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1 *Introduction.* Given any functor of 2-valued logic there is a corresponding unit of computing machinery which is capable of completely representing the behaviour of that functor. These units are called decision elements. The introduction of this term is due to Goodell [3].

Sobociński [12] has shown that there exists a functor of four arguments which may define any of the functors of one or two arguments by the substitution of variables P, Q, etc., or constants 0, 1 in its arguments, this functor only being used once in any definition. The latter clause is important since it is well known that any Sheffer function can be used to define any functor, but there is no restriction on the number of occurrences of the function. Such a functor as that defined by Sobociński is said to "generate" all the functors of two arguments. Defining such a functor corresponds to constructing a decision element which, by suitable setting of the inputs, can represent any of the 2-place functors. Decision elements of this type are called universal decision elements.

Following Sobociński's work, Rose [9] gives several functors of four arguments which correspond to universal decision elements and suggests a method to determine all such functors. Pugmire and Rose [8] suggest a very different approach to the same problem and Foxley [2] combined the advantages of both methods to actually determine the set of all four-variable formulae which correspond to universal decision elements. More recently Rose [11] has investigated three-valued universal decision elements.

In the present paper we are concerned with generating functors which will generate some particular subset of the set of functors of two arguments, but not the whole set. For this purpose we shall be considering three-place functors $\Phi(X, Y, Z)$. Such a functor cannot correspond to a universal decision element since it can easily be shown that it cannot generate a sufficient number of binary functors (see Sobociński [12]).

The particular subsets which will be considered are defined in section **3** but basically it is shown that:

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(i) no three-place functor can generate more than two of conjunction, disjunction, incompatibility (NAND) and joint denial (NOR);
(ii) there is essentially one functor which will generate conjunction, disjunction, exclusive or, equivalence, implication and non-implication in addition to negation;
(iii) there is essentially one functor which will generate joint denial, incompatibility, exclusive or, equivalence, implication and non-implication in addition to negation.

Such a functor will be said to correspond to a quasi-universal decision element which will be abbreviated to QUDE in the following sections. We shall speak loosely of a functor $\Phi(X, Y, Z)$ being a QUDE to mean that the functor corresponds to a quasi-universal decision element.

2 *Notation and Definitions*. For a two-place functor $F(X, Y)$ we define its value sequence to be $\langle klmn \rangle$ where $F(0, 0) =_{\mathrm{T}} k$, $F(0, 1) =_{\mathrm{T}} l$, $F(1, 0) =_{\mathrm{T}} m$, $F(1, 1) =_{\mathrm{T}} n$ and $k, l, m, n \in \{0, 1\}$. It is clear from the context whether or not 0, 1 are being used to denote the two logical constants or the truth values they assume. In discussing a particular two-place functor it will often be convenient to identify it by its value sequence.

For the three-place functor $\Phi(X, Y, Z)$ which is considered in the later sections it is supposed that $\Phi(0, 0, 0) =_{\mathrm{T}} a$, $\Phi(0, 0, 1) =_{\mathrm{T}} b$, $\Phi(0, 1, 0) =_{\mathrm{T}} c$, $\Phi(0, 1, 1) =_{\mathrm{T}} d$, $\Phi(1, 0, 0) =_{\mathrm{T}} e$, $\Phi(1, 0, 1) =_{\mathrm{T}} f$, $\Phi(1, 1, 0) =_{\mathrm{T}} g$, $\Phi(1, 1, 1) =_{\mathrm{T}} h$, where $a, b, c, d, e, f, g, h \in \{0, 1\}$. Clearly the value sequence of $\Phi(X, Y, Z)$ is $\langle abcdefgh \rangle$.

It is possible to obtain a maximum of nine binary functors from $\Phi(X, Y, Z)$ by substitution of the variables P, Q or the constants 0, 1 subject to the conditions:

(i) the resulting functor contains both P and Q (to ensure a binary functor results);
(ii) the first substitution of P into $\Phi(X, Y, Z)$ is in a place preceding the first substitution of Q;
(iii) the latter condition is to avoid repetitions caused by the labelling of the variables.

