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LOCAL RECURSIVE THEORY

VLADETA VUČKOVIĆ

1 *Introduction.* The purpose of this paper is to outline a generalization of the recursive theory, which is quite different from existing generalizations. Instead of an axiomatic treatment as an ultimate goal, we proceed in the opposite direction, considering sets which are, as far as recursive notions are in question, well behaved only locally. Our hope is that such a point of view will not end in an imitation of the recursive theory and that it will produce new, interesting and non-trivial problems and results.

In order to give a definite picture of such a *Local Recursive Theory* we do not present the most general case possible. Already in this case the number of problems which arise is overwhelming.

Methodically, local recursive theory is a development of Malcev's general theory of enumerations. However, in this paper we use only the simplest enumerations, *indexings*, i.e., bijective maps $\alpha : N \to U_\alpha$, where N is the set of non-negative integers and U_α a denumerable set.

2 *Recursive Manifolds.* If $\alpha : N \to U_\alpha$ is an indexing, we can identify U_α with N and pursue the recursive theory on U_α in a trivial way. However, if $\mathfrak{A}$ is a family of indexings $\alpha : N \to U_\alpha$ and $M = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$, the introduction of recursive notions on M, by use of sets U_α, becomes a problem whose outcome is not obvious.

Definition 2.1. A set M is called a *recursive manifold* (an RM) iff:

(i) There is a family $\mathfrak{A}$ (an *atlas* on M) of indexings $\alpha : N \to U_\alpha$, where each U_α is a subset of M (a *local neighborhood*), such that $M = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$;

(ii) For all pairs $\langle \alpha, \beta \rangle \in \mathfrak{A}^2$, the numerical map $\alpha^{-1} \circ \beta$ is a partial recursive function with recursive domain (inclusive $\emptyset$, the empty set, as a possible domain).

Example 2.1. Let M be an infinite set and $\alpha : N \to U$ an indexing of a subset U of M. If $M = U$, M is an RM with atlas $\{\alpha\}$. If $M \neq U$, to every $x \in M - U$ correspond the local neighborhood $U_x = U \cup \{x\}$ and the indexing $\alpha_x : N \to U_x$, defined by

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$$\alpha_x(i) = \begin{cases} x \text{ for } i = 0, \\ \alpha(i-1) \text{ for } i \geq 1. \end{cases}$$

Then M becomes an RM, with atlas $\mathfrak{A} = \{\alpha_x \mid x \in M - U\}$. Remark that every $\alpha_x^{-1} \circ \alpha_y$, for $x \neq y$, is the identity on its domain (which is $N^+ = N - \{0\}$). This recursive manifold will be called the *trivial* RM on M.

An atlas $\mathfrak{B}$ on M will be called *maximal* iff, for every atlas $\mathfrak{A}$ on M, the inclusion $\mathfrak{B} \subset \mathfrak{A}$ implies the equality $\mathfrak{B} = \mathfrak{A}$. By Zorn Lemma,[1] to every atlas $\mathfrak{A}$ on M corresponds a maximal atlas $\mathfrak{B} \supset \mathfrak{A}$; in some particular cases, we are able to construct a maximal atlas containing a given atlas.

Example 2.2. Let $M = N$ and let $I : N \to N$ be the identity of N. Then N is an RM with atlas $\{I\}$. The maximal atlas $\mathfrak{B} \supset \{I\}$ consists of all recursive functions $\beta : N \to N$, which are injective and have recursive ranges.

Example 2.3. Let $\langle \alpha_i \rangle_{i \in N}$ be any sequence of increasing recursive functions. Denote by U_i the range of α_i, and let $M = \bigcup_{i=0}^{\infty} U_i$. Then M is an RM with the atlas $\mathfrak{A} = \{\alpha_i \mid i \in N\}$.

