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$\mathcal{K}$ AND $\mathcal{Z}$

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Goldblatt has shown in [1] that a system belongs to Sobociński's family $\mathcal{Z}$ if and only if it is the intersection of S5 and a certain member of the family $\mathcal{K}$. Offered here is an alternative proof of this same result which I obtained independently a short while after the appearance of the $\mathcal{Z}$ systems. The strategy is very similar to that used by Goldblatt, but there are enough differences of detail so that the present argument may still be of some interest.

We begin by establishing that if

$$\beta = CLMApqCLMNqCMKApqNqLMKApqNq$$
$$\gamma = CLMApNqCLMqCMKApNqqLMKApNqq$$
$$\xi = CLMCqLqCLMCNqLNqCMKCqLqCNqLNqLMKCqLqCNqLNq,$$

then $CK\xi K\gamma\beta ALCMpLMpLCLMqMLq$ is a thesis of S4. Suppose not. Then there exists a reflexive and transitive Kripke-style model $\mathfrak{A} = \langle w, W, R\rangle$ and valuation V on $\mathfrak{A}$ such that

$$V(CK\xi K\gamma\beta ALCMpLMpLCLMqMLq, w) = 0,$$

whence

$$V(\xi, w) = 1 \quad (1)$$
$$V(K\gamma\beta, w) = 1 \quad (2)$$
$$V(LCMpLMp, w) = 0 \quad (3)$$
$$V(LCLMqMLq, w) = 0. \quad (4)$$

There are now two cases to be considered.

Case 1. Suppose

$$V(MALqLNq, w) = 1,$$

whence it follows that there is some $x \in W$ such that wRx and either $V(Lq, x) = 1$ or $V(LNq, x) = 1$. But then $V(KCqLqCNqLNq, x) = 1$ and therefore

$$V(MKCqLqCNqLNq, w) = 1. \quad (5)$$

Moreover, from (4) it follows that there is some $y \in W$ such that wRy and

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$$V(LMq, y) = V(LMNq, y) = 1.$$

But this yields

$$V(KCqLqCNqLNq, z) = 0$$

for every $z \in W$ such that yRz, whence it follows that

$$V(MKCqLqCNqLNq, y) = 0$$

and so we have

$$V(LMKCqLqCNqLNq, w) = 0. \tag{6}$$

Of course, we also have

$$V(LMCqLq, w) = V(LMCNqLNq, w) = 1 \tag{7}$$

since $LMCqLq$ and $LMCNqLNq$ are theses of S4. But now from (5), (6), and (7) it follows that $V(\xi, w) = 0$, contrary to (1).

Case 2. Suppose, on the other hand, that

$$V(MALqLNq, w) = 0,$$

whence it follows that $V(LKMNqMq, w) = 1$. Therefore, for every $x \in W$ such that wRx, we have $V(MNq, x) = V(Mq, x) = 1$ and hence also $V(MApq, x) = V(MApNq, x) = 1$. But then

$$V(LMApq, w) = V(LMNq, w) = 1 \tag{8}$$
$$V(LMApNq, w) = V(LMq, w) = 1. \tag{9}$$

Moreover, from (3) we know that there is some $x \in W$ such that wRx and $V(CMpLMp, x) = 0$, whence it follows that

$$V(Mp, x) = 1 \tag{10}$$
$$V(LMp, x) = 0. \tag{11}$$

By (11) there is some $y \in W$ such that xRy and $V(Mp, y) = 0$. But this yields $V(p, z) = 0$ for every $z \in W$ such that yRz, whence we easily get

$$V(KApqNq, z) = V(KApNqq, z) = 0.$$

Therefore

$$V(MKApqNq, y) = V(MKApNqq, y) = 0,$$

whence, since wRy,

$$V(LMKApqNq, w) = V(LMKApNqq, w) = 0. \tag{12}$$

On the other hand, it follows from (10) that there is some $u \in W$ such that xRu and $V(p, u) = 1$. However, either $V(q, u) = 1$ or $V(Nq, u) = 1$, and consequently we have either

$$V(KApqNq, u) = 1 \text{ or } V(KApNqq, u) = 1$$

and hence, since wRu, either

$$\mathsf{V}(MKApqNq,\ w) = 1 \text{ or } \mathsf{V}(MKApNqq,\ w) = 1. \tag{13}$$

But (8), (9), (12), and (13) entail that $\mathsf{V}(K\gamma\beta, w) = 0$, contrary to (2).

We have obtained a contradiction in each of the two cases, thereby completing the proof that $CK\xi K\gamma\beta ALCMpLMpLCLMqMLq$ is validated by every reflexive and transitive Kripke-style model, and so is a thesis of S4. But now β, γ, and ξ are just substitution instances of Zeman's axiom, from which it follows that $ALCMpLMpLCLMqMLq$ is a thesis of every system in the family Z.

Now it is not difficult to show that if S is any normal system between S4 and S4.4 axiomatized with substitution, detachment, and necessitation as sole primitive rules of inference, and ξ and ζ are formulas having no propositional variables in common, then $S + AL\xi L\zeta$ is the intersection of $S + \xi$ and $S + \zeta$. Therefore, in particular, we know that

$$(S + CMpLMp) \cap (S + CLMqMLq) = S + ALCMpLMpLCLMqMLq.$$

However, we have just seen that $ALCMpLMpLCLMqMLq$ is a thesis of every Z system and thus, since it is known that Zeman's axiom is a thesis of the above system, we also have

$$S + ALCMpLMpLCLMqMLq = S + CLMpCLMqCMKpqLMKpq$$

and hence

$$(S + CMpLMp) \cap (S + CLMqMLq) = S + CLMpCLMqCMKpqLMKpq,$$

that is, since $S5 = S + CMpLMp$,

$$S5 \cap (S + CLMqMLq) = S + CLMpCLMqCMKpqLMKpq.$$

Consequently, since the members of Z are gotten by adding Zeman's axiom to a Gödel-style base for a system between S4 and S4.4, a system belongs to Z if and only if it is the intersection of S5 and the K system resulting from the addition of $CLMqMLq$ as an axiom to that same base. And this, of course, is Goldblatt's result.

REFERENCE

[1] Goldblatt, R. I., "A study of Z modal systems," *Notre Dame Journal of Formal Logic*, vol. XV (1974), pp. 289-294.

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