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COMBINATORIAL SYSTEMS WITH AXIOM

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*Introduction:** The purpose of this paper is to investigate the general decision problems associated with a number of combinatorial systems with axiom. In particular, we shall show the many-one equivalence of the general halting problem for Turing machines, the general decision problem for Thue systems with axiom, the general decision problem for semi-Thue systems with axiom, and the general decision problem for Post normal systems with axiom. This, combined with a recent result of Overbeek [5], shows that every recursively enumerable (r.e.) many-one degree (of unsolvability) is represented by each of these general problems for systems with axiom. Finally, this latter result is proven to be best possible in that it does not hold for every r.e. one-one degree.

Historical Background: Semi-Thue systems, Thue systems and Post normal systems were defined by Post as proper subsets of canonical forms. Decision problems associated with these systems have been studied by various authors, e.g., [1], [2], [3], [4], [6], [7], and [8]. In particular, W. E. Singletary [7] has combined results of his own and those of others in such a way as to provide an effective proof of the (r.e.) equivalence of the general decision problems which are of concern to us here. The stronger results to be proven here were announced in [4] and form part of an extensive study into the equivalence of general combinatorial decision problems.

Preliminary Definitions: A semi-Thue system T is a pair (Σ, R) where Σ is a finite alphabet and R is a finite set of productions of the form $\alpha \rightarrow \beta$, for α and β words over Σ. T is said to be a Thue system if $\alpha \rightarrow \beta$ belongs to R implies $\beta \rightarrow \alpha$ is also in R. For any arbitrary pair of words W, W' over Σ, we say that W' is an immediate successor of W in T, denoted $T(W, W')$ if $W \equiv P \alpha Q$, $W' \equiv P \beta Q$, P and Q are words over Σ, and $\alpha \rightarrow \beta$ is in R.

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A Post normal system N is a pair (Σ, R) where Σ is a finite alphabet and R is a finite set of productions of the form $\alpha \to \beta$, for α and β words over Σ. For any arbitrary pair of words W, W' over Σ, we say that W' is an immediate successor of W in N, denoted $N(W, W')$, if $W \equiv \alpha P$, $W' \equiv P\beta$, P is a word over Σ, and $\alpha \to \beta$ is in R.

Let M be a semi-Thue system, Thue system or Post normal system and let A be a word over the alphabet of M. Then M_A shall denote a system with axiom. For arbitrary words W and W' over the alphabet of M, we say W' is derivable from W in M, denoted $M[W, W']$, if either $W \equiv W'$, $M(W, W')$, or W' is derivable from an immediate successor of W. The decision problem for M_A is the problem of determining, for any arbitrary word W over the alphabet of M, whether or not W is derivable from A in M.

A Turing machine M is a triple (Σ, Q, S) where $\Sigma = \{a_1, \ldots, a_n\}$ is a finite alphabet; $Q = \{q_1, \ldots, q_m\}$ is a finite set of symbols, called states; a_0, $\mathcal{R}$ and $\mathcal{L}$ are new symbols; and S is a non-empty set of quadruples of the form $qbDs$ where q and s are states, b is a member of $(\Sigma \cup \{a_0\})$ and $D \in (\{\mathcal{R}, \mathcal{L}, a_0\} \cup \Sigma)$ and with the property that no two distinct quadruples of S, $qbDs$ and $q'b'D's'$ are such that $qb \equiv q'b'$. We shall call members of $Q \times (\Sigma \cup \{a_0\})$ discriminants. A terminal discriminant is defined to be any discriminant qa such that there is no quadruple of S with the initial subword qa. A configuration of M is any string of symbols W over $(\Sigma \cup \{a_0\} \cup Q)$ such that W contains exactly one occurrence of a member of Q, the leftmost symbol of W is not a_0, the rightmost symbol of W is a member of $(\Sigma \cup \{a_0\})$ and is a_0 only if the symbol occurrence immediately to its left is a member of Q. We say that W' is the immediate successor of $W \equiv a_{i_1} \ldots a_{i_k} q_u a_{j_1} \ldots a_{j_h}$, denoted $M(W, W')$, if and only if any of the following cases obtain:

(i) $q_u a_{j_1} a_p q_v \in S$ and $W' \equiv a_{i_1} \ldots a_{i_k} q_v a_p a_{j_2} \ldots a_{j_h}$;
(ii) $q_u a_{j_1} \mathcal{R} q_v \in S$, $h > 1$, either $k > 0$ or $j_1 > 0$, and $W' \equiv a_{i_1} \ldots a_{i_k} a_{j_1} q_v a_{j_2} \ldots a_{j_h}$;
(iii) $q_u a_{j_1} \mathcal{R} q_v \in S$, $h > 1$, $k = 0$, $j_1 = 0$ and $W' \equiv q_v a_{j_2} \ldots a_{j_h}$;
(iv) $q_u a_{j_1} \mathcal{R} q_v \in S$, $h = 1$, either $k > 0$ or $j_1 > 0$, and $W' \equiv a_{j_1} \ldots a_{j_k} a_{j_1} q_v a_0$;
(v) $q_u a_{j_1} \mathcal{R} q_v \in S$, $h = 1$, $k = 0$, $j_1 = 0$ and $W' \equiv q_v a_0$;
(vi) $q_u a_{j_1} \mathcal{L} q_v \in S$, $k > 0$, either $j_1 > 0$ or $h > 1$, and $W' \equiv a_{i_1} \ldots a_{i_{k-1}} q_v a_{i_k} a_{j_1} \ldots a_{j_h}$;
(vii) $q_u a_{j_1} \mathcal{L} q_v \in S$, $k > 0$, $j_1 = 0$, $h = 1$ and $W' \equiv q_{i_1} \ldots a_{i_{k-1}} q_v a_{i_k}$;
(viii) $q_u a_{j_1} \mathcal{L} q_v \in S$, $k = 0$, either $j_1 > 0$ or $h > 1$, and $W' \equiv q_v a_0 a_{j_1} \ldots a_{j_h}$;
(ix) $q_u a_{j_1} \mathcal{L} q_v \in S$, $k = 0$, $j_1 = 0$, $h = 1$ and $W' \equiv q_v a_0$.

Let M be a Turing machine. For arbitrary configuration C of M, we say C is terminal if and only if there is no C' such that $M(C, C')$. C' is said to be derivable from C, denoted $M[C, C']$, if either $C \equiv C'$, $M(C, C')$ or C' is derivable from an immediate successor of C. C is said to be mortal if there is a terminal C' such that $M[C, C']$. The halting problem for M is the problem of determining for an arbitrary configuration C of M whether or not C is mortal.

Let G and G' be two general decision problems. Then we say that G is many-one reducible to G' if there exists a one-one effective mapping Ψ of the problems p associated with G into the problems associated with G' such that p is of the same many-one degree as $\Psi(p)$. G and G' are said to be many-one equivalent if each is many-one reducible to the other. Every r.e. many-one degree of unsolvability is said to be represented by G' if the general decision problem for r.e. sets is many-one reducible to G'.

Part 1: In this part we shall show that the general halting problem for Turing machines, denoted $\mathcal{M}_H$, is many-one reducible to the general decision problem for Thue systems with axiom, denoted $\mathcal{T}_A$.

Let M be an arbitrary Turing machine with alphabet $\Sigma = \{a_1, \ldots, a_n\}$, state set $Q = \{q_1, \ldots, q_p\}$ and Turing table Z. Let q_H, q'_H, q''_H and h be new symbols. We define a Thue system T with axiom hq_Hh, denoted T_{hq_Hh}. T will have alphabet $\Sigma(T) = \Sigma \cup \{a_0\} \cup Q \cup \{q_H, q'_H, q''_H, h\}$ and the productions of T are as follows where $\alpha \longleftrightarrow \beta$ means $\alpha \to \beta$ and $\beta \to \alpha$ are both productions.

