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MODAL SYSTEM S3 AND THE PROPER AXIOMS OF S4.02 AND S4.04

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In [1], p. 392, a proof attributed to G. E. Hughes is published which stated that in the field of S4 the proper axiom of S4.04, *cf.*, e.g., [8],

L1 ℭ*LMLpCpLp*

and the formula

L2 ℭ*pLCMLpp*

are inferentially equivalent. It is self-evident that in the field of S4 also a formula

L3 ℭ*LMLpLCpLp*

is inferentially equivalent to **L1** (and to **L2**).

In this note the effects of the addition of **L1**, **L2**, **L3** and of the formulas

Ł1 ℭℭℭ*pLppCLMLpp*

i.e., of the proper axiom of S4.02, *cf.* [6], and

Ł2 ℭℭℭ*pLppLCLMLpp*

which, obviously, in the field of S4 is inferentially equivalent to **Ł1** in the system S3 respectively will be investigated.

1 In the discussions presented below the following matrices

𝔐1

p	1	2	3	4
Mp	1	1	1	3
Lp	2	4	4	4

𝔐2

p	1	2	3	4	5	6	7	8
Mp	1	1	1	1	5	5	5	8
Lp	1	4	4	4	8	8	8	8

𝔐3

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Mp	1	1	1	1	5	6	7	8	1	1	1	1	5	6	7	8
Lp	1	10	11	12	16	16	16	16	9	10	11	12	16	16	16	16

will be used. These matrices are given here only for the functors *M* and *L*.

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An acquaintance with 4, 8 and 16 valued ordinary logical Boolean matrices for the functors C and N is presupposed. Matrix $\mathfrak{M}1$, in which 1 and 2 are the designated values, is the familiar Group I of Lewis-Langford, *cf.* [2], p. 493. Concerning matrices $\mathfrak{M}2$ and $\mathfrak{M}3$, in which 1 is the designated value, *cf.* [7], pp. 350-351, matrices $\mathfrak{M}7$ and $\mathfrak{M}9$.

2 Matrix $\mathfrak{M}1$ verifies system S3 and the formulas **Ł1**, **Ł2** and **L1**, but falsifies:

(i) the proper axiom of S4, i.e., $\mathfrak{C}LpLLp$ for $p/1$: $\mathfrak{C}L1LL1 = LC2L2 = LC24 = L3 = 4$;
(ii) **L2** for $p/2$: $\mathfrak{C}2LCML22 = \mathfrak{C}2LCM42 = \mathfrak{C}2LC32 = \mathfrak{C}2L2 = LC24 = L3 = 4$;
(iii) **L3** for $p/1$: $\mathfrak{C}LML1LC1L1 = \mathfrak{C}LM2LC12 = \mathfrak{C}L1L2 = LC24 = L3 = 4$.

Hence, the addition of **Ł1**, **Ł2** and **L1**, as the new axioms, to S3 does not generate system S4.04. On the other hand, an addition of **L2** or of **L3**, as a new axiom, gives S4.04. *Proof:*

2.1 Assume S3 and the formula **L2**. Then:

Z1	$\mathfrak{C}\mathfrak{C}pq\mathfrak{C}LpLq$	[S3°]
Z2	$\mathfrak{C}LpLLCMLpp$	[*Z1*, $q/LCMLpp$; **L2**]
Z3	$LLCMLCpLCMLppCpLCMLpp$	[*Z2*, $p/CpLCMLpp$; **L2**]

Since, *cf.* [3], p. 148, the addition of any formula of the form $LL\alpha$ to S3 gives S4 and since we proved *Z3*, it follows from the definition of system S4.04, *cf.* [8], that the proof is complete.

