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A STUDY OF $\mathcal{Z}$ MODAL SYSTEMS

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In [10], Sobociński has shown that the addition to various S4-extensions of Zeman's formula

Z1 $\quad LMp \,.\, LMq \rightarrow (M(p \,.\, q) \rightarrow LM(p \,.\, q))$

generates a new family that he refers to as the $\mathcal{Z}$ modal systems. In this paper a completeness proof is given for the system Z1 = S4 + **Z1**, and the finite model property is established. Since the system is finitely axiomatisable its decidability follows. Furthermore it is shown that each $\mathcal{Z}$ modal system is the intersection of S5 with some system from family $\mathcal{K}$.

In the field of S4, **Z1** is inferentially equivalent to

Z2 $\quad L(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$

the formula added to S4.4 by Schumm to obtain the system now called S4.9 (*cf.* Sobociński [9], p. 361)

(1)	$LM(p \vee q) \,.\, LM(\sim p \vee q) \rightarrow (M((p \vee q) \,.\, (\sim p \vee q)) \rightarrow LM((p \vee q) \,.\, (\sim p \vee q)))$	**Z1**, $p/p \vee q$, $q/\sim p \vee q$
(2)	$(p \vee q) \,.\, (\sim p \vee q) \leftrightarrow q$	PC
(3)	$Mq \rightarrow M((p \vee q) \,.\, (\sim p \vee q))$	(2), C2
(4)	$LM((p \vee q) \,.\, (\sim p \vee q)) \rightarrow LMq$	(2), C2
(5)	$LMp \rightarrow LM(p \vee q)$	C2
(6)	$\sim MLp \rightarrow LM(\sim p \vee q)$	C2
(7)	$LMp \,.\, \sim MLp \rightarrow (Mq \rightarrow LMq)$	(1), (3), (4), (5), (6), PC
(8)	$(LMp \rightarrow MLp) \vee (Mq \rightarrow LMq)$	(7), PC
(9)	$M(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$	(8), C2
(10)	$M(LMp \rightarrow MLp) \rightarrow (LMp \rightarrow MLp)$	S4
(11)	$(LMp \rightarrow MLp) \rightarrow L(LMp \rightarrow MLp)$	**Z1**, S4 (*cf.* [1])
Z2	$L(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$	(9), (10), (11), PC

This shows that Z1 contains the system S4 + **Z2**. We shall subsequently establish the converse in two different ways.

Definitions and discussion of the model-theoretic concepts used below are given in Segerberg [5], [6], and [7] (*cf.* also the Metatheorem of [1]).

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Proposition 1: *A is a theorem of* S4 + **Z2** *if and only if it is verified by every* S4*-frame satisfying*

$$\forall x \forall y((xRy \to yRx) \vee \exists z(yRz \,.\, \forall w(zRw \to z = w))) \qquad \text{(a)}$$

Proof: Necessity. We leave it to the reader to check that any S4-frame satisfying (a) verifies **Z2**.

Sufficiency. Let A be any wff not derivable in S4 + **Z2**. Then A is false at some point t in the canonical model for S4 + **Z2**, and hence false at t in the submodel $\mathfrak{U}$ generated from the canonical model by t. Let Ψ be the closure under modalities of the set of all subwff of A, and $\mathfrak{U}'$ a Lemmon filtration of $\mathfrak{U}$ through Ψ. Then $\mathfrak{U}'$ is finite, reflexive, and transitive, and hence is an S4-frame (Segerberg [5], Section 3, and [6], Chapter I, Theorem 7.6). Furthermore by the Filtration Theorem ([7], p. 303) A is false in $\mathfrak{U}'$ at $[t]$. It remains only to show that $\mathfrak{U}'$ satisfies (a).

Let $[x]$ be any point in $\mathfrak{U}'$ and suppose there is some point $[y]$ in $\mathfrak{U}'$ such that

$$[x]\,R'\,[y] \text{ and not } [y]\,R'\,[x] \text{ in } \mathfrak{U}' \qquad \text{(b)}$$

We have to prove the second disjunct of (a).

Since $\mathfrak{U}'$ is finite, reflexive, and transitive there is some final cluster Y in $\mathfrak{U}'$ that either contains $[y]$ or succeeds $[y]$ ([5], p. 19). From (b) it follows that $[x]$ precedes Y and so Y is a non-initial final cluster. Now if we can show that Y is a simple final cluster, i.e., it consists of a single element with no alternative except itself, then the proof will be complete.