1. $q_ia_j \longleftrightarrow q_va_k$, whenever $q_ia_ja_kq_v \in Z$
2. $hq_ia_0h \longleftrightarrow hq_va_0h$, whenever $q_ia_0\mathcal{L}q_v \in Z$
3. $hq_ia_0a_k \longleftrightarrow hq_va_0a_0a_k$, for all k, $0 \le k \le n$, whenever $q_ia_0\mathcal{L}q_v \in Z$
4. $ha_kq_ia_0h \longleftrightarrow hq_va_kh$, for all k, $1 \le k \le n$, whenever $q_ia_0\mathcal{L}q_v \in Z$
5. $a_ja_kq_ia_0h \longleftrightarrow a_jq_va_kh$, for all pairs (k, j), $0 \le k \le n$, $0 \le j \le n$, whenever $q_ia_0\mathcal{L}q_v \in Z$
6. $ha_kq_ia_0a_j \longleftrightarrow hq_va_ka_0a_j$, for all pairs (k, j), $1 \le k \le n$, $0 \le j \le n$, whenever $q_ia_0\mathcal{L}q_v \in Z$
7. $a_ma_kq_ia_0a_j \longleftrightarrow a_mq_va_ka_0a_j$, for all triples (k, m, j), $0 \le k \le n$, $0 \le m \le n$, $0 \le j \le n$, whenever $q_ia_0\mathcal{L}q_v \in Z$
8. $hq_ia_j \longleftrightarrow hq_va_0a_j$, whenever $q_ia_j\mathcal{L}q_v \in Z$, with $j \ne 0$
9. $ha_kq_ia_j \longleftrightarrow hq_va_ka_j$, for all k, $1 \le k \le n$, whenever $q_ia_j\mathcal{L}q_v \in Z$, with $j \ne 0$
10. $a_ma_kq_ia_j \longleftrightarrow a_mq_va_ka_j$, for all pairs (m, k), $0 \le m \le n$, $0 \le k \le n$, whenever $q_ia_j\mathcal{L}q_v \in Z$ with $j \ne 0$
11. $hq_ia_0h \longleftrightarrow hq_va_0h$, whenever $q_ia_0\mathcal{R}q_v \in Z$
12. $hq_ia_0a_k \longleftrightarrow hq_va_k$, for all k, $1 \le k \le n$, whenever $q_ia_0\mathcal{R}q_v \in Z$
13. $hq_ia_0a_0a_k \longleftrightarrow hq_va_0a_k$, for all k, $0 \le k \le n$, whenever $q_ia_0\mathcal{R}q_v \in Z$
14. $a_kq_ia_jh \longleftrightarrow a_ka_jq_va_0h$, for all k, $0 \le k \le n$, whenever $q_ia_j\mathcal{R}q_v \in Z$
15. $a_mq_ia_ja_0a_k \longleftrightarrow a_ma_jq_va_0a_k$, for all pairs (m, k), $0 \le m \le n$, $0 \le k \le n$, whenever $q_ia_j\mathcal{R}q_v \in Z$
16. $a_mq_ia_ja_k \longleftrightarrow a_ma_jq_va_k$, for all pairs (m, k), $0 \le m \le n$, $1 \le k \le n$, whenevery $q_ia_j\mathcal{R}q_v \in Z$
17. $hq_ia_jh \longleftrightarrow ha_jq_va_0h$, whenever $q_ia_j\mathcal{R}q_v \in Z$, with $j \ne 0$
18. $hq_ia_ja_0a_k \longleftrightarrow ha_jq_va_0a_k$, for all k, $0 \le k \le n$, whenever $q_ia_j\mathcal{R}q_v \in Z$, with $j \ne 0$
19. $hq_ia_ja_k \longleftrightarrow ha_jq_va_k$, for all k, $1 \le k \le n$, whenever $q_ia_j\mathcal{R}q_v \in Z$ with $j \ne 0$
20. $q_ia_j \longleftrightarrow q''_Ha_j$, whenever $1 \le i \le p$, $0 \le j \le n$ and q_ia_j is a terminal discriminant
21. $q''_Ha_jh \longleftrightarrow q_Hh$, for all j, $0 \le j \le n$
22. $q''_Ha_ja_k \longleftrightarrow q'_Ha_k$, for all pairs (j, k), $0 \le j \le n$, $0 \le k \le n$
23. $q'_Ha_ja_k \longleftrightarrow q'_Ha_k$, for all pairs (j, k), $0 \le j \le n$, $0 \le k \le n$
24. $q'_Ha_jh \longleftrightarrow q_Hh$, for all j, $1 \le j \le n$
25. $a_ja_kq_H \longleftrightarrow a_jq_H$, for all pairs (j, k), $0 \le j \le n$, $0 \le k \le n$
26. $ha_jq_H \longleftrightarrow hq_H$, for all j, $1 \le j \le n$

We shall first show that the decision problem for T_{hq_Hh} is of the same many-one degree as the halting problem for M. As an initial step in this proof, we shall investigate the properties of the semi-Thue system S with alphabet $\Sigma(T)$ and productions comprised of the above 26 rule sets of T without inverses.

Lemma 1. *Define a normal word over $\Sigma(T)$ to be any word of the form hCh where C contains no h and exactly one occurrence of a symbol from $Q \cup \{q_H, q'_H, q''_H\}$. Let W be an arbitrary normal word. Then W_1 and W_2 are immediate successors of W in S only if $W_1 \equiv W_2$ and W_1 is a normal word. That is, either W is terminal in S or W has an unique immediate successor. Using this we have, by transitivity, that if W_1 and W_2 are both derivable from W via n applications of productions of S, then $W_1 \equiv W_2$ and W_1 is normal.*

Proof: Let W be an arbitrary normal word. If W is terminal, then the lemma is trivially true. If W contains a q_H, q'_H or q''_H, then observation of production sets 21-26 shows that at most one rule may apply to W and it can be easily verified that if one of these is applied to a normal word then the resultant word must be both unique and normal. If W is not terminal and does not have an occurrence of q_H, q'_H or q''_H, then it must have a subword $q_i a_j$ for $1 \leq i \leq p$ and $0 \leq j \leq n$. Assume W is of this form, then observation of rule sets 1-20 will verify the lemma.

