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EXTENSIONS OF GÖDEL'S COMPLETENESS THEOREM AND THE LÖWENHEIM-SKOLEM THEOREM

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This note has a modest purpose: to present in the language of consequence operations some arguments, found in the theory of universal models, which extend the famous theorems of the title. This formulation provides further, but natural, motivations for the study of universal models besides the usual one, given for example in [3], chapter 10.

Let $\mathcal{L}$ be (the set of formulas of) a countable first-order language. A theory is a set of sentences of $\mathcal{L}$. Each theory T defines a consequence operation $\mathbf{C}_T$:

$$X \in \mathbf{C}_T(S) \textit{ iff } T \cup S \vdash X$$

where $X \in \mathcal{L}$, $S \subseteq \mathcal{L}$ and $\vdash$ is the standard provability relation. Each $\mathcal{L}$-structure $\mathfrak{A}$ also defines a consequence operation $\mathbf{C}_{\mathfrak{A}}$ as follows:

$$X \in \mathbf{C}_{\mathfrak{A}}(S) \textit{ iff } \forall h(\mathfrak{A} \models S[h] \Longrightarrow \mathfrak{A} \models X[h])$$

where h ranges over all valuations of $\mathfrak{A}$ and where "$\mathfrak{A} \models S[h]$" means that h simultaneously satisfies all formulas in S. Note that $\mathbf{C}_{\mathfrak{A}}(\emptyset)$ is the set of formulas true in $\mathfrak{A}$.

A special case of Gödel's completeness theorem may now be written as follows:[1]

T is a complete[2] *theory iff* $\mathbf{C}_T(\emptyset) = \mathbf{C}_{\mathfrak{A}}(\emptyset)$, *for some structure* $\mathfrak{A}$.

The Löwenheim-Skolem theorem (for complete theories) becomes in this notation:

T is a complete theory iff $\mathbf{C}_T(\emptyset) = \mathbf{C}_{\mathfrak{A}}(\emptyset)$ *for some countable structure* $\mathfrak{A}$.

1. The general completeness theorem is:
$\mathbf{C}_{\emptyset}(S) = \bigcap_{\mathfrak{A}} \mathbf{C}_{\mathfrak{A}}(S)$.

2. A complete theory is assumed consistent.

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The reader will notice that the above famous theorems involve the values of the consequence operations C_T and $\mathsf{C}_{\mathfrak{A}}$ on the empty set only! This observation immediately leads to the following questions:

1. For which complete theories T does there exist an $\mathfrak{A}$ such that $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$ (i.e., $\mathsf{C}_T(S) = \mathsf{C}_{\mathfrak{A}}(S)$, all $S \subseteq \mathcal{L}$)?

2. For which complete theories T does there exist a countable $\mathfrak{A}$ such that $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$?

Remark. We are restricting our attention to complete theories because, if $\mathsf{C}_T(\emptyset) = \mathsf{C}_{\mathfrak{A}}(\emptyset)$, T is complete.

Our first theorem generalizes the completeness theorem and answers question 1.

Theorem 1. *T is a complete theory iff $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$, for some structure $\mathfrak{A}$.*

Proof. In view of the above remark, we need only prove the "only if" part. We use the following easily proved lemma:

Lemma. *$\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$ iff whenever $T \cup S$ is consistent, then for some h, $\mathfrak{A} \models T \cup S[h]$.*

Now let $(S_i: i \in I)$ be the collection of all sets of formulas consistent with T. For each i in I, let $d^i = \langle d_1^i, d_2^i, \ldots \rangle$ be a distinct list of new constant symbols, and let S_i^* be the set of formulas obtained by replacing each free occurrence of the variable v_n by d_n^i in every formula in S_i. Now let $S = \bigcup S_i^*$ and let $T' = T \cup S$. Assume that we have shown T' consistent. Let $\mathfrak{A}' = \langle \mathfrak{A}, \langle d^i: i \in I \rangle \rangle$ be a model of T'. Then clearly S_i is satisfiable in $\mathfrak{A}$ by d^i. From the lemma it follows that $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$.