The nine possible substitutions, with the resulting binary value sequences, are listed in Table 1. The binary value sequences are numbered at the right for easy identification.

substitution	binary value sequence	
$X/P,\ Y/P,\ Z/Q$	$\langle abgh \rangle$	(1)
$X/P,\ Y/Q,\ Z/P$	$\langle acfh \rangle$	(2)
$X/P,\ Y/Q,\ Z/Q$	$\langle adeh \rangle$	(3)
$X/0,\ Y/P,\ Z/Q$	$\langle abcd \rangle$	(4)
$X/P,\ Y/0,\ Z/Q$	$\langle abef \rangle$	(5)
$X/P,\ Y/Q,\ Z/0$	$\langle aceg \rangle$	(6)
$X/P,\ Y/Q,\ Z/1$	$\langle bdfh \rangle$	(7)
$X/P,\ Y/1,\ Z/Q$	$\langle cdgh \rangle$	(8)
$X/1,\ Y/P,\ Z/Q$	$\langle efgh \rangle$	(9)

Table 1

3 *Classification of the 16 binary functors.* There are 16 binary functors which are listed in Table 2. Of these, six are trivial in the sense that they simplify to unary functors (1, 4, 6, 11, 13 and 16) and we group them into a class K_4. The remaining ten functors are divided into three classes as follows:

$$\mathsf{K}_1 = \{P \wedge Q,\ P \vee Q,\ P/Q,\ P \downarrow Q,\ P \oplus Q\},$$
$$\mathsf{K}_2 = \{P \equiv Q,\ F_1(P, Q),\ G_1(P, Q)\},$$
$$\mathsf{K}_3 = \{F_2(P, Q),\ G_2(P, Q)\},$$

where $F_1(P, Q), F_2(P, Q) \in \{P \supset Q, Q \supset P\}$, $F_1(P, Q) \neq F_2(P, Q)$, and $G_1(P, Q)$, $G_2(P, Q) \in \{P \not\supset Q, Q \not\supset P\}$, $G_1(P, Q) \neq G_2(P, Q)$.

The functors which are included in K_1 are those which seem to be the most useful from the standpoint of logical design (XOR is included because of its frequent occurrence in functional units such as adders). Use will be made of a class which is a subclass of K_1, viz. $\mathsf{K}_1' = \{P \wedge Q, P \vee Q, P/Q, P \downarrow Q\}$.

We shall write that $\Phi(X, Y, Z)$ generates K_j $(1 \leq j \leq 4)$ to

	value sequence	functor	notation
1	$\langle 0000 \rangle$		
2	$\langle 0001 \rangle$	AND, conjunction	$P \wedge Q$
3	$\langle 0010 \rangle$	non-implication	$P \not\supset Q$
4	$\langle 0011 \rangle$	P	
5	$\langle 0100 \rangle$	non-implication	$Q \not\supset P$
6	$\langle 0101 \rangle$	Q	
7	$\langle 0110 \rangle$	XOR, exclusive or	$P \oplus Q$
8	$\langle 0111 \rangle$	OR, disjunction	$P \vee Q$
9	$\langle 1000 \rangle$	NOR, joint denial	$P \downarrow Q$
10	$\langle 1001 \rangle$	equivalence	$P \equiv Q$
11	$\langle 1010 \rangle$	$\sim Q$	
12	$\langle 1011 \rangle$	implication	$Q \supset P$
13	$\langle 1100 \rangle$	$\sim P$	
14	$\langle 1101 \rangle$	implication	$P \supset Q$
15	$\langle 1110 \rangle$	NAND, incompatibility	P/Q
16	$\langle 1111 \rangle$		

Table 2

mean that $\Phi(X, Y, Z)$ generates each functor in K_j.

4 *An initial result.*

Theorem 4.1. *If $\Phi(X, Y, Z)$ generates $P \wedge Q$ and $P \vee Q$ then it cannot generate either of P/Q or $P \downarrow Q$.*

Any functor $F(P, Q)$ which can be generated by $\Phi(X, Y, Z)$ must be such that either $F(0, 0) =_{\mathrm{T}} a$ or $F(1, 1) =_{\mathrm{T}} h$. Consequently, to generate $\langle 0001 \rangle$ and $\langle 0111 \rangle$ there are three possible pairs of values for (a, h), viz.

