Notre Dame Journal of Formal Logic
Volume XIV, Number 2, April 1973
NDJFAM

ON AN ALLEGED CONTRADICTION LURKING IN FREGE'S *BEGRIFFSSCHRIFT*

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Jean van Heijenoort, in his introduction [1] to Bauer-Mengelberg's translation of Frege's *Begriffsschrift* [2], claims to see a contradiction lurking in the logical system of that work:*

> Frege allows a functional letter to occur in a quantifier. . . . The result is that the difference between function and argument is blurred . . . in the derivation of formula (77) he substitutes $\mathfrak{F}$ [a quantificationally bound function letter] for $\mathfrak{a}$ [a quantificationally bound individual variable] in $f(\mathfrak{a})$, at least as an intermediate step. If we also observe that in the derivation of formula (91) he substitutes $\mathfrak{F}$ for f [a "free" function letter], we see that he is on the brink of a paradox. He will fall into the abyss when (1891) he introduces the course-of-values of a function as something "complete in itself," which may be taken as an argument.[1]

Van Heijenoort is mistaken in supposing that any paradox can arise from the derivations he cites in the *Begriffsschrift*. In that early work, Frege is pioneering the development of quantificational logic. While he does not yet have all the machinery or the terminology to precisely spell out the distinction between what he would later call "first-level" and "second-level" functions, he never confuses the two. And because his functions occur in "levels," Frege's functional calculus (including that in the *Begriffsschrift* of 1879) is free of the kind of paradox which, beginning in 1891,[2] does afflict his set theory. Frege himself points this out in a letter to Russell in June of 1902 [4] when responding to Russell's letter [5] about the discovery of a paradox.

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*I am indebted to Peter Geach for helpful discussions and some of the points raised in this paper.

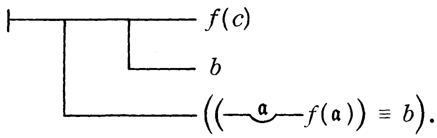
1. See [1], p. 3.

2. In that year, in [3] Frege first introduced the notion of the "course-of-values" {Wertverlauf} of a function.

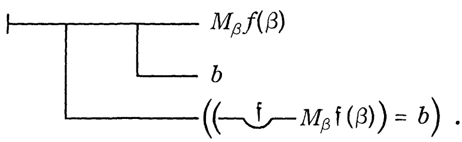
Received August 13, 1971

Consider now the "suspicious" derivations that van Heijenoort mentions:

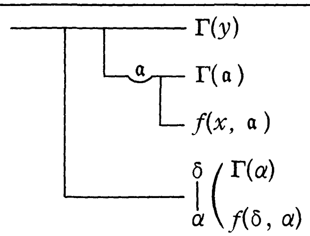
In the first one, while proving formula (77), Frege cites the following principle (68):

68 

Rather than this principle, he actually needs—but has not yet developed the machinery to express—an analogous second-order principle (call it 68′) involving quantification over functions. In the later notation of the *Grundgesetze* [6] (ignoring the, for this purpose, irrelevant switch from '≡' to '=') it would look like this:

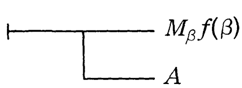
68′ 

The appropriate substitution table (placed horizontally for convenience) would then run as follows:

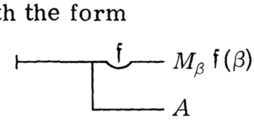
$\mathfrak{f}$	$M_\beta \Gamma(\beta)$	b	f
$\mathfrak{F}$		γ $\sim f(x_\gamma, y_\beta)$ β	F

These substitutions in formula (68′), and the detachment of the definitionally true equivalence (76) in the *Begriffsschrift*, yields formula (77) with flawless correctness.

Similarly, the second "suspicious" derivation—that of formula (91)—requires a second-order principle (a confinement rule) involving quantification over functions. Such a principle would be: Given a formula with the form



we can derive a formula with the form



if 'A' is an expression in which f does not occur and if f stands only in the argument places of $M_\beta f(\beta)$.

The "substitution" of $\mathfrak{F}$ for f in the derivation of formula (91) referred to by van Heijenoort is actually the two-step procedure of applying such a second-order confinement and then substitution $\mathfrak{F}$ for $\mathfrak{f}$. In a footnote to that derivation, Frege himself calls attention to his use of a confinement rule, but van Heijennort mistakenly interprets the footnote as a mere acknowledgement of quantifying over functions.[3]

Thus, at the time he wrote the *Begriffsschrift,* Frege was not yet able to express some needed second-order principles for the derivations which van Heijenoort mentions. Nevertheless, there is no contradiction lurking in them, and machinery that Frege would later develop can easily clear up any difficulties.

REFERENCES

[1] van Heijenoort, Jean, "Editor's introduction," in G. Frege, *Begriffsschrift,* (trans. S. Bauer-Mengelberg), in J. van Heijenoort, ed., *From Frege to Gödel,* Harvard University Press, Cambridge, Massachusetts (1967).

[2] Frege, G., *Begriffsschrift,* Nebert, Halle (1879).

[3] Frege, G., *Funktion und Begriff,* Pohle, Jena (1891).

[4] Frege, G., "Letter to Russell," in van Heijenoort, ed., *op. cit.*, pp. 127-128.

[5] Russell, B., "Letter to Frege," *ibid*., pp. 123-124.

[6] Frege, G., *Grundgesetze der Arithmetik,* Band I, Pohle, Jena (1893).

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3. See van Heijenoort's addition to Frege's footnote to the derivation of formula (91); [1], p. 66.