Definition 2.2. Let M be an RM with atlas $\mathfrak{A}$ and let M_1 be an RM with atlas $\mathfrak{A}_1$. Then:

(i) A set $X \subset M$ is *recursively enumerable* (*recursive*) iff, for every $\alpha \in \mathfrak{A}$, the set $\alpha^{-1}(X)$ is a recursively enumerable (recursive) subset of N.
(ii) Let $X \subset M$. A map $f : X \to M_1$ is *partial recursive* iff X is a r.e. (recursively enumerable) set and, for all pairs $\langle \alpha, \alpha_1 \rangle \in \mathfrak{A} \times \mathfrak{A}_1$, the numerical map $f_{\alpha,\alpha_1} = \alpha_1^{-1} \circ f \circ \alpha$ is a partial recursive function. If f is p.r. (partial recursive) and total it is called *recursive*.

In considering *functionals*, i.e., maps $f : X \to N$, $X \subset M$, and *anti-functionals*, i.e., maps $f : D \to M$, $D \subset N$, we shall agree to consider N as an RM with the atlas $\{I\}$, where I is the identity on N.

It is obvious how Definition 2.2 can be generalized to subsets of M^p and to maps $f : X \to M_1$, $X \subset M^p$. In (i), instead of $\alpha^{-1}(X)$ one has to consider all sets

$$X^{-1}_{\alpha_1, \ldots, \alpha_p} = \{\langle \alpha_1^{-1}(x_1), \ldots, \alpha_p^{-1}(x_p) \rangle \mid \langle x_1, \ldots, x_p \rangle \in X\},$$

and in (ii) one has to consider all numerical maps

$$f_{\alpha_1, \ldots, \alpha_p, \beta}(n_1, \ldots, n_p) = \beta^{-1}(f(\alpha_1(n_1), \ldots, \alpha_p(n_p))),$$

for all $\langle \alpha_1, \ldots, \alpha_p, \beta \rangle \in \mathfrak{A}^p \times \mathfrak{A}_1$.

In this paper, M will be a fixed recursive manifold with atlas $\mathfrak{A}$. In general, $\langle M, \mathfrak{A} \rangle$ denotes that M is a recursive manifold with the atlas $\mathfrak{A}$. One should remark that, under Definition 2.2, all local neighborhoods U_α are recursive sets, that every $\alpha : N \to M$, as a map of N into M, is a

1. This remark is due to Professor T. H. Payne, University of California, Riverside.

recursive anti-functional, and that every $\alpha^{-1}: U_\alpha \to N$, as a map from M onto N, is a partial recursive functional with recursive domain. As an application of this remark we shall prove a theorem on atlases. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two atlases on the set M. We say that they are *weakly equivalent* iff they induce the same "r.e.," "p.r." and "recursive" notions for sets and functionals in $\langle M, \mathfrak{A}\rangle$ and $\langle M, \mathfrak{B}\rangle$.

Theorem 2.1. *Two atlases $\mathfrak{A}$ and $\mathfrak{B}$ on a set M are weakly equivalent iff their union $\mathfrak{C} = \mathfrak{A} \cup \mathfrak{B}$ is an atlas on M, which is weakly equivalent with both $\mathfrak{A}$ and $\mathfrak{B}$.*

Proof. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are weakly equivalent. Take any $\alpha \in \mathfrak{A}$. By equivalence and by the remark above, α^{-1} is a p.r. function with recursive domain in $\langle M, \mathfrak{B}\rangle$. Thus, for every $\beta \in \mathfrak{B}$, $\alpha^{-1} \circ \beta$ is a p.r. function with recursive domain. Similarly, every $\beta^{-1} \circ \alpha$, $\beta \in \mathfrak{B}$, $\alpha \in \mathfrak{A}$, is a p.r. function with recursive domain. This proves that $\mathfrak{C}$ is an atlas on M. Obviously it is weakly equivalent with both $\mathfrak{A}$ and $\mathfrak{B}$. Converse trivial.