(i) $(a, h) = (0, 0)$; (ii) $(a, h) = (1, 1)$; (iii) $(a, h) = (0, 1)$.

Now (i) requires

$$\langle 0001\rangle, \langle 0111\rangle \in \{\langle abcd\rangle, \langle abef\rangle, \langle accg\rangle\}$$

and for (ii)

$$\langle 0001\rangle, \langle 0111\rangle \in \{\langle bdfh\rangle, \langle cdgh\rangle, \langle efgh\rangle\}$$

which lead respectively to the conditions

$$(0, 0), (1, 1) \in \{(b, c), (b, e), (c, e)\}$$

and

$$(0, 0), (1, 1) \in \{(d, f), (d, g), (f, g)\},$$

neither of which is possible. Consequently to generate $\langle 0001\rangle$ and $\langle 0111\rangle$ we have $(a, h) = (0, 1)$. However this precludes the possibility of generating either $\langle 1110\rangle$ or $\langle 1000\rangle$ since both of these require either $a = 1$ or $h = 0$. The theorem follows.

The converse result which we give below follows immediately by interchanging 1, 0 in the above proof.

Corollary 4.2. *If $\Phi(X, Y, Z)$ generates P/Q and $P \downarrow Q$ then it cannot generate either of $P \wedge Q$ or $P \vee Q$.*

Corollary 4.3. *$\Phi(X, Y, Z)$ cannot generate four distinct functors from $\mathbf{K}_1$.*

This is a trivial consequence of the theorem since it would require $\Phi(X, Y, Z)$ to generate three distinct functors from $\mathbf{K}_1'$ contrary to the theorem.

5 *The two decision elements.* The two main results of this section show that, excluding permutations, there are just two distinct three-place QUDEs. It will be seen that, by permutations of the variables, five related QUDEs can be obtained from each. This can be proved independently since it can be shown that a necessary condition for a three-place functor to generate three functors from $\mathbf{K}_1$ and two from $\mathbf{K}_2$ is that the functor is fully conjugated, (see [7]).

Let $\mathbf{K}_{11} = \{P \wedge Q, P \vee Q, P \oplus Q\}$, and $\mathbf{K}_{12} = \{P/Q, P \downarrow Q, P \oplus Q\}$.

Theorem 5.1. *There is essentially only one QUDE which generates $\mathbf{K}_{11} \cup \mathbf{K}_2$.*

Initially we deduce the necessary conditions for $\Phi(X, Y, Z)$ to generate $\mathbf{K}_{11}$. It is easily seen that $a = 0$ and $h = 1$ are necessary since otherwise $(0, 0), (1, 1) \in \{(d, f), (d, g), (f, g)\}$ (to generate $\langle 0001\rangle$ and $\langle 0111\rangle$ from (7), (8) and (9)) and $(0, 0), (1, 1) \in \{(b, c), (b, e), (c, e)\}$ (to generate $\langle 0111\rangle$ and $\langle 0001\rangle$ from (4), (5) and (6)), respectively, and neither of these is possible. At this stage the nine binary functors which can be generated have the following value sequences: (1) $\langle 0bg1\rangle$, (2) $\langle 0cf1\rangle$, (3) $\langle 0de1\rangle$, (4) $\langle 0bcd\rangle$, (5) $\langle 0bef\rangle$, (6) $\langle 0ceg\rangle$, (7) $\langle bdf1\rangle$, (8) $\langle cdg1\rangle$, (9) $\langle efg1\rangle$.

