a substitution instance of Axiom 2, 3, 6 or 7, then B is of Form 4 as is apparent. If B_r is a substitution instance of Axiom 5 or Axiom 7 then B is clearly of Form 2 or Form 3, respectively.
Case 2c. Assume B_q is of Form 4 and B_r is of Form 2. Then from the conditions on Forms 4 and 2 and from the fact that if $W_1 \vdash_{\mathrm{T}} W_2$ and $W_3 \vdash_{\mathrm{T}} W_4$ then $W_1W_3 \vdash_{\mathrm{T}} W_3W_4$ we see that B is of Form 4.
Case 2d. Assume B_q is of Form 4 and B_r is of Form 3. Then from the conditions on Forms 4 and 3 and rule 5 for semi-Thue systems we see that B is of Form 4.

This takes care of the operative cases. We argue that the other thirteen cases are vacuus as follows. If B_q is of Form 1, 2 or 3 and B_r is of Form 4 the conclusion follows by Lemma 1(b). If B_q and B_r are both of Form 4, the conclusion follows by Lemma 1(a). If B_q is of Form 1, 2 or 3 and B_r is also of Form 1, 2 or 3 we consider the antecedent of the minor premiss and the antecedent of the antecedent of the major premiss and the conclusion again follows by Lemma 1(a). □

Lemma 3 *If A is a regular wff, then* $\vdash_{\mathrm{PT}} [A \vee h] \supset [A \vee h]$.

Proof: The proof of Lemma 3 is by mathematical induction on n, the number of occurrences of 1* and b* in A.

Case 1. If $n = 1$, the conclusion follows by Axiom 1. If $n = 2$, the conclusion follows by Axioms 1 and 2. If $n = 3$ the result may be obtained by using Axioms 1, 2, and 5.
Case 2. Assume that $n > 3$ and that the lemma holds for all positive integers less than n. Then A is of the form $A_1 \vee A_2$ and the proof may be outlined as follows:

$$[A_1 \vee h] \supset [A_1 \vee h] \qquad \text{by hyp. ind.}$$
$$[A_2 \vee h] \supset [A_2 \vee h] \qquad \text{by hyp. ind.}$$
$$[[A_1 \vee A_2] \vee h] \supset [[A_1 \vee A_2] \vee h] \qquad \text{by Axiom 5}$$
$$\text{i.e., } [A \vee h] \supset [A \vee h] \ \square$$

If A is a regular wff there are only finitely many ways in which the occurrences of 1* and b* in A may be grouped by brackets and v symbols to form a regular wff. We shall write $\{A\}_i$ to represent the i'th such grouping in some assumed canonical ordering.

Lemma 4 *If A is a regular wff, then* $\vdash_{\mathrm{PT}} [\{A\}_i \vee h] \supset [\{A\}_j \vee h]$ *for any positive integers i and j such that* $\{A\}_i$ *and* $\{A\}_j$ *are defined.*

Proof: The proof of Lemma 4 is by mathematical induction on n, the number of occurrences of 1* and b* in A. If $n = 1$ or $n = 2$, then $\{A\}_i$ is $\{A\}_j$ and the result follows from Lemma 3 and Axiom 5 or Axiom 7. If X is a regular wff, *the length of X* is the number of occurrences of 1* and b* in X. We shall write $||X||$ for the length of X. Assume that $n > 3$ and the lemma holds for all positive integers less than n. Let $\{A\}_i$ be $[A_1 \vee A_2]$ and let $\{A\}_j$ be $[B_1 \vee B_2]$. We consider the following cases.

Case 1. $||A_1|| = ||B_1||$. Then $||A_2|| = ||B_2||$ and the argument may be outlined as follows:

$$[A_1 \vee h] \supset [B_1 \vee h] \qquad \text{by hyp. ind.}$$
$$[A_2 \vee h] \supset [B_2 \vee h] \qquad \text{by hyp. ind.}$$
$$[[A_1 \vee A_2] \vee h] \supset [[B_1 \vee B_2] \vee h] \qquad \text{by Axiom 5}$$
$$\text{i.e., } [\{A\}_i \vee h] \supset [\{A\}_j \vee h]$$

Case 2a. $||A_1|| = ||B_1|| + k$. Let A_{11} be a disjunction of the first $||A_1|| - k$ occurrences of 1* and b* in A_1 and let A_{12} be a disjunction of the last k occurrences of 1* and b* in B_2 and let B_{22} be a disjunction of the last $||B_2|| - k$ occurrences of 1* and b* in B_2. Then

$$||A_{11}|| = ||B_1||, \ ||A_{12}|| = ||B_{21}|| \text{ and } ||A_2|| = ||B_{22}||.$$

