Notre Dame Journal of Formal Logic
Volume XIV, Number 4, October 1973
NDJFAM

ON THE REAL LOGICAL STRUCTURE OF LEWIS' *INDEPENDENT PROOF*

J. E. WIREDU

C. I. Lewis introduced strict implication into modern logic as a concept which "expresses precisely that relation which holds when valid deduction is possible and fails to hold when valid deduction is not possible", *cf.* [2], p. 247. This he did out of a certain dissatisfaction with material implication which he thought other logicians (such as Russell) had claimed to be capable of expressing the relation which interested him. Lewis argued that material implication fell short of the deducibility relation principally on account of the so-called paradoxes of material implication. As is well known, Lewis himself discovered early that strict implication was subject to analogous "paradoxes," and this at first shook his faith in the capabilities of his brain child. On recovering his faith, Lewis offered his famous proof for the thesis that the "paradoxes" of strict implication pointed to inescapable facts about deducibility "which are easily overlooked," *cf.* [2], p. 248. Lewis, of course, had no interest in just showing that rules could be specified such that the "paradoxes" might be seen to be logistically derivable. That would have been extremely unremarkable. What he wished to show is that there are rules of inference which are *intuitively* unexceptionable and which nevertheless unavoidably commit us to the "paradoxes." And this is the point of calling the proof "independent."

In the controversy which Lewis' claim has provoked, every single step in his argument has been disputed. It is not my intention in this paper to discuss the individual steps of the argument. I do, in fact, believe that Lewis did offer an acceptable proof for his claim, provided that deducibility is interpreted in such a way as to distinguish it from inferability,[1] see

1. It is interesting to note that Lewis himself considered the possibility of drawing a distinction between deducibility and inferability though, surprisingly, he did not propose it, *cf.* [2], p. 514.

Received May 1, 1972

my [3]. My aim here, however, is only to discuss the logical structure of Lewis' argument.

It is a surprising fact that alike in adverse as in favourable discussions of Lewis' thesis only a truncated version of the argument is given. The most widely discussed of the "paradoxes" of strict implication is the one usually formulated as that an impossible proposition strictly implies any proposition whatever. The attention which this "paradox" has attracted is deserved since it quite clearly provides a test case and is also intrinsically striking. Lewis' proof for the contention that this "paradox" reveals an unavoidable property of deducibility is generally represented as follows (using Copi's style of Natural Deduction, see [1]):

→ 1. $p \cdot -p$.	
2. p	1, Simplification.
3. $-p$	1, Simplification.
4. $p \vee q$	2, Addition.
5. q	3, 4, Disjunctive Syllogism.
6. $(p \cdot -p) \prec q$	1-5, Conditional Proof.

As far as it goes, this representation differs from the corresponding argument in Lewis only in unessential ways. For example, because Lewis' manner of argumentation is not fully of a natural deduction type, not conditional proof but rather the principle of the transitivity of strict implication would seem to be the justification involved in the last step. It is, however, of special moment to note the following consideration. Even if this derivation is accepted as valid, it may only be taken as showing that *a contradiction* strictly implies any proposition whatever, which is not the same as showing that *an impossible proposition* strictly implies any proposition whatever. Actually, this latter proposition is a misleading way of alluding to the theorem of the calculus of strict implication which Lewis wished to commend to our intuitions. Symbolically, the theorem is $-\Diamond p \prec (p \prec q)$. In prose this is better rendered in some such manner as "that p is impossible strictly implies that p strictly implies q." Were the formulation "an impossible proposition implies any proposition whatever" appropriate, its correct symbolisation would have been not $-\Diamond p \prec (p \prec q)$ but rather $-\Diamond p \prec q$ which, assuredly, is not a thesis of strict implication. Lewis and his commentators have all been guilty of this linguistic laxity.

At all events, however, there is a clear difference between $(p \cdot -p) \prec q$, which the derivation given above proves (granting that it, in fact, proves it) and $-\Diamond p \prec (p \prec q)$, which Lewis wished to establish. Even in a piece of intuitive argumentation the step from the first to the second ought to be justified. Lewis did, in fact, offer a justification or the essentials of one, but it has been generally ignored. As a preliminary to the derivation which is usually given as the whole of the "independent proof," Lewis explained:

> To say 'p is necessary' means 'p is implied by its own denial' or 'the denial of p is not self-consistent' . . . . To say 'p is impossible' means 'p implies its own denial' or 'p is not self-consistent.' Necessary truths so defined coincide with the class of tautologies or truths which can be certified by logic alone; and impossible propositions coincide with the class of those which deny some tautology.
>
> Every tautology is expressible as some proposition of the general form $p \vee -p$. . . . The negation of any proposition of the form $p \vee -p$ is a corresponding proposition of the form $p \cdot -p$. (*Cf.* [2], pp. 248-449).

Thus to say that p is impossible, in Lewis' symbolism, $-\Diamond p$, is to say that p is logically equivalent to a proposition of the form $p \cdot -p$. It was in virtue of this relation between $-\Diamond p$ and $p \cdot -p$ that, immediately on deducing $(p \cdot -p) \prec q$, Lewis felt able to conclude—all too compressedly: "The theorem $-\Diamond p \cdot \prec \cdot p \prec q$ states a fact about deducibility." I shall show directly how a symbolic formulation of Lewis' explanation of impossibility can be incorporated into the derivation so as to make the formal completeness of the "independent" proof obvious to inspection.

The following formulation of the definition of impossibility is clearly indicated in the extended quotation just given:

$$-\Diamond p =_{df} [p = (r \cdot -r)] \qquad \text{(Def. of '}-\Diamond\text{')}$$

The equality symbol in the *definiens* is used in Lewis' sense according to which $p = q$ means $(p \prec q) \cdot (q \prec p)$. Note that p, q, r . . . are employed here as general, so-called metalogical variables. The proof now proceeds as follows:

1. $-\Diamond p$	
2. $p = (r \cdot -r)$	1, Def. of '$-\Diamond$'.
3. $[p \prec (r \cdot -r)] \cdot [(r \cdot -r) \prec p]$	2, Def. of '='.
4. $p \prec (r \cdot -r)$	3, Simplification.
5. $r \cdot -r$	
6. r	5, Simplification.
7. $-r$	5, Simplification.
8. $r \vee q$	6, Addition.
9. q	7, 8, Disjunctive Syllogism.
10. $(r \cdot -r) \prec q$	5-9, Conditional Proof.
11. $p \prec q$	4, 10, Hypothetical Syllogism.
12. $-\Diamond p \prec (p \prec q)$	1-11, Conditional Proof.

If all the rules of deduction employed in this derivation are granted to be intuitively unobjectionable, then it constitutes a formally complete proof of the claim that $-\Diamond p \prec (p \prec q)$ states a fact about deducibility. But whether or not this is conceded, my point is that the above derivation reveals the real logical structure of Lewis' argument which is usually inadequately represented.

REFERENCES

[1] Copi, Irving M., *Symbolic Logic*, Third edition, The Macmillan Company, New York (1967).

[2] Lewis, C. I., and H. L. Langford, *Symbolic Logic*, Second edition, Dover (1959).

[3] Wiredu, J. E., "Deducibility and Inferability," *Mind*, vol. LXXXII (1973), pp. 31-55.

University of Ghana
Legon, Accra, Ghana