Now for $P \oplus Q$:

$$\langle 0110\rangle \in \{\langle 0bcd\rangle, \langle 0bef\rangle, \langle 0ceg\rangle\}$$

so at least two of b, c, e will be 1. Further $0 \epsilon \{b, c, e\}$ since otherwise $P \wedge Q$ cannot be generated. Suppose $b = c = 1$, $e = 0$. The other two cases follow by interchanging X, Y or X, Z in $\Phi(X, Y, Z)$ in the succeeding work. The only difference that this causes is that in some cases we shall generate $P \supset Q$, $P \not\supset Q$ instead of $Q \supset P$, $Q \not\supset P$ respectively. If $b = c = 1$, $c = 0$ then $d = 0$ to generate $P \oplus Q$. To generate $P \vee Q$:

$$\langle 0111 \rangle \epsilon \{\langle 01g1 \rangle, \langle 01f1 \rangle, \langle 0fg1 \rangle\} \text{ so } 1 \epsilon \{g, f\}.$$

This gives the conditions to generate $\mathbf{K}_{11}$ and now we consider $\mathbf{K}_2$. To generate $P \equiv Q$ we require

$$\langle 1001 \rangle \epsilon \{\langle 10f1 \rangle, \langle 10g1 \rangle\} \text{ so } 0 \epsilon \{g, f\}.$$

Suppose $g = 1, f = 0$. The alternative case follows by an interchange of the variables X, Z in $\Phi(X, Y, Z)$. The nine functors which can be generated by $\Phi(X, Y, Z)$ now take the following form:

(1) $\langle 0111 \rangle$, (2) $\langle 0101 \rangle$, (3) $\langle 0001 \rangle$, (4) $\langle 0110 \rangle$, (5) $\langle 0100 \rangle$,
(6) $\langle 0101 \rangle$, (7) $\langle 1001 \rangle$, (8) $\langle 1011 \rangle$, (9) $\langle 0011 \rangle$.

(1), (3), (4), (5), (7), (8) are $P \vee Q$, $P \wedge Q$, $P \oplus Q$, $Q \not\supset P$, $P \equiv Q$, $Q \supset P$ respectively showing that $\Phi(X, Y, Z)$ can generate $\mathbf{K}_{11} \cup \mathbf{K}_2$. The other three functors generated by $\Phi(X, Y, Z)$ are unary functors.

The exact conditions governing $\Phi(X, Y, Z)$ in order to generate $\mathbf{K}_{11} \cup \mathbf{K}_2$ are:

(i) $(a, h) = (0, 1)$;
(ii) either (a) $b = c = 1$; $d = e = 0$; $1, 0 \epsilon \{f, g\}$;
or (b) $c = e = 1$; $g = b = 0$; $1, 0 \epsilon \{d, f\}$;
or (c) $e = b = 1$; $f = c = 0$; $1, 0 \epsilon \{g, d\}$.

These conditions lead to the six distinct functors that we expect.

Theorem 5.2. *There is essentially only one QUDE which generates* $\mathbf{K}_{12} \cup \mathbf{K}_2$.

This follows from Theorem 5.1 by interchanging 1, 0 in the proof. This gives us the generated sets required since $\mathbf{K}_{11} \cup \mathbf{K}_2 \rightarrow \mathbf{K}_{12} \cup \mathbf{K}_2$ on the interchange of 1, 0.

To avoid confusion we shall use $\Phi(X, Y, Z)$ for the QUDE of Theorem 5.1 and $\Psi(X, Y, Z)$ for that of Theorem 5.2. The exact specification for $\Psi(X, Y, Z)$ is as follows:

(i) $(a, h) = (1, 0)$;
(ii) either (a) $b = c = 0$; $d = e = 1$; $1, 0 \epsilon \{f, g\}$;
or (b) $c = e = 0$; $g = b = 1$; $1, 0 \epsilon \{d, f\}$;
or (c) $e = b = 0$; $f = c = 1$; $1, 0 \epsilon \{g, d\}$.

As expected these conditions yield six functors.

The next theorem summarizes the two previous results and shows how negation may be defined. Let $\mathbf{K}_5 = \{\sim P\}$.

Theorem 5.3. *There are, excluding permutations, only two three-place functors which can generate* $\mathbf{K}_2 \cup \mathbf{K}_5$ *in addition to three functors from* $\mathbf{K}_1$.

