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ON COMPACTNESS IN MANY-VALUED LOGIC. I

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Our purpose in this paper is to formulate and prove a rather general compactness property for finitely many-valued logics, from which more familiar forms of compactness are derivable. The proof employs a slight generalization of Robinson's Special Valuation Lemma [1], p. 13. Although other such results are available in the literature,[1] the present version shows in an interesting way the effect of many-valuedness on the compactness proof, and also avoids reliance on "high-powered" results such as Tychonoff's theorem. Finally, our version makes no assumptions about the expressive power or designated values of the system.

1 An *n-valued logic* L_n is a system $\langle A, O, M, D\rangle$ such that (i) A is a (finite or infinite) set of objects called the *atomic formulas* or *atoms* of L_n; (ii) O is a finite set of objects, discrete from A, called *operations*, with each of which is associated an unique non-negative integer called its *degree*; (iii) M contains for each member of O of degree m an unique m-ary mapping of $\{1, \ldots, n\}$ into itself,[2] called the *matrix* of the operation; and (iv) D is $\{1, \ldots, m\}$ for some $m < n$; the elements of D are called the *designated values* of L_n.

The set W of *well-formed formulas* (wffs) of L_n is the least set containing the atoms and such that if $\mathsf{p}_1, \ldots, \mathsf{p}_k$ are elements of W and $\mathbf{o}^k$ is an operation of degree k, then the concatenation $\mathbf{o}^k\mathsf{p}_1 \ldots \mathsf{p}_k$ is in W. If B is any set of atoms, W_B is the set of wffs formed from B as W is from A. If p is any wff, $\overline{\mathsf{p}}$ is the least set B of atoms such that $\mathsf{p} \in W_B$; intuitively, $\overline{\mathsf{p}}$ comprises the atoms which have occurrences in p.

A mapping α of A into $\{1, \ldots, n\}$ is called an *assignment of values* for L_n. Each assignment determines an unique *interpretation* I_α of W as

1. The most general results are those of [2]; a simplified version of these is available in [3]. A definition of compactness close to ours appears in [4].

2. The choice of integers as truth-values is for convenience only; any n-element set would do.

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follows: if p is atomic, $\mathsf{I}_\alpha \mathrm{p} = \alpha \mathrm{p}$; otherwise, p is of the form $\mathbf{o}^k \mathrm{p}_1 \ldots \mathrm{p}_k$ and we may suppose $\mathsf{I}_\alpha \mathrm{p}_i$ defined for $i \leq k$. Let f be the matrix of $\mathbf{o}^k$; then $\mathsf{I}_\alpha \mathrm{p} = f(\mathsf{I}_\alpha \mathrm{p}_1, \ldots, \mathsf{I}_\alpha \mathrm{p}_k)$. It is clear that I_α is locally determined, in the sense that if $\alpha|\overline{\mathrm{p}} = \beta|\overline{\mathrm{p}}$, then $\mathsf{I}_\alpha \mathrm{p} = \mathsf{I}_\beta \mathrm{p}$.

Two wffs are *equivalent* iff they have the same value on every assignment. As is well known, if B contains k atoms, then a subset of W_B can contain at most n^{n^k} pairwise non-equivalent wffs.

An n-tuple $\mathcal{K} = \langle K_1, \ldots, K_n \rangle$ of subsets of W is *satisfied* by an assignment α iff for some $i \leq n$ and $\mathrm{p} \in K_i$, $\mathsf{I}_\alpha \mathrm{p} = i$. $\mathcal{K}$ is *valid* iff satisfied by every assignment. $\mathcal{K}$ is finite iff each K_i is finite; $\mathcal{L} = \langle L_1, \ldots, L_n \rangle$ is a sub-n-tuple of $\mathcal{K}$ iff each L_i is a subset of K_i.

Lemma 1.1 *$\mathcal{K}$ is valid iff it contains a valid sub-n-tuple $\mathcal{L}$ with the following properties:* (i) *each* $\mathrm{p} \in K_i$ *is equivalent to some* $\mathrm{q} \in L_i$; (ii) *the elements of L_i are pairwise non-equivalent—for all $i \leq n$.*

Proof: Equivalence is clearly an equivalence relation on W. Let L_i consist of one representative from each equivalence class represented in K_i (using the axiom of choice if K_i contains infinitely many non-equivalent wffs). The lemma is now immediate.