To get a contradiction we suppose Y is a proper cluster and therefore contains at least two distinct elements, say $[z]$ and $[w]$. Since $[z] \neq [w]$ there is some wff B in Ψ such that

$$B \text{ is true in } \mathfrak{U}' \text{ at } [z], \text{ but false at } [w] \qquad \text{(c)}$$

Since the relation R' is universal over Y, it follows from (c) that

$$MB \text{ is true, and } LB \text{ is false, in } \mathfrak{U}' \text{ at every point in } Y \qquad \text{(d)}$$

Now if zRu in $\mathfrak{U}$, $[z]\,R'\,[u]$ in $\mathfrak{U}'$. But Y is final, so $[u] \in Y$, whence by (d), MB is true but LB is false in $\mathfrak{U}'$ at $[u]$. But MB and LB are in Ψ, so by the Filtration Theorem MB is true and LB is false in $\mathfrak{U}$ at u. This shows that LMB is true and MLB is false in $\mathfrak{U}$ at z, hence

$$(LMB \to MLB) \text{ is false at } z \qquad \text{(e)}$$

Now the model $\mathfrak{U}$ is transitive and generated by t, and so

$$tRu, \text{ for all } u \text{ in } \mathfrak{U} \qquad \text{(f)}$$

Then (e) and (f) together yield

$$L(LMB \to MLB) \text{ is false in } \mathfrak{U} \text{ at } t \qquad \text{(g)}$$

As in [6], Chapter II, Lemma 2.1 we can construct a Boolean combination C of members of Ψ such that

$$C \text{ is true in } \mathfrak{U} \text{ at } u \text{ iff } [u] \notin Y \qquad \text{(h)}$$

Now from (b), $[x] \notin Y$, so by (h), C is true in $\mathfrak{U}$ at x. Thus by (f)

$$MC \text{ is true at } t \tag{i}$$

Now if zRu, $[u] \in Y$ (see above) so by (h), C is false at u. Hence MC is false at z, so by (f)

$$LMC \text{ is false at } t \tag{j}$$

The reflexivity of $\mathfrak{U}$, together with (i) and (j) show that

$$L(MC \rightarrow LMC) \text{ is false at } t \tag{k}$$

But (g) and (k) contradict the fact that every substitution-instance of **Z2** is true at every point in $\mathfrak{U}$, and in particular at t. This ends our proof.

It is an easy matter to check that **Z1** is verified by every S4-frame satisfying condition (a) above, and so by Proposition 1 is derivable in S4 + **Z2**. This, with our earlier result establishes that Z1 = S4 + **Z2**.

Corollary ([5], section 3) Z1 *has the finite model property.*

Proof. The model $\mathfrak{U}'$ in Proposition 1 has at most 2^{14n} elements, where n is the number of subwff of the non-theorem A that it falsifies (Z1 has the same fourteen distinct modalities as S4).

If the element $[t]$ of Proposition 1 is contained in a final cluster then, since $[t]$ generates $\mathfrak{U}'$, the underlying frame of $\mathfrak{U}'$ consists of a single cluster and therefore verifies S5. If, on the other hand, $[t]$ is not contained in a final cluster then every final cluster in $\mathfrak{U}'$ is non-initial and therefore by the proof of Proposition 1 is simple. But a finite S4-frame in which every final cluster is simple is a frame for K1 (Segerberg [5]). Thus $\mathfrak{U}'$ is either an S5-model or a K1-model and since it falsifies the non-Z1-theorem A, A must be a non-theorem of either S5 or K1, and hence of S5 $\cap$ K1. But Z1 is contained in both S5 and K1, and so we conclude that Z1 = S5 $\cap$ K1.

This intersection result, and the fact that **Z1** is a consequence in S4 of **Z2**, may alternatively be deduced from Theorem 2 of Halldén [2] (*cf.* also Theorem 2 of Kripke [3]). This theorem shows that if S is a system containing **PC** and having modus ponens as its only primitive rule of inference, and if A and B are two formulae with no propositional variable in common, then

$$(S + (A \vee B)) = (S + A) \cap (S + B).$$

Now Simons [8] has given an axiomatisation of S4 that has modus ponens as its only primitive rule. Furthermore, the same system results when the same extra axioms are added to this basis and to Lewis' basis for S4. This observation, and Halldén's theorem show immediately that S4 + **Z2** is the intersection of S4 + $L(Mq \rightarrow LMq)$ and S4 + $L(LMp \rightarrow MLp)$, i.e., of S5 and K1. But **Z1** is easily derivable in both S5 and K1, whence it is derivable in S4 + **Z2**.

Proposition 2. *Let S be a system contained in* S5 *and obtained by adding some axioms S (which may be a finite conjunction of wff) to* S4. *Then* (S + **Z1**) = S5 ∩ (S + **K1**).

Proof: Since **Z1** is a theorem of S5, and deducible from **K1**, it is immediate that (S + **Z1**) ⊆ S5 ∩ (S + **K1**). For the converse we use again an axiomatisation of S4 with modus ponens the only primitive rule. Letting $B = L(Mq \rightarrow LMq)$ and $K = L(LMp \rightarrow MLp)$ we see by Halldén's theorem that the system S5 ∩ (S + **K1**) can be axiomatised as S4 + $(B \vee (S\,.K))$, where B and $(S\,.K)$ have no variable in common. Now if A is a theorem of this system then by the Deduction Theorem there are finitely many instances $(B_i \vee (S_i\,.K_i))$, $(1 \leqslant i \leqslant n)$, such that $(B_1 \vee (S_1\,.K_1)) \ldots (B_n \vee (S_n\,.K_n)) \rightarrow A$ is derivable in S4. Hence $(B_1 \vee S_1)\,.(B_1 \vee K_1) \ldots (B_n \vee S_n)\,.(B_n \vee K_n) \rightarrow A$ and so $S_1\,.(B_1 \vee K_1) \ldots S_n\,.(B_n \vee K_n) \rightarrow A$ is derivable in S4. But each conjunct in the antecedent of this last formula is derivable in S + **Z1**, and therefore so is A. QED

From Proposition 2 we read off the following connections between $\mathcal{Z}$ and $\mathcal{K}$ modal systems.