Let us point that in the definition of the weak equivalence we cannot omit the condition on functionals. To see this, consider any denumerable set M and two recursive manifolds $\langle M, \{\alpha\}\rangle$ and $\langle M, \{\beta\}\rangle$ whose atlases are singletones. By Theorem 2.1 $\{\alpha\}$ and $\{\beta\}$ are weakly equivalent on M iff there is a recursive permutation $p: N \to N$ such that $\beta = \alpha \circ p$. By a theorem of Kent there exists a non-recursive permutation $f: N \to N$, such that, for every r.e. set $E \subset N$, both $f(E)$ and $f^{-1}(E)$ are r.e. Thus, if $\beta: N \to M$ is defined by $\beta = \alpha \circ f$, β and α induce the same notions "r.e." and "recursive" for subsets of M. However, for functionals this is not true. Define $F: M \to N$ by $F(\beta(n)) = f(n)$. F is not recursive in $\langle M, \{\beta\}\rangle$. However, since $F(\alpha(n)) = F(\beta(f^{-1}(n))) = n$, F is recursive in $\langle M, \{\alpha\}\rangle$, i.e., $\{\alpha\}$ and $\{\beta\}$ are not weakly equivalent.

Definition 2.3. Let A be a subset of M. χ_A, the *characteristic functional of* A, is the map $\chi_A: M \to N$, defined by

$$\chi_A(x) = \begin{cases} 0 \text{ if } x \in A, \\ 1 \text{ if } x \in \complement A = M - A. \end{cases}$$

Theorem 2.2.(i) *A set $A \subset M$ is recursive iff both A and $\complement A$ are r.e. sets.*

(ii) *A set $A \subset M$ is recursive iff χ_A is recursive.*

Proof. (i) If A is recursive, then all $\alpha^{-1}(A)$, $\alpha \in \mathfrak{A}$, are recursive. Then all $\alpha^{-1}(\complement A) = \complement\alpha^{-1}(A) = N - \alpha^{-1}(A)$ are r.e. Converse similar. (ii) Remark that $\chi_A \circ \alpha$ is the characteristic function of $\alpha^{-1}(A)$.

Theorem 2.2 should not suggest that local recursive theory will be a verbal counterpart of the classical recursive theory. Already the problem of enumeration of all r.e. subsets of a recursive manifold presents some new and unpleasant aspects. (However, as it will be shown in section **4**, a very general class of recursive manifolds has very definite enumerations of all r.e. subsets.)

Let $w_i = \left\{ n \in N \,\middle|\, \bigvee_y T_1(i, n, y) \right\}$ be the i-th r.e. subset of N, in the Kleene enumeration of such subsets. For every $\alpha \in \mathfrak{A}$ let $w_i^\alpha = \alpha(w_i)$. Consider, for every $\varphi \in N^{\mathfrak{A}}$ (the family of all maps of $\mathfrak{A}$ into N), the set

$$(2.1) \qquad w_\varphi = \bigcup_{\alpha \in \mathfrak{A}} w^\alpha_{\varphi(\alpha)} .$$

One cannot conclude that w_φ is a r.e. subset of M, since, for every $\beta \in \mathfrak{A}$,

$$(2.2) \qquad \beta^{-1}(w_\varphi) = w_{\varphi(\beta)} \cup \bigcup_{\substack{\alpha \in \mathfrak{A} \\ \alpha \neq \beta}} \beta^{-1} \circ \alpha \, (w_{\varphi(\alpha)})$$

is not necessarily a r.e. subset of N. However, if $A \subset M$ is a r.e. set then, choosing a $\varphi \in N^{\mathfrak{A}}$ such that $\alpha^{-1}(A) = w_{\varphi(\alpha)}$, we obtain

Theorem 2.3. *For every* r.e. *set* $A \subset M$ *there is a* $\varphi \in N^{\mathfrak{A}}$ *such that* $A = w_\varphi$, *where* w_φ *is as in* (2.1).

Similarily, if

$$w_i^{(p)} = \left\{ \langle n_1, \ldots, n_p \rangle \in N^p \,\middle|\, \bigvee_y T_p(i, n_1, \ldots, n_p, y) \right\},$$

and, for $\langle \alpha_1, \ldots, \alpha_p \rangle \in A^p$,

$$w_i^{\alpha_1, \ldots, \alpha_p} = \{ \langle \alpha_1(n_1), \ldots, \alpha_p(n_p) \rangle \mid \langle n_1, \ldots, n_p \rangle \in w_i^{(p)} \},$$

for every r.e. *set* $A \subset M^p$ *then there is a* $\varphi \in N^{\mathfrak{A}^p}$ *such that* $A = w_\varphi^{(p)}$, *where*

$$(2.3) \qquad w_\varphi^{(p)} = \bigcup_{\langle \alpha_1, \ldots, \alpha_p \rangle \in \mathfrak{A}^p} w^{\alpha_1, \ldots, \alpha_p}_{\varphi(\alpha_1, \ldots, \alpha_p)} .$$

As we have pointed out there exist special recursive manifolds in which one can enumerate all r.e. sets. (See section **4**.)

