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A FORMAL CHARACTERIZATION OF ORDINAL NUMBERS

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In this paper we present the axioms for a first-order finitely axiomatized theory **ORD**, some of whose models are relational systems $\mathcal{S}$ with the following particular characteristics:

(i) S, the domain of discourse of $\mathcal{S}$, is any ordinal number;

and

(ii) each primitive relation symbol of the alphabet of **ORD** is interpreted in $\mathcal{S}$ in the standard manner.

Of special importance is the fact, demonstrated below, that **ORD** is an example of a theory in which the proof-theoretic notions of explicit and implicit definability, as stated in Beth [1], [2] and Smullyan [3], may be illustrated.

1 Basic Concepts. Let T be a first-order theory whose non-logical axioms are the set of sentences denoted by Γ_0. Let P, P_1, P_2 . . . be the relation symbols of the alphabet of T which occur in at least one member of Γ_0. In addition, P will be assumed to be an n-place relation symbol for some positive integer n.

P is *explicitly definable* from P_1, P_2 . . . in T if there exists a wff $U(x_1, x_2, \ldots, x_n)$, all of whose relation symbols occur in the list P_1, P_2 . . ., such that

$$\Gamma_0 \vdash (\forall x_1)(\forall x_2) \ldots (\forall x_n) [P(x_1, x_2, \ldots, x_n) \rightleftarrows U(x_1, x_2, \ldots, x_n)].$$

Let P' be a relation symbol of the alphabet of T having the same number of places as P. Assume P' does not occur in Γ_0, and let Γ_0' be the result of substituting P' for P in every sentence of Γ_0 in which P appears.

P is *implicitly definable* from P_1, P_2 . . . in T if

$$\Gamma_0 \cup \Gamma_0' \vdash (\forall x_1)(\forall x_2) \ldots (\forall x_n) [P(x_1, x_2, \ldots, x_n) \rightleftarrows P'(x_1, x_2, \ldots, x_n)].$$

2 The Theory **ORD**. The first-order theory **ORD** is, basically, a theory with equality, such that the four binary relation symbols, $\approx$, $\subset$, $\subseteq$, and ϵ

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exhaust the list of non-logical symbols in its alphabet. The set Γ_0 of non-logical axioms of **ORD** consists of the following ten members:

ORD 1 $(\forall x)(\forall y)(\forall z)\ [((x \subset y) \wedge (y \subset z)) \rightarrow (x \subset z)]$;[1]
ORD 2 $(\forall x)\ [\sim(x \subset x)]$;
ORD 3 $(\forall x)(\forall y)\ [(x \subset y) \rightarrow \sim(y \subset x)]$;
ORD 4 $(\forall x)(\forall y)\ [(x \subset y) \vee (y \subset x) \vee (x \approx y)]$;
ORD 5 $(\forall x)(\forall y)\ [(x \subseteq y) \rightleftarrows ([(\forall z)\ [(z \subset x) \rightarrow (z \subset y)] \wedge \sim(x \approx y)] \vee (x \approx y))]$;
ORD 6 $(\forall x)(\forall y)(\forall z)(\forall u)\ [(x \approx y) \rightarrow [(z \approx u) \rightarrow [(x \subset z) \rightarrow (y \subset u)]]]$;
ORD 7 $(\forall x)(\forall y)(\forall z)(\forall u)\ [(x \approx y) \rightarrow [(z \approx u) \rightarrow [(x \subseteq z) \rightarrow (y \subseteq u)]]]$;
ORD 8 $(\forall x)(\forall y)(\forall z)(\forall u)\ [(x \approx y) \rightarrow [(z \approx u) \rightarrow [(x \approx z) \rightarrow (y \approx u)]]]$;
ORD 9 $(\forall x)\ [x \subseteq x]$;
ORD 10 $(\forall x)\ [x \approx x]$.