Lemma 2. *Let C and C' be arbitrary configurations of M, then $M(C, C')$ if and only if $S(hCh, hC'h)$, and hence, $M[C, C']$ if and only if $S[hCh, hC'h]$. In addition, C is mortal in M if and only if $S[hCh, hq_Hh]$.*

Proof: Observation of the productions of S will show this to be true.

As a second step in the proof, we shall now investigate the properties of the semi-Thue system $\overline{S}$ whose alphabet is $\Sigma(T)$ and whose productions are exactly the inverses of those of S.

Lemma 3. *The decision problem for $\overline{S}_{hq_Hh}$ is of the same many-one degree as the halting problem for M.*

Proof: Let W be an arbitrary word over $\Sigma(T)$. Then observation of the productions of $\overline{S}$ will show that $\overline{S}[hq_Hh, W]$ if and only if W is of one of the following forms:

(i) hXq_Hh where X is a word over $(\Sigma \cup \{a_0\})$, the initial symbol of which is not a_0.

(ii) hXq'_HYh such that X and Y are words over $(\Sigma \cup \{a_0\})$ where the initial symbol of X is not a_0, Y is not the empty word and the final symbol of Y is not a_0.

(iii) hXq''_HYh such that X and Y are words over $(\Sigma \cup \{a_0\})$ where the initial symbol of X is not a_0, Y is not the empty word and the final symbol of Y is a_0 only if $Y \equiv a_0$.

(iv) hCh for C a mortal configuration of M.

But then the decision problem for $\overline{S}_{hq_Hh}$ many-one reduces to the halting problem for M, since it is decidable if a word is an instance of (i), (ii), or (iii). Finally, let C be an arbitrary configuration of M. Then, by (iv), C is seen to be mortal if and only if $\overline{S}[hq_Hh, hCh]$. Hence, the lemma is proven.

The following lemma which is essentially due to Post [6] shows that the decision problem for T_{hq_Hh} is of the same many-one degree as that for $\overline{S}_{hq_Hh}$.

Lemma 4. *For an arbitrary word W over $\Sigma(T)$, $T[hq_Hh, W]$ if and only if $\overline{S}[hq_Hh, W]$.*

Proof: If $\overline{S}[hq_Hh, W]$, then clearly $T[hq_Hh, W]$. Suppose $T[hq_Hh, W]$. Let $hq_Hh \equiv W_1$, $W \equiv W_u$, and let $T(W_1, W_2)$, $T(W_2, W_3)$, . . ., $T(W_{u-1}, W_u)$ be the shortest derivation of W in T. Then each W_j, $2 \leq j \leq u$, is a result of W_{j-1} with respect to one of the rules of S or $\overline{S}$. If only rules of $\overline{S}$ were used, we would be through. Hence, we may assume that for some j, $2 \leq j \leq u$, W_j is the result of a rule σ of S applied to W_{j-1} and each of the previous steps involves rules from $\overline{S}$. Now W_{j-1} is not hq_Hh since none of the rules of S is applicable to hq_Hh. Hence, W_{j-1} is the result of applying a rule σ' of $\overline{S}$ to W_{j-2}. But then W_{j-2} is the result of applying the inverse of σ', call it $\overline{\sigma}'$, to W_{j-1}. This shows that W_{j-2} and W_j are the result of applying rules $\overline{\sigma}'$, and σ of S, respectively, to W_{j-1}. Now, since W_{j-1} is a normal word, there is only one word X, such that $S(W_{j-1}, X)$. Hence, $W_{j-2} \equiv W_j$ and we have that $T(W_1, W_2)$, . . ., $T(W_{j-2}, W_{j+1})$, . . ., $T(W_{u-1}, W_u)$ is a shorter derivation of W. But this is a contradiction. Therefore, $\overline{S}[hq_Hh, W]$.

Lemma 5. *The decision problem for T_{hq_Hh} is of the same many-one degree as the halting problem for M.*

Proof: Lemmas 3 and 4.