Theorem 2.4. *There exist injective and recursive numerical functions* h *and* g *such that, for all* $\varphi, \psi \in N^{\mathfrak{A}}$ *for which* w_φ *and* w_ψ *are* r.e. *sets,*

$$(2.4) \qquad w_\varphi \cup w_\psi = w_{h(\varphi, \psi)} \text{ and } w_\varphi \cap w_\psi = w_{g(\varphi, \psi)} .$$

Proof. By an application of the classical Iteration Theorem from "above."

However, we are unable to establish a satisfactory form of the Iteration Theorem for r.e. subsets of M. Since the identity I_M on M is a recursive map, we obtain easily

Theorem 2.5. (i) *A set* $X \subset M$ *is* r.e. *iff it is the domain of a* p.r. *map from* M *into* M.

(ii) *Every* r.e. *set* $X \subset M$ *is the range of a* p.r. *map from* M *into* M. (The map in question is $I_X = I_M | X$.)

Remark that ranges of p.r. maps $f : X \to M$, $X \subset M$, where X is r.e., are not necessarily r.e. subsets of M. We have

$$(2.5) \qquad f(X) = \bigcup_{\alpha \in A} f(X \cap U_\alpha),$$

where each $f(X \cap U_\alpha)$ is a r.e. subset of M; however, this does not imply that the union in (2.5) is a r.e. subset of M. It is important, therefore, to point that the Graph Theorem holds.

Theorem 2.6. (The Graph Theorem) *A partial map $f: X \to M$, where X is a* r.e. *subset of M, is partial recursive iff its graph G_f is a* r.e. *subset of M^2.*

Proof. Let f be partial recursive. For $\alpha, \beta \in \mathfrak{A}$ consider the set

$$\begin{aligned}(G_f^{-1})_{\alpha,\beta} &= \{\langle n, m\rangle \mid \langle \alpha(n), \beta(m)\rangle \in G_f\}\\ &= \{\langle n, m\rangle \mid m = \beta^{-1}\circ f\circ\alpha\ (n)\}.\end{aligned}$$

Since $\beta^{-1}\circ f\circ\alpha$ is a p.r. numerical function, it follows that each $(G_f^{-1})_{\alpha,\beta}$ is a r.e. subset of N, i.e., that G_f is a r.e. subset of M^2. Conversely, if G_f is r.e., each $(G_f^{-1})_{\alpha,\beta}$ is a r.e. subset of N and so each $\beta^{-1}\circ f\circ\alpha$ is a p.r. function, since $m = \beta^{-1}\circ f\circ\alpha\ (n) \longleftrightarrow \langle n, m\rangle \in (G_f^{-1})_{\alpha,\beta}$.

Theorem 2.7. (i) *Let $f: M \to M$ be a recursive map and let $g: M \to N$ be a recursive functional. Then the composition $g\circ f: M \to N$ is a recursive functional.*
(ii) *Let $h: N^p \to N$ be recursive and let each functional $f_i: M \to N$ be recursive. Then $g: M \to N$, defined by*

$$g(x) = h(f_1(x), \ldots, f_p(x)),$$

is a recursive functional.

We conclude this section with two examples.

Example 2.4. Let S be a denumerable set and let $\hat{S}$ be the semigroup generated by S, with concatenation as the semigroup operation. Let $\alpha: N \to S$ be an indexing of S. We shall use the existence of effective codings to introduce a corresponding indexing $\hat{\alpha}: N \to \hat{S}$. This existence can be formulated as follows.