3 Illustration of Explicit and Implicit Definability in ORD. Using the notation of the last section, take for P the relation symbol $\subset$ and for P' the relation symbol ϵ. Then the set Γ_0' consists of

ORD′ 1 $(\forall x)(\forall y)(\forall z)\ [((x \epsilon y) \wedge (y \epsilon z)) \rightarrow (x \epsilon z)]$;
ORD′ 2 $(\forall x)\ [\sim(x \epsilon x)]$;
ORD′ 3 $(\forall x)(\forall y)\ [(x \epsilon y) \rightarrow \sim(y \epsilon x)]$;
ORD′ 4 $(\forall x)(\forall y)\ [(x \epsilon y) \vee (y \epsilon x) \vee (x \approx y)]$;
ORD′ 5 $(\forall x)(\forall y)\ [(x \subseteq y) \rightleftarrows ([(\forall z)\ [(z \epsilon x) \rightarrow (z \epsilon y)] \wedge \sim(x \approx y)] \vee (x \approx y))]$;
ORD′ 6 $(\forall x)(\forall y)(\forall z)(\forall u)\ [(x \approx y) \rightarrow [(z \approx u) \rightarrow [(x \epsilon z) \rightarrow (y \epsilon u)]]]$;

and where ORD′ n = ORD n for $n = 7, 8, 9, 10$.

We then have the following

Theorem I: *With P, P', Γ_0, and Γ_0' so described, P is implicitly definable in* **ORD** *by $U(x, y)$, where $U(x, y)$ is the wff* $[(x \subseteq y) \wedge \sim(x \approx y)]$.

Theorem II: *With P, P', Γ_0, and Γ_0' so described, P is implicitly definable in* **ORD**, *i.e., from $\Gamma_0 \cup \Gamma_0'$ it is possible to deduce* $(\forall x)(\forall y)\ [(x \subset y) \rightleftarrows (x \epsilon y)]$.

In proving each of these theorems, we omit the details of formal logic, and merely indicate how each step follows from preceding ones by invoking the appropriate member of Γ_0 or Γ_0'. It should, however, be pointed out that a proof of each completely within the syntax of **ORD** is possible.

For the proof of Theorem I, first assume that $(x \subset y)$. If, in addition, $(x \approx y)$ is assumed, then these two would yield $(x \subset x)$ by ORD 6; but $(x \subset x)$ is impossible by ORD 2. Thus, $(x \subset y)$ implies $\sim(x \approx y)$. On the other hand, suppose $(x \subset y)$ did not imply $(x \subseteq y)$, i.e., suppose both $(x \subset y)$ and $\sim(x \subseteq y)$ were true. Since $\sim(x \subseteq y)$ is the case, $\sim([(\forall z)[(z \subset x) \rightarrow (z \subset y)] \wedge \sim(x \approx y)] \vee (x \approx y))$ follows by ORD 5. That is, $([\sim(\forall z)\ [(z \subset x) \rightarrow (z \subset y)] \vee (x \approx y)] \wedge \sim(x \approx y))$ results from $\sim(x \subseteq y)$. Since $\sim(x \approx y)$ has already been established, it must also follow that $\sim(\forall z)\ [(z \subset x) \rightarrow (z \subset y)]$ is true, i.e.,

1. Throughout this paper we adopt the convention of placing the binary relation symbol between the symbols being related.

there must exist some w such that $(w \subset x)$ but $\sim(w \subset y)$. Hence, some w exists such that $(w \subset x)$ and either $(y \subset w)$ or $(y \approx w)$, by ORD 4. Suppose $(y \approx w)$ were true. Since $(w \subset x)$, it would follow that $(y \subset x)$, which is impossible by the original assumption that $(x \subset y)$ and ORD 3. Furthermore, if $(y \subset w)$ were so, then $(y \subset w)$ coupled with $(w \subset x)$ would again yield $(y \subset x)$ by ORD 1. Since all possibilities have been exhausted, the conclusion is that it is impossible for $(x \subset y)$ and $\sim(x \subseteq y)$ to hold jointly. Thus if $(x \subset y)$ is true, then $(x \subseteq y)$ follows, i.e., $(x \subset y)$ implies $(x \subseteq y)$. Therefore, if $(x \subset y)$, then both $(x \subseteq y)$ and $\sim(x \approx y)$, i.e., $(x \subset y)$ implies $[(x \subseteq y) \wedge \sim(x \approx y)]$.