2.2 Now, let us assume S3 and the formula **L3**. Then:

Z1	$\mathfrak{C}pMp$	[S1]
Z2	$\mathfrak{C}\mathfrak{C}pq\mathfrak{C}pCpq$	[S2°, *cf.* in [4] the proof of *Z3*]
Z3	$\mathfrak{C}\mathfrak{C}LpMqLMCpq$	[S2°, *cf.* in [4] the proof of *Z9*]
Z4	$\mathfrak{C}LMCMpLqLMLCpq$	[S2, *cf.* in [4] the proof of *Z8*]
Z5	$\mathfrak{C}LCpqLCLpLq$	[S3°]
Z6	$\mathfrak{C}LMLpLCLpLLp$	[**L3**; *Z5*, q/Lp; S1°]
Z7	$\mathfrak{C}LMLpMLCLpLLp$	[*Z6*; *Z1*, $p/LCLpLLp$; S1°]
Z8	$LMCMLpLCLpLLp$	[*Z3*, p/MLp, $q/LCLpLLp$; *Z7*]
Z9	$LMLCLpCLpLLp$	[*Z4*, p/Lp, $q/CLpLLp$; *Z8*]
Z10	$\mathfrak{C}LCLpCLpLLpLLCLpCLpLLp$	[*Z6*, $p/CLpCLpLLp$; *Z9*]
Z11	$\mathfrak{C}LCLpLLpLLCLpCLpLLp$	[*Z2*, p/Lp, $q/CLpLLp$; *Z10*; S1°]
Z12	$\mathfrak{C}LMLpLLCLpCLpLLp$	[*Z6*; *Z11*; S1°]
Z13	$LLCLCLpCLpLLpCLCLpCLpLLpLLCLpCLpLLp$	[*Z12*, $p/CLpCLpLLp$; *Z9*]

Since the proven formula *Z13* has the form $LL\alpha$, the proof, *cf.* section **2.1** above, is complete.

2.3 Thus, we have:

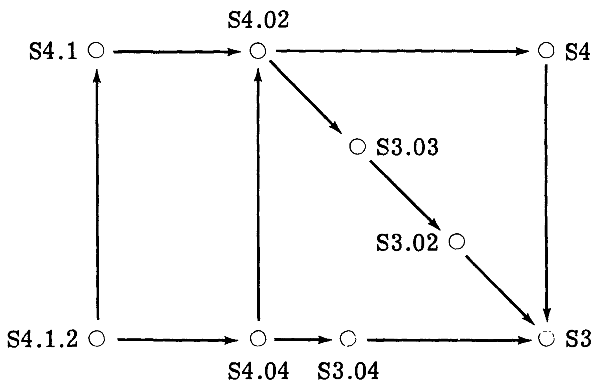
$$\{S4.04\} \rightleftarrows \{S4; \mathbf{L1}\} \rightleftarrows \{S4; \mathbf{L2}\} \rightleftarrows \{S4; \mathbf{L3}\} \rightleftarrows \{S3; \mathbf{L2}\} \rightleftarrows \{S3; \mathbf{L3}\}$$

3 In section **2** above it is shown that the addition of **L1**, as a new axiom, to

S3 does not generate S4.04. On the other hand, matrix $\mathfrak{M}3$ which verifies S3, Ł1 and Ł2 falsifies **L1**, *cf.* [5], p. 374, section **4.4**. Hence, we have a system, viz. S3.04 = {S3; **L1**}, which is a proper extension of S3, a proper subsystem of S4.04 and contains neither S4.02 nor S4. It remains an open problem whether in the field of S3 **L1** implies Ł1 or Ł2.

4 Since matrix $\mathfrak{M}2$ verifies S3, but falsifies Ł1, *cf.* [6], p. 381, section **1**, and Ł2 for $p/3$: $\mathfrak{C}\mathfrak{C}\mathfrak{C}3L33LCLML33 = \mathfrak{C}\mathfrak{C}LC343LCLM43 = \mathfrak{C}\mathfrak{C}L23LCL13 = \mathfrak{C}LC43LC13 = \mathfrak{C}L1L3 = LC14 = L4 = 4$, we can distinguish two proper extensions of S3, viz. S3.02 = {S3; Ł1} and S3.03 = {S3; Ł2}. The reasonings given in sections **2** and **3** above imply that both these systems are proper subsystems of S4.02 and neither of them contains S3.04 or S4. On the other hand, it is self-evident that in the field of S3, S3.03 contains S3.02. But, I was unable to prove that the former system contains the latter properly.

5 The discussion presented in this note can be visualized by the following diagram:



in which an arrow occurring between two systems indicates that a tail system contains an edge system, supposing that S3.04 contains neither S3.02 nor S3.03 and that S3.03 contains S3.02 properly.

6 Remark: In [1], p. 342, Goldblatt says that from a modal-theoretical stand-point **L2** is the "right" axiom of S3.04. But, since in the field of S3 **L1** is weaker than **L2**, from the logical (syntactical) point of view **L1** can be considered as a more suitable axiom for S4.04.

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