Lemma 2.1. *There exists a bijective map $\tau: N \to \bigcup_{p=1} N^p$ with following properties:*

(i) *There is a recursive numerical function l, such that for all $n \in N$ and $p \geq 1$ $\tau(n) \in N^p \longleftrightarrow l(n) = p$.*
(ii) *There exists a recursive $\sigma: N^2 \to N$ such that, for all $n \in N$*

$$\tau(n) = \langle \sigma(1,n), \sigma(2,n), \ldots, \sigma(l(n),n)\rangle .$$

With notations of this lemma define $\hat{\alpha}: N \to \hat{S}$ by

$$\hat{\alpha}(n) = \alpha(\sigma(1,n))\,\alpha(\sigma(2,n))\ \ldots\ \alpha(\sigma(l(n),n)). \tag{2.6}$$

Remark that, by Theorem 2.1, two singletone atlases $\{\alpha\}$ and $\{\beta\}$ on a denumerable set M are equivalent iff $\beta^{-1}\circ\alpha$ is a recursive permutation. We can use this to prove

Theorem 2.9. *Let $\{\alpha\}$ and $\{\beta\}$ be two atlases on S and let $\{\hat{\alpha}\}$ and $\{\hat{\beta}\}$ be the corresponding atlases on $\hat{S}$, defined by* (2.6). *Then $\{\hat{\alpha}\}$ and $\{\hat{\beta}\}$ are weakly equivalent on $\hat{S}$ iff $\{\alpha\}$ and $\{\beta\}$ are weakly equivalent on S.*

Example 2.5. One could be tempted to absolutize the local recursive notions by demanding that they hold on all possible atlases. For example, let us say that a set $A \subset M$ is *absolutely* r.e. iff it is r.e. in every recursive manifold $\langle M, \mathfrak{A}\rangle$. Such sets exist; M, $\emptyset$, all finite subsets of M and all sets whose complements are finite. However, there does not exist any set $E \subset M$, such that both E and its complement are infinite, which is absolutely recursively enumerable. To see this it is enough to consider all recursive manifolds $\langle N, \{\alpha\}\rangle$, with one-element atlases. A set $E \subset N$ (infinite, with infinite complement) will be absolutely r.e. iff, for all permutations p of N, $p(E)$ is a r.e. subset of N. Trivially, such sets do not exist.

3 *Lifting and Relative Recursiveness.* Let $\langle M, \mathfrak{A}\rangle$ be an RM. We know few recursive or partial recursive maps from M into M, or into N. The identity I_M on M and every constant map are recursive; their restrictions to r.e. subsets of M are partial recursive. And that is about all we know! This imposes the problem of *lifting* recursive and partial recursive functions (and sets, as well) from N into M. We can give only some very restrictive positive results about such liftings.

Theorem 3.1. (The Local Lifting Theorem for Functionals) *Let* $f: S \to N$, $S \subset N$, *be a* p.r. *function, and let* $\alpha_0 \in \mathfrak{A}$ *be a fixed element of the atlas* $\mathfrak{A}$ *on* M. *Define* $\hat{f}_0 : \alpha_0(S) \to N$ *by*

$$\hat{f}_0(x) = f(\alpha_0^{-1}(x)).$$

Then f_0 *is a* p.r. *functional.*

Proof. $\alpha_0(S)$ is a r.e. subset of M, since, for every $\beta \in \mathfrak{A}$, $\beta^{-1}(\alpha_0(S)) = \beta^{-1} \circ \alpha_0(S)$ is the direct image of a r.e. subset of N under a p.r. function. Also, for every $\beta \in \mathfrak{A}$, $\hat{f}_0(\beta(n)) \simeq f(\alpha_0^{-1}(\beta(n))) = (f \circ (\alpha_0^{-1} \circ \beta))(n)$, which proves that every $\hat{f}_0 \circ \beta$ is a p.r. function. Thus, f_0 is a p.r. functional.