The argument can then be outlined as follows:

$[[A_1 \vee A_2] \vee h] \supset [[[A_{11} \vee A_{12}] \vee A_2] \vee h]$	by Case 1
$[A_{12} \vee h] \supset [B_{21} \vee h]$	by Case 1
$[A_{11} \vee h] \supset [A_{11} \vee h]$	by Lemma 3
$[[A_{11} \vee A_{12}] \vee h] \supset [[A_{11} \vee B_{21}] \vee h]$	by Axiom 5
$[A_2 \vee h] \supset [A_2 \vee h]$	by Lemma 3
$[[[A_{11} \vee A_{12}] \vee A_2] \vee h] \supset [[[A_{11} \vee B_{21}] \vee A_2] \vee h]$	by Axiom 5
$[A_2 \vee h] \supset [B_{22} \vee h]$	by Case 1
$[[A_{11} \vee B_{21}] \vee h] \supset [[A_{11} \vee B_{21}] \vee h]$	by Lemma 3
$[[[A_{11} \vee B_{21}] \vee A_2] \vee h] \supset [[[A_{11} \vee B_{21}] \vee B_{22}] \vee h]$	by Axiom 5
$[[[A_{11} \vee B_{21}] \vee B_{22}] \vee h] \supset [[A_{11} \vee [B_{21} \vee B_{22}]] \vee h]$	by Axiom 6
$[A_{11} \vee h] \supset [B_1 \vee h]$	by Case 1
$[[B_{21} \vee B_{22}] \vee h] \supset [[B_{21} \vee B_{22}] \vee h]$	by Lemma 3
$[[A_{11} \vee [B_{21} \vee B_{22}]] \vee h] \supset [[B_1 \vee [B_{21} \vee B_{22}]] \vee h]$	by Axiom 5
$[[B_1 \vee [B_{21} \vee B_{22}]] \vee h] \supset [[B_1 \vee B_2] \vee h]$	by Case 1
$[[A_1 \vee A_2] \vee h] \supset [[B_1 \vee B_2] \vee h]$	by Axiom 8
i.e., $[\{A\}_i \vee h] \supset [\{A\}_j \vee h]$	

Case 2b. $||A_1|| + k = ||B_1||$. By the symmetry of the axioms for P_T it should be clear that this case follows from an argument similar to that for Case 2a. We omit the proof. □

The following lemma is the converse of Lemma 2 in the sense that it, together with one of the clauses of 2, shows that the word problem for T is one-one reducible to the decision problem for P_T.

Lemma 5 *If* W_1 *and* W_2 *are words over* A_T *and* $W_1 \vdash_\mathsf{T} W_2$, *then* $\vdash_{\mathsf{P}_\mathsf{T}} W_1' \supset W_2'$.

Proof: The proof is by mathematical induction on n, the number of lines in a given proof of $W_1 \vdash_\mathsf{T} W_2$. Let $X_1 \vdash_\mathsf{T} Y_1, X_2 \vdash_\mathsf{T} Y_2, \ldots, X_n \vdash_\mathsf{T} Y_n$, where X_1 is W_1 and Y_n is W_2 be a proof in T.

Case 1. $n = 1$. Then $X_n \vdash_\mathsf{T} Y_n$ is justified by rule 3 or rule 4 for semi-Thue systems; i.e., W_1 is W_2 or $W_1 \rightarrow W_2$ is a defining relation of U_T. If W_1 is W_2 the lemma holds by Lemma 3, if $W_1 \rightarrow W_2$ it follows from Axiom 4.

Case 2. Assume $n > 1$ and the result holds for all positive integers less than n.

Case 2a. $X_n \vdash_\mathsf{T} Y_n$ is justified by rule 1. Then X_n is AX_j and Y_n is AY_j for some $j < n$, and some word A. The proof is easily outlined as follows:

$[X_j^* \vee h] \supset [Y_j^* \vee h]$	by hyp. ind.
$[A^* \vee h] \supset [A^* \vee h]$	by Lemma 3
$[[A^* \vee X_j^*] \vee h] \supset [[A^* \vee Y_j^*] \vee h]$	by Axiom 5
$[(AX_j)^* \vee h] \supset [[A^* \vee X_j^*] \vee h]$	by Lemma 4
$[[A^* \vee Y_j^*] \vee h] \supset [(AY_j)^* \vee h]$	by Lemma 4
$[(AX_j)^* \vee h] \supset [(AY_j)^* \vee h]$	by Axiom 8
i.e., $W_1' \supset W_2'$	