Z1 = S5 ∩ K1
Z2 = S5 ∩ K1.2
Z3 = S5 ∩ K1.1
Z4 = S5 ∩ K2
Z5 = S5 ∩ K2.1
Z6 = S5 ∩ K3
Z7 = S5 ∩ K3.1
Z8 = S5 ∩ K3.2
Z9 = S5 ∩ K4 = S4.9[1]

There is some interest in the relationship between **Z2** and the formula

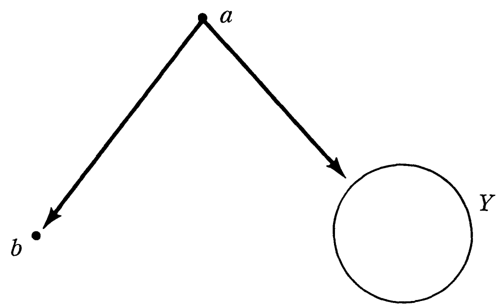
Δ $(LMp \rightarrow MLp) \vee (Mq \rightarrow LMq)$

which is obviously derivable from **Z2**. In showing that **Z2** was a consequence of **Z1** we first derived **Δ** from **Z1** and then used the S4.01 axiom

Γ4 $(LMp \rightarrow MLp) \rightarrow L(LMp \rightarrow MLp)$

which is also derivable from **Z1** in S4, to obtain **Z2**. Thus **Δ** and **Z2** are equivalent in the field of S4.01 and therefore in the field of every proper S4-extension except S4.02 and S4.04 (*cf.* [1]). However in S4 itself **Δ** is weaker than **Z2**. This follows from consideration of the matrix 𝔐11 of Sobociński [9], p. 350. By the methods of Lemmon [4], section **5**, 𝔐11 is seen to be the matrix representation of the four-element reflexive frame that can be displayed graphically as

1. A similar, and independent, proof that the systems Z9 and S4.9 are identical has been obtained by K. Fine (*cf. Notre Dame Journal of Formal Logic*, vol. XIII (1972), p. 118).



with the circle Y denoting a two-element cluster. Now $(LMp \rightarrow MLp)$ cannot be falsified at a or b, since each of these points has an alternative (*viz* b) that has no alternative except itself (Segerberg [5], p. 18). On the other hand $(Mq \rightarrow LMq)$ cannot be falsified on the cluster Y. Thus the frame, and hence the matrix must verify Δ. But this frame does not satisfy condition (a) of Proposition 1 and so there will be some assignment on it that falsifies **Z2**. In fact for $p = 6$, $q = 6$, $L(M6 \rightarrow LM6) \vee L(LM6 \rightarrow ML6) = L(5 \rightarrow 13) \vee L(13 \rightarrow 16) = L9 \vee L4 = 9 \vee 12 = 9$. Thus the addition to S4 of Δ yields a new system properly contained in Z1.

Concerning other matrices of Sobociński [9] we make the following comments:

1) $\mathfrak{M}11$ also rejects S4.01 (*cf.* [1]). But $\mathfrak{M}8$ verifies S4.01 while rejecting Δ. For $p = 6$, $q = 4$, $(M4 \rightarrow LM4) \vee (LM6 \rightarrow ML6) = (4 \rightarrow 8) \vee (1 \rightarrow 8) = 5 \vee 8 = 5$. Thus S4 + Δ is independent of S4.01.
2) $\mathfrak{M}6$ rejects S4.02 and S4.04 but verifies **K1** and hence Δ. But $\mathfrak{M}8$ verifies S4.02 and S4.04, so these two systems are independent of S4 + Δ.
3) Since $\mathfrak{M}11$ verifies S4.04 and Δ but rejects **Z2** we obtain two new systems, S4.02 + Δ and S4.04 + Δ, properly contained in Z3 and Z2 respectively. The matrix $\mathfrak{M}4$ verifies S4.02 and Δ but rejects S4.04, showing that S4.02 + Δ is a proper subsystem of S4.04 + Δ.

I conjecture that S4 + Δ is characterised by the class of S4-frames satisfying

$$\forall x \forall y ((xRy \rightarrow yRx) \vee \exists z (xRz \,.\, \forall w (zRw \rightarrow z = w))),$$

or alternatively by the class of finite S4-frames in which every non-final cluster is succeeded by at least one simple final cluster.

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