In case in which the function f in Theorem 3.1 is recursive, the lifting f_0 has U_{α_0} as domain and it is a p.r. functional with recursive domain. Thus, it can be extended to a recursive functional $\hat{f}: M \to N$, defining it to be a constant on $M - U_{\alpha_0}$. Similarly to Theorem 3.1 one can prove

Theorem 3.2. (The Local Lifting Theorem for Maps) *Let* $f: S \to N$, $S \subset N$, *be a* p.r. *function and let* $\alpha_0 \in \mathfrak{A}$ *be a fixed element of the atlas* $\mathfrak{A}$. *Define* $\hat{f}_0 : \alpha_0(S) \to M$ *by*

$$\hat{f}_0(\alpha_0(n)) \simeq \alpha_0(f(n)).$$

Then $\hat{f}_0$ *is a* p.r. *map. In case* f *is recursive,* f_0 *can be extended to a recursive map of* M *into* M.

Local liftings are possible also for r.e. and recursive sets. However, the problem of global lifting depends essentially on the structure of the atlas $\mathfrak{A}$. Let us call an RM $\langle M, \mathfrak{A}\rangle$ *special* iff, for all $\langle \alpha, \beta\rangle \in \mathfrak{A}^2$, $\beta^{-1} \circ \alpha$ is the identity on its domain (in case this domain is non-empty).

Theorem 3.3. (Global Lifting Theorem for Special Manifolds) *Let* $\langle M, \mathfrak{A}\rangle$ *be* *a special manifold. Let E be any* r.e. *subset of N. Then* $\hat{E}$, *defined by* $\hat{E} = \bigcup_{\alpha \in \mathfrak{A}} \alpha(E)$ *is a* r.e. *subset of M.*

Theorem 3.3 shows that one can develop the Post theory of r.e. sets on special manifolds. In the following example we exhibit some frustrations inherent in such an enterprise.

Example 3.1. By our principle of localization, we have to adopt the following definition: a set $A \subset M$ is *immune* iff, for every $\alpha \in \mathfrak{A}$, the set $\alpha^{-1}(A)$ is an immune subset of N.

Let U denote the set of all odd members of N, and let $\alpha : N \to U$ be defined by $\alpha(n) = 2n + 1$. For every $i \in N$ let $U_i = \{2i\} \cup U$. Define the indexings $\alpha_i : N \to U_i$ by

$$\alpha_i(n) = \begin{cases} 2i & \text{if } n = 0 \\ \alpha(n-1) & \text{if } n \geq 1. \end{cases}$$

Then $\mathfrak{A} = \{\alpha_i | i \in N\}$ is an atlas on $M = \bigcup_{i=0}^{\infty} U_i = N$. Let H be any immune subset of N, and let $E = N - U$. Set $A = \alpha(H) \cup E$. Then, for every $i \in N$, $\alpha_i^{-1}(A) = H \cup \{0\}$ is an immune subset of N. Thus, A is an immune subset of M. Let now $A_1 = A \cup \{\mathrm{I}\}$, where I is the identity on N. The pair $\langle N, \mathfrak{A}_1 \rangle$ is anew an RM. However, the set A is no more an immune subset of N, since $\mathrm{I}^{-1}(A)$ contains all even integers.

One could try to avoid the difficulties of a localized Post theory by the introduction of *global definitions*, which, in case of the manifold $\langle N, \{\mathrm{I}\} \rangle$ will reduce to usual definitions. In this case, the questions of cardinalities and the adequate formulations of notions "finite" and "infinite" start to play a significant role. Every r.e. set in $\langle M, \mathfrak{A} \rangle$ has the form

$$w_\varphi = \bigcup_{\alpha \in \mathfrak{A}} w^{\alpha}_{\varphi(\alpha)}, \quad \varphi \in N^{\mathfrak{A}},$$

where each $w^{\alpha}_{\varphi(\alpha)}$ is an at most denumerable set. Let $\overline{\overline{X}}$ denote the cardinal number of X. If $\aleph_0 \leq \overline{\overline{\mathfrak{A}}}$ then $\overline{\overline{w_\varphi}} \leq \overline{\overline{\mathfrak{A}}}$, and if $\overline{\overline{\mathfrak{A}}} \leq \aleph_0$ then each w_φ is at most denumerable. Since each U_α is a denumerable set and $M = \bigcup_{\alpha \in \mathfrak{A}} U_\alpha$, we obtain for $\overline{\overline{M}}$ the same estimates as for $\overline{\overline{w_\varphi}}$. Thus, we can say that a w_φ is *globaly infinite* iff $\overline{\overline{w_\varphi}} = \overline{\overline{M}}$. Also, let us say that a set $A \subset M$ is *globaly immune* iff it is globaly infinite and does not contain any globaly infinite r.e. set. We can prove:

Theorem 3.4. *Let* $\langle M, \mathfrak{A} \rangle$ *have the property that* $\overline{\overline{M}} = \overline{\overline{\mathfrak{E}}}$, *where* $\mathfrak{E}$ *is the family of all globaly infinite* r.e. *subsets of M. Then there exist* $2^{\overline{\overline{M}}}$ *sets* $A \subset M$, *such that both A and* $\complement A$ *are globaly immune.*

Proof. ($\overline{\sigma}$ denotes the cardinal of the ordinal σ). Let σ be the smallest ordinal such that $\overline{\sigma} = \overline{\overline{M}}$ and such that, for every $\eta < \sigma$, $\overline{\eta} < \overline{\overline{M}}$. Well-order $\mathfrak{E}$ into an ordinal sequence $\langle w_\xi \rangle_{\xi < \sigma}$. To each $\xi < \sigma$ correspond the ordered pair $\langle x_\xi, y_\xi \rangle$ of elements of M so that:

(i) $x_\xi \neq y_\xi$, $x_\xi \in w_\xi$ and $y_\xi \in w_\xi$; and (ii) both x_ξ and y_ξ are not elements of $\bigcup_{\eta<\xi} \{x_\eta, y_\eta\}$. Let A consist of exactly one element of each of pairs $\langle x_\xi, y_\xi \rangle$. Then $\overline{\overline{A}} = \overline{\overline{CA}} = \overline{\overline{M}}$ and both A and CA do not contain any w_ξ. This choice can be done in $2^{\overline{\overline{M}}}$ different ways.

In general, non-constructive proofs of the Post theory can be modified in a global version of this theory for RM's. One can vary such global versions, introducing conditions on cardinalities of sets in question, in such a way that all global notions reduce to usual ones on $\langle N, \{I\} \rangle$.

A theory of Turing degrees can easily be developed on any RM. Let $\mathfrak{F}$ denote the family of all total functionals $f: M \to N$.

Definition 3.1. (i) A functional $f \in \mathfrak{F}$ is *recursive in* the functional $g \in \mathfrak{F}$, in symbol $f \leq_M g$, iff, for every $\alpha \in \mathfrak{A}$, $f \circ \alpha \leq_T g \circ \alpha$, i.e., $f \circ \alpha$ is recursive in $g \circ \alpha$. (ii) A set $A \subset M$ is *recursive in* the set $B \subset M$, iff χ_A is recursive in χ_B.

Obviously, $f \leq_M g$ iff there is $\varphi \in N^{\mathfrak{A}}$ such that for all $\langle \alpha, n \rangle \in \mathfrak{A} \times N$

$$f \circ \alpha(n) = U(\mu_y T_1^{g \circ \alpha}(\varphi(\alpha), n, y)). \tag{3.1}$$

Defining *M-degrees* in the obvious way, we conclude easily that each M-degree $[f]$ is either denumerable (in case $\overline{\overline{\mathfrak{A}}} \leq \aleph_0$) or satisfies $\overline{\overline{[f]}} \leq 2^{\overline{\overline{\mathfrak{A}}}}$ (in case $\aleph_0 \leq \overline{\overline{\mathfrak{A}}}$). Similar estimates are valid for the number of predecessors of each M-degree. One proves easily that $\mathfrak{U}_M$, the class of all M-degrees, is an upper semi-lattice.