Case 2b. $X_n \vdash_{\mathsf{T}} Y_n$ is justified by rule 2. Then X_n is $X_j A$ and Y_n is $Y_j A$ for some $j < n$, and some word A. The proof is analogous to that for Case 2a and is therefore omitted.
Case 2c. $X_n \vdash_{\mathsf{T}} Y_n$ is justified by rule 3 or rule 4. Then the result follows from Case 1.
Case 2d. $X_n \vdash_{\mathsf{T}} Y_n$ is justified by rule 5. Then X_n is X_j, Y_n is Y_k, and Y_j is X_k for some j and k, $1 \leq j < n$, $1 \leq k < n$. The result follows from the induction hypothesis, Axiom 8 and *modus ponens*. □

Lemma 6 *Every wff A of* $\mathsf{P_T}$ *which can be abbreviated into a formula of* Form 1, 2, 3 *or* 4 *of* Lemma 2 *is a theorem of* $\mathsf{P_T}$.

Proof: We shall consider the forms separately.

Form 1. Clearly the result holds in this case as all substitution instances of the axioms are theorems.
Form 2 and Form 3. The result holds here by the conditions on these forms and the presence of Axiom 5 and Axiom 8, respectively.
Form 4. The restriction on Form 4 requires that W_1 and W_2 be regular and that $\langle W_1 \rangle \vdash_{\mathsf{T}} \langle W_2 \rangle$. Now by Lemma 5 if $W_1 \vdash_{\mathsf{T}} W_2$ then $\vdash_{\mathsf{P_T}} W_1' \supset W_2'$ and the result follows from Lemma 4. □

Lemma 7 *For any two words X and W on* A_{T}, $X \vdash_{\mathsf{T}} W$ *if and only if* $\vdash_{\mathsf{P_T}} X' \supset W'$; *hence the word problem for* T *is one-one reducible to the decision problem for* $\mathsf{P_T}$.

Proof: This is an easy consequence of Lemma 2 and Lemma 5. □

Lemma 8 *The decision problem for* $\mathsf{P_T}$ *is many-one reducible to the word problem for* T.

Proof: Assume that we have a decision procedure $\mathcal{R}$ for solving the word problem for T. Let A be a wff of $\mathsf{P_T}$. Test whether A can be abbreviated into a formula of Form 1. If so A is a theorem of $\mathsf{P_T}$. If not test whether A can be abbreviated into a wff of Form 2 or Form 3. This will require testing whether or not the well defined formula specified in the condition of Form 2 or 3 as the case may be is of Form 4. Assume, for the moment, this can be done by a well specified appeal to $\mathcal{R}$. Then if A is of Form 2 or Form 3 it is a theorem of $\mathsf{P_T}$. If not test whether or not A is of Form 4. By the condition on Form 4 this requires one precisely defined appeal to $\mathcal{R}$. If A is of Form 4 then it is a theorem of $\mathsf{P_T}$. If not A is not a theorem of $\mathsf{P_T}$. □

Lemmas 7 and 8, along with the result of Overbeek cited above, are sufficient to complete the proof of Theorem 1. For completeness we state the following corollary.

Corollary *There exists a uniformly effective procedure* P *such that the result of applying* P *to any semi-Thue system* T *is a* **PIPC** (**PPC**) $\mathsf{P_T}$ *such that the decision problem for* $\mathsf{P_T}$ *is of the same many-one* r.e. *degree of unsolvability as the word problem for* T.

In order to show that Theorem 1 is "best possible" we need only prove that there exists a one-one r.e. degree of unsolvability which is not representable by the decision problem for a **PIPC (PPC)**. This is accomplished by the following theorem.

Theorem 2 *There is no* **PIPC (PPC)** *which is of the same one-one* r.e. *degree of unsolvability as a simple set.*

Proof: In order to establish the result we need only show that given any **PIPC (PPC)**, P, with an unsolvable decision problem there exists an infinite recursively enumerable set of wffs of P which are non-theorems. This is easy, for, since the decision problem for P is unsolvable, there exists a tautology A which is not a theorem of P. Let $\phi_1, \phi_2, \ldots, \phi_n$ be the set of distinct variables occurring in A. Then for any set of n distinct variables of P say $\psi_1, \psi_2, \ldots, \psi_n$ the substitution instance of A gotten by substituting ψ_1 for ϕ_1, ψ_2 for ϕ_2, . . ., ψ_n for ϕ_n is not a theorem of P. □

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