2 By a *partial valuation* of a set S in a set U we mean any partial map from S to U; a *total* valuation is one whose domain is all of S. We write as usual $D\phi$ for the domain of ϕ and $\phi|T$ for the restriction of ϕ to T. We shall require the following lemma, whose proof differs from that of Robinson's Special Valuation Lemma only in the minor detail that U may contain any finite number of elements, rather than just two. Details are left to the reader.

Lemma 2.1 *Let $\{\phi_\nu\}$ be an indexed set of partial valuations of S in a finite set U, such that for any finite $T \subseteq S$, there is a ν such that $T \subseteq D$. Then there is a total valuation ψ such that for each finite T, there is a ν with $T \subseteq D\phi_\nu$ and $\psi | T = \phi_\nu | T$.*

3 We can now state and prove our main result.

General Compactness Theorem: *Every valid n-tuple of sets of wffs contains a valid finite sub-n-tuple.*

Proof: Suppose no finite sub-n-tuple of $\mathcal{K} = \langle K_1, \ldots, K_n \rangle$ is valid. By Lemma 1.1 we may assume without loss of generality that the elements of K_i are pairwise non-equivalent. Let ν be any finite set of atoms, and let $L_{\nu i} = W_\nu \cap K_i$ for each $i \leq n$. Then as previously noted, $L_{\nu i}$ must be finite. Hence $\mathcal{L}_\nu = \langle L_{\nu 1}, \ldots, L_{\nu n} \rangle$ is a finite sub-n-tuple of $\mathcal{K}$ and therefore some assignment α fails to satisfy it (by hypothesis). Set $\phi_\nu = \alpha|\nu$. Then $\{\phi_\nu\}$ is a set[3] of partial valuations of A in $\{1, \ldots, n\}$ which satisfies

3. There is a surreptitious but avoidable use of the axiom of choice at this point.

the antecedent of 2.1, hence there is a total valuation of A in $\{1, \ldots, n\}$ (i.e., an assignment) α^* which coincides with some ϕ_ν on each finite subset of A. Let p be any element of K_i. $\overline{\mathrm{p}}$ is finite, so for some ν, $\overline{\mathrm{p}} \subseteq D\phi_\nu = \nu$ and $\alpha^*|\overline{\mathrm{p}} = \phi_\nu|\overline{\mathrm{p}}$. By construction, $\phi_\nu = \alpha|\nu$ for some assignment α which fails to satisfy $\mathcal{L}_\nu$. Since $\overline{\mathrm{p}} \subseteq \nu$, we have $\alpha^*|\overline{\mathrm{p}} = \phi_\nu|\overline{\mathrm{p}} = \alpha|\overline{\mathrm{p}}$, and hence by local determination, $\mathsf{I}_{\alpha^*}\mathrm{p} = \mathsf{I}_\alpha \mathrm{p}$. Furthermore, since $\overline{\mathrm{p}} \subseteq \nu$, $\mathrm{p} \in W_\nu$ and hence $\mathrm{p} \in W_\nu \cap K_i = L_{\nu i}$, so since α does not satisfy $\mathcal{L}_\nu$, $\mathsf{I}_\alpha \mathrm{p} \neq i$. It follows that α^* fails to satisfy $\mathcal{K}$, so that $\mathcal{K}$ is not valid. The theorem now follows by contraposition.

4 A more usual notion of compactness involves a concept of satisfiability defined as follows: a set of wffs is *satisfiable$_D$* iff some assignment gives every element of the set a designated value. As a corollary to our general theorem we have the

Standard Compactness Theorem: *If every finite subset of K is satisfiable$_D$, then K is also.*

Proof: Suppose K is not satisfiable$_D$, and set $\mathcal{K} = \langle K_1, \ldots, K_n \rangle$, where if $i \in D$, $K_i = \emptyset$, and otherwise $K_i = K$. Then $\mathcal{K}$ is clearly valid, so contains a valid finite sub-n-tuple $\mathcal{L} = \langle L_1, \ldots, L_n \rangle$. Let $L = \bigcup_{i \notin D} L_i$. Then L is a finite subset of K, and is not satisfiable$_D$, since any assignment which satisfied it would *ipso facto* fail to satisfy $\mathcal{L}$. The theorem follows by contraposition.

5 Extension of the above theorems to quantificational many-valued logics will be discussed in a sequel to this paper.

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