Clearly these are the functors of the two above theorems as long as negation can be defined. For this we may use

$$\sim P =_{\mathrm{T}} \Phi(0, 1, P) \text{ and } \sim P =_{\mathrm{T}} \Psi(P, 1, 1).$$

Theoretical justifications of these are included in the following sections in which formulae corresponding to the QUDEs are obtained, using the conditioned disjunction functor of Church [1] which is defined by

$$[X, Y, Z] =_{df} X \wedge Y \vee \sim Y \wedge Z.$$

6 *Description of the QUDEs.* As previously noted, both Φ and Ψ are fully conjugated, i.e., there are six distinct functors for each. Details of these are given in Table 3, the abbreviation of Φ for $\Phi(X, Y, Z)$, etc. being used in the headings.

X	Y	Z	Φ	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Ψ	Ψ_1	Ψ_2	Ψ_3	Ψ_4	Ψ_5
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	0	0	1	0	0	0	1	0
0	1	0	0	1	1	0	1	1	0	0	1	0	0	1
0	1	1	0	0	0	1	1	0	0	1	1	1	1	0
1	0	0	1	0	0	1	1	1	0	1	0	1	0	0
1	0	1	0	0	1	0	0	1	1	1	1	0	0	1
1	1	0	1	1	0	0	0	0	1	0	0	1	1	1
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0

Table 3

The following formulae corresponding to these functors can easily be deduced:

$$\begin{aligned}
\Phi(X, Y, Z) &=_{\mathrm{T}} [X, Y, X \oplus Z], & \Psi(X, Y, Z) &=_{\mathrm{T}} [Y \oplus Z, X, \sim Y],\\
\Phi_1(X, Y, Z) &=_{\mathrm{T}} [Y, X, Y \oplus Z], & \Psi_1(X, Y, Z) &=_{\mathrm{T}} [X \oplus Y, Z, \sim Y],\\
\Phi_2(X, Y, Z) &=_{\mathrm{T}} [Z, X, Y \oplus Z], & \Psi_2(X, Y, Z) &=_{\mathrm{T}} [X \oplus Y, Z, \sim X],\\
\Phi_3(X, Y, Z) &=_{\mathrm{T}} [Z, Y, X \oplus Z], & \Psi_3(X, Y, Z) &=_{\mathrm{T}} [X \oplus Z, Y, \sim X],\\
\Phi_4(X, Y, Z) &=_{\mathrm{T}} [Y, Z, X \oplus Y], & \Psi_4(X, Y, Z) &=_{\mathrm{T}} [X \oplus Z, Y, \sim Z],\\
\Phi_5(X, Y, Z) &=_{\mathrm{T}} [X, Z, X \oplus Y], & \Psi_5(X, Y, Z) &=_{\mathrm{T}} [Y \oplus Z, X, \sim Z].
\end{aligned}$$

Taking $\Phi(X, Y, Z)$, $\Psi(X, Y, Z)$ as typical the following definitions of the binary functors may be made:

$$\begin{aligned}
P \wedge Q &=_{df} \Phi(P, Q, P), & P/Q &=_{df} \Psi(P, P, Q),\\
P \vee Q &=_{df} \Phi(P, P, Q), & P \downarrow Q &=_{df} \Psi(P, Q, 0),\\
P \oplus Q &=_{df} \Phi(P, 0, Q), & P \oplus Q &=_{df} \Psi(1, P, Q),\\
P \equiv Q &=_{df} \Phi(P, Q, 1), & P \equiv Q &=_{df} \Psi(P, Q, 0),\\
P \supset Q &=_{df} \Phi(1, Q, P), & P \supset Q &=_{df} \Psi(P, 0, Q),\\
P \not\supset Q &=_{df} \Phi(0, Q, P), & P \not\supset Q &=_{df} \Psi(P, 1, Q),\\
\sim P &=_{df} \Phi(1, 0, P), & \sim P &=_{df} \Psi(1, P, 1).
\end{aligned}$$