Theorem 1. *$\mathcal{M}_H$ is many-one reducible to $\mathcal{T}_A$.*

Proof: Immediate from Lemma 5.

Part 2: In this section we shall complete the necessary steps to show the many-one equivalence of $\mathcal{M}_H$, $\mathcal{T}_A$, $\mathcal{S}_A$ (the general decision problem for semi-Thue systems with axiom), and $\mathcal{N}_A$ (the general decision problem for Post normal systems with axiom).

Lemma 6. *$\mathcal{T}_A$ is many-one reducible to $\mathcal{S}_A$.*

Proof: This follows directly from the fact that every Thue system is a semi-Thue system.

Lemma 7. *$\mathcal{S}_A$ is many-one reducible to $\mathcal{N}_A$.*

Proof: Let $S = (\Sigma, R)$ be an arbitrary semi-Thue system and let A be a word over Σ. Define the Post normal system $N = (\Sigma(N), R(N))$ as follows:

$\Sigma(N) = \Sigma \cup \{h\}$, where h is a symbol not contained in Σ.

$R(N)$ is comprised of the productions: $h \rightarrow h$, $a \rightarrow a$, for every $a \in \Sigma$, and $\alpha \rightarrow \beta$, whenever $\alpha \rightarrow \beta$ is a member of R.

The proof may now be completed by showing that the decision problem for S_A is of the same many-one degree as that for N_{hA}.

Let W be an arbitrary word over Σ and let W' be an arbitrary word over $\Sigma(N)$. Observation of $R(N)$ shows that $S[A, W]$ if and only if $N[hA, hW]$. Now, clearly $N[hA, W']$ only if W' contains exactly one occurrence of the symbol h. Assume W' is of this form. Then $W' \equiv W_1hW_2$, for some words W_1 and W_2 over Σ, and $N[hA, W']$ if and only if $S[A, W_2W_1]$.

Lemma 8. *$\mathcal{N}_A$ is many-one reducible to $\mathcal{M}_H$.*

Proof: This may be shown by a series of reductions. First, observation of proofs presented by Cudia and Singletary [2] shows that $\mathcal{N}_A$ is many-one reducible to the general decision problem for r.e. sets. And finally, Overbeek [5] has demonstrated that the general decision for r.e. sets is many-one reducible to $\mathcal{M}_H$.

Theorem 2. *$\mathcal{M}_H$, $\mathcal{T}_A$, $\mathcal{S}_A$ and $\mathcal{N}_A$ are many-one equivalent.*

Proof: Follows from Theorem 1 and Lemmas 6, 7, and 8.

Part 3: As a final step we now show that every r.e. many-one degree is represented by each of $\mathcal{T}_A$, $\mathcal{S}_A$ and $\mathcal{N}_A$ and further that this result is best possible with regard to degree representation.

Lemma 9. *Every r.e. many-one degree is represented by $\mathcal{M}_H$.*

Proof: This has been shown by Overbeek [5].

Theorem 3. *Every r.e. many-one degree is represented by $\mathcal{T}_A$, $\mathcal{S}_A$ and $\mathcal{N}_A$.*

Proof: Immediate from Theorem 2 and Lemma 9.

Lemma 10. *No instance of $\mathcal{T}_A$, $\mathcal{S}_A$ or $\mathcal{N}_A$ is of the same many-one degree as an r.e. simple set.*

Proof: Let P_A be a Thue, semi-Thue or Post normal system with axiom whose decision problem is unsolvable. Then there must exist a word W_0 over the alphabet of P such that A does not derive W_0 and $\{X | P[X, W_0]\}$ is infinite. For assume that is not so, then the decision problem for P_A is solvable by the following algorithm: Let W be an arbitrary word over the alphabet of P. Using the productions of P, generate the set of words which may derive W until either A is encountered or all members of the set have been listed. This listing procedure is carried out in the following manner. At stage 0, list W. At stage $n+1$, list all immediate predecessors of words listed at stage n. That is, all words W_1, such that $P(W_1, W_2)$ and W_2 was listed at stage n. Clearly, by our assumption, A must eventually be listed if the set of words which derive W is infinite. Hence, this procedure is finite and $P[A, W]$ if and only if A is listed.

Now, $\{X | P[X, W_0]\}$ is an infinite r.e. set in the complement of the set of words derivable from A. Thus, any set of the same one-one degree as the decision problem for P_A must be non-simple.

Theorem 4. *Not every r.e. one-one degree is represented by $\mathcal{T}_A$, $\mathcal{S}_A$ or $\mathcal{N}_A$.*

Proof: This is an immediate consequence of Lemma 10.

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