Conversely, suppose it is the case that both $(x \subseteq y)$ and $\sim(x \approx y)$. In addition, suppose it were false that $(x \subset y)$. Then, by ORD 4, either $(x \approx y)$ or $(y \subset x)$. But it is immediate that $(x \approx y)$ is impossible, since it has been assumed that $\sim(x \approx y)$. Furthermore, suppose $(y \subset x)$. Since $(x \subseteq y)$ has been assumed, $[(\forall z)\ [(z \subset x) \rightarrow (z \subset y)] \wedge \sim(x \approx y)] \vee (x \approx y)$ holds by ORD 5. Since $(y \subset x)$, y is a candidate for z, i.e., $[(y \subset x) \rightarrow (y \subset y)]$ is possible; but since $(y \subset x)$ is assumed, we obtain the conclusion that $(y \subset y)$, which is impossible by ORD 2. Hence the assertion that $(y \subset x)$ produces a contradiction. The only remaining alternative is $(x \subset y)$, which must hold by ORD 4. Therefore, $[(x \subseteq y) \wedge \sim(x \approx y)]$ implies $(x \subset y)$, completing the equivalence and hence the proof of Theorem I.

The proof of Theorem II follows along similar lines. First assume $(x \subset y)$ is the case. In order to prove $(x \in y)$ is a consequence, ORD′ 4 will be used to eliminate the possibilities $(x \approx y)$ and $(y \in x)$. Indeed, suppose $(x \approx y)$ were true; then $(x \subset y)$ would become $(x \subset x)$, which violates ORD 2. On the other hand, if it were true that $(y \in x)$, then $(y \subseteq x)$ would follow, for suppose $(z \in y)$ for any z. Then $(z \in y)$ together with $(y \in x)$ would yield $(z \in x)$ by ORD′ 1, and hence, by ORD′ 5, $(y \subseteq x)$, since $\sim(x \approx y)$ has also been established. Applying ORD 5 with $(y \subseteq x)$ established produces the fact that for all z, $[([(z \subset y) \rightarrow (z \subset x)] \wedge \sim(x \approx y)) \vee (x \approx y)]$. But, by virtue of the fact that $\sim(x \approx y)$ is true, it would follow that $(x \subset x)$, since we have assumed that $(x \subset y)$. Thus, by ORD′ 4, the only remaining alternative is $(x \in y)$, and so $(x \subset y)$ implies $(x \in y)$.

Conversely, suppose $(x \in y)$. Then it cannot be the case that $(x \approx y)$, for if so, $(x \in y)$ would become $(x \in x)$, which is impossible by ORD′ 2. Furthermore, suppose $(y \subset x)$ were so. Then $(y \subseteq x)$ would follow, for suppose $(z \subset y)$ for any z. Then, using ORD 1 with $(z \subset y)$ and $(y \subset x)$, we get $(z \subset x)$; by ORD 5, $(y \subseteq x)$ follows, since it is also known that $\sim(x \approx y)$. Using ORD′ 5 with $(y \subseteq x)$ established, it is the case that for all z, $[([(z \in y) \rightarrow (z \in x)] \wedge \sim(x \approx y)) \vee (x \approx y)]$, i.e., for all z, $(z \in y)$ implies $(z \in x)$. Since this implication holds for all z, it must certainly hold for z set equal to x; that is, $(x \in y)$ implies $(x \in x)$. Thus, $(x \in x)$ is deduced from $(y \subset x)$, and by ORD 4, the only remaining alternative is $(x \subset y)$. Hence $(x \in y)$ implies $(x \subset y)$, completing the proof of Theorem II.

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