Direct verification of these definitions follows easily since

$$P \wedge Q =_T [P, Q, 0], \qquad P/Q =_T [P \oplus Q, P, \sim P],$$
$$P \vee Q =_T [P, P, P \oplus Q], \qquad P \downarrow Q =_T [Q \oplus Q, P, \sim Q],$$
$$P \oplus Q =_T [P, 0, P \oplus Q] =_T [P \oplus Q, 1, \sim P],$$
$$P \equiv Q =_T [P, Q, P \oplus 1] =_T [Q \oplus 0, P, \sim Q],$$
$$P \supset Q =_T [1, Q, P \oplus 1] =_T [Q \oplus 0, P, 1],$$
$$P \not\supset Q =_T [0, Q, P \oplus 0] =_T [Q \oplus 1, P, 0],$$
$$\sim P =_T [1, 0, P \oplus 1] =_T [P \oplus 1, 1, \sim P].$$

7 *Various results involving QUDEs.* In this section we prove four elementary theorems.

Theorem 7.1. *The undefined functors of* $\mathbf{K}_1$ *can be defined using two QUDEs.*

This follows since:

$$P/Q =_T \Phi(1, 0, \Phi(P, Q, P)),$$
$$P \downarrow Q =_T \Phi(1, 0, \Phi(P, P, Q)),$$
$$P \wedge Q =_T \Psi(0, \Psi(P, P, Q), 0),$$
$$P \vee Q =_T \Psi(0, \Psi(P, Q, Q), 0).$$

Theorem 7.2. *$\Psi(X, Y, Z)$ is a Sheffer function though $\Phi(X, Y, Z)$ is not.*

$\Psi(X, Y, Z)$ is a Sheffer function since we have

$$X/Y =_T \Psi(X, X, Y).$$

However $\Phi(0, 0, 0) =_T 0$ so Φ possesses the proper closure property and is consequently not a Sheffer function (see [5] or [6]).

So far we have not established the connection between $\Phi(X, Y, Z)$ and $\Psi(X, Y, Z)$. The following theorem accomplishes this:

Theorem 7.3. $\Phi(\sim X, \sim Y, \sim Z) =_T \Psi(Y, X, Z)$.

The proof is very straightforward since

$$\begin{aligned}\Phi(\sim X, \sim Y, \sim Z) &=_T [\sim X, \sim Y, \sim X \oplus \sim Z]\\ &=_T [\sim X \oplus \sim Z, Y, \sim X]\\ &=_T [X \oplus Z, Y, \sim X]\\ &=_T \Psi(Y, X, Z).\end{aligned}$$

Negated output from QUDEs can be obtained simply by negating a single input as the following theorem shows.

Theorem 7.4. (i) $\sim \Phi(X, Y, Z) =_T \Phi(\sim X, Y, Z)$;
(ii) $\sim \Psi(X, Y, Z) =_T \Psi(X, \sim Y, Z)$.

The proof is based on the result that $\sim [P, Q, R] =_T [\sim P, Q, \sim R]$. We have

$$\begin{aligned}\text{(i)}\quad \sim \Phi(X, Y, Z) &=_T \sim [X, Y, X \oplus Z]\\ &=_T [\sim X, Y, \sim (X \oplus Z)]\\ &=_T [\sim X, Y, \sim X \oplus Z] \text{ since } \sim (X \oplus Z) =_T \sim X \oplus Z\\ &=_T \Phi(\sim X, Y, Z);\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad \sim\Psi(X, Y, Z) &=_{\mathrm{T}} \sim[Y \oplus X, X, \sim Y]\\ &=_{\mathrm{T}} [\sim Y \oplus X, X, \sim(\sim Y)]\\ &=_{\mathrm{T}} \Psi(X, \sim Y, Z).\end{aligned}$$

If functional units are constructed out of these Φ-gates and Ψ-gates (a Φ-gate being a decision element corresponding to $\Phi(X, Y, Z)$) then a considerable reduction in the number of gates required for a particular unit can be obtained, when compared with the number of NAND gates which would be used for the same circuit. The penalty, of course, is that each Φ-gate is considerably more complex than a basic NAND gate. For example, two-level full adder circuits can be made using four Φ-gates or four Ψ-gates (compared with the eight or nine NAND gates normally used).

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