We show T' is consistent. If not, for some finite subset I' of I with n members, $T \cup \bigcup (S_i^*: i \in I')$ is inconsistent. For notational convenience, we assume that $n = 2$ and that the sets S_i^* are closed under conjunction. Then for some formulas X, Y in S_a^* and S_b^* respectively ($a, b \in I$)

$$T \cup \{X(v/d^a), Y(v/d^b)\}$$

is inconsistent. ($X(v/d^a)$ has the obvious meaning.) But then $T \cup \{(\exists v) X, (\exists v) Y\}$ is inconsistent. However, since T is complete, $T \vdash (\exists v) X$ and $T \vdash (\exists v) Y$. (Indeed, otherwise, $T \vdash (\forall v) \neg X$, so that X would be inconsistent with T). This contradicts the consistency of T and completes the proof.

Remarks. 1. Note that the cardinality of the model $\mathfrak{A}$ in the previous theorem may be chosen $\leqslant 2^{\aleph_0}$.
2. By a slight modification of the above proof we can show: every structure $\mathfrak{A}$ has an elementary extension $\mathfrak{A}'$ such that $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}'}$, where $T = \mathsf{C}_{\mathfrak{A}}(\emptyset) = \mathsf{C}_{\mathfrak{A}'}(\emptyset)$. One need only replace $T = \mathsf{C}_{\mathfrak{A}}(\emptyset)$ in the above proof by what Shoenfield calls $\mathsf{D}_e(\mathfrak{A})$ (see [1], p. 74). It is easy to see that $\mathsf{D}_e(\mathfrak{A}) \cup S$ is consistent if $T \cup S$ is, and any model of $\mathsf{D}_e(\mathfrak{A})$ is an expansion of an elementary extension of $\mathfrak{A}$.

We answer question 2 in the next theorem which may be considered an extension of the Löwenheim-Skolem theorem. The terms "universal model" and "n-type" are defined in [1], pp. 89 and 102.

Theorem 2. *Let T be a complete theory.* $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$ *for some countable* $\mathfrak{A}$ *iff for each n, T has $\leqslant\aleph_0$ n-types.*

Proof. The condition is clearly necessary. Conversely, if T has $\leqslant\aleph_0$ n-types for each n, T has a (countable) universal model $\mathfrak{A}$ (see [1], p. 102). $\mathfrak{A}$ is the desired model. Indeed, suppose that h satisfies $T \cup S$ is the countable structure $\mathfrak{A}'$. Since $\mathfrak{A}$ is universal, we may assume that $\mathfrak{A}'$ is an elementary submodel of $\mathfrak{A}$. Hence h satisfies $T \cup S$ in $\mathfrak{A}$. It follows from the lemma that $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$.

Remark. From our lemma (or from the techniques of [2]) one obtains the following characterization of those structures $\mathfrak{A}$ such that $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$ (where $T = \mathsf{C}_{\mathfrak{A}}(\emptyset)$): $\mathsf{C}_T = \mathsf{C}_{\mathfrak{A}}$ iff every countable model of T is isomorphic to an elementary submodel of $\mathfrak{A}$.

Problem. Let O be the set of open formulas of $\mathcal{L}$ and let T be a subset of O. What are necessary and sufficient conditions on T such that there is a countable structure $\mathfrak{A}$ with the property

$$(*) \qquad O \cap \mathsf{C}_T(S) = O \cap \mathsf{C}_{\mathfrak{A}}(S), \text{ all } S \subseteq O.?$$

It can be shown that there is some (perhaps uncountable) $\mathfrak{A}$ satisfying $(*)$ iff T has the joint embedding property.

REFERENCES

[1] Shoenfield, J. R., *Mathematical Logic,* Addison Wesley, Reading, Mass. (1967).

[2] Mycielski, J., and C. Ryll-Nardzewski, "Equationally compact algebras II," *Fundamenta Mathematicae*, vol. LXI (1968), pp. 271-281.

[3] Bell, J., and A. B. Slomson, *Models and Ultraproducts,* North Holland, Amsterdam (1969).

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