The *completion* f' of an $f \in \mathfrak{F}$ can be defined as the characteristic functional of the set C_f, where, for every $\alpha \in \mathfrak{A}$,

$$\alpha^{-1}(C_f) = \left\{ n \,\middle|\, \bigvee_y T_1^{f \circ \alpha}(n, n, y) \right\}. \tag{3.2}$$

Then $[f] < [f']$. In general, all pathological examples from the theory of Turing degrees can be lifted locally to examples for M-degrees. As an example we give:

Theorem 3.5. (Friedberg-Muchnik Theorem) *There exist* r.e. *sets* $A \subset M$ *and* $B \subset M$ *such that* $A \not\leq_M B$ *and* $B \not\leq_M A$.

Proof. If X, $Y \subset N$ are the sets from the Friedberg-Muchnik theorem, any local lifting $A = \alpha_0(X)$ and $B = \alpha_0(Y)$ will satisfy our theorem.

Existence of M-degrees raises so many problems relative to Turing degrees that we cannot list them without a further and more detailed study.

4 *Locally Finite* RM's. The variety of conditions which can be imposed upon an atlas $\mathfrak{A}$ cannot be compared with the corresponding variety in topology. However, some analogies are possible. We discuss briefly one which seems to be very promising.

Definition 4.1. A recursive manifold $\langle L, \mathfrak{A} \rangle$ is *locally finite* iff, for every $\alpha \in \mathfrak{A}$, the family

(4.1) $$\mathfrak{A}_\alpha = \{\beta \in \mathfrak{A} \mid U_\alpha \cap U_\beta \neq \varphi\}$$

is finite.

Two theorems show at once the advantages in working with locally finite RM's.

Theorem 4.1. (The Enumeration Theorem) *Let* $\langle L, \mathfrak{A}\rangle$ *be a locally finite* RM. *A set* $A \subset L$ *is* r.e. *iff* $A = w_\varphi$, *for some* $\varphi \in N^{\mathfrak{A}}$, *where* w_φ *is as in* (2.1).

Theorem 4.2. *In every locally finite* RM $\langle L, \mathfrak{A}\rangle$, *a set* $A \subset L$ *is* r.e. *iff it is the range of a* p.r. *map from L into L.*

In every recursive manifold $\langle M, \mathfrak{A}\rangle$ the inverse map f^{-1} of an injective p.r. map f from M into M is a p.r. map, provided its range is a r.e. set, since, for all α, $\beta \in \mathfrak{A}$, $\beta^{-1} \circ f^{-1} \circ \alpha = (\alpha^{-1} \circ f \circ \beta)^{-1}$. Thus, we have:

Corollary 4.2.1. *In every locally finite* RM, *the inverse of a* p.r. *injective map is a* p.r. *map.*

We cannot say anything about direct or inverse images of r.e. sets under p.r. maps without further restrictions. We do this by

Definition 4.2. Let $\langle M, \mathfrak{A}\rangle$ be an RM. A map $f: X \to M$, $X \subset M$, is *compact* iff, for every $\alpha \in \mathfrak{A}$, $f^{-1}(U_\alpha)$ can be covered by finite many local neighborhoods U_β, $\beta \in \mathfrak{A}$.

On the recursive manifold $\langle N, \{I\}\rangle$, where I is the identity on N, all maps are compact.

Theorem 4.3. *Let* $\langle M, \mathfrak{A}\rangle$ *be an* RM, *and let* $f: X \to M$, $X \subset M$, *be a compact* p.r. *map. Then*:

(i) *If* $E \subset M$ *is a* r.e. *set then both* $f(E)$ *and* $f^{-1}(E)$ *are* r.e. *sets.*
(ii) *If* $g: Y \to M$, $Y \subset M$, *is any* p.r. *map, the composition* $g \circ f$ *is a* p.r. *map.*

A theory of reducibility of sets can be built in a locally finite RM, using compact recursive maps. Essentially, such a theory is a global theory and its connections with M-degrees are not clear for the moment to us.

5 *Atlases on N.* In order to exhibit possible rewards of the local recursive theory for the classical recursive theory, we shall now consider the notions "simple" and "immune" under various atlases on N. *Classical recursive theory* can be considered as the local recursive theory on $\langle N, \{I\}\rangle$, where I is the identity on N. We have pointed out that the maximal atlas $\mathfrak{C}$ on N which contains $\{I\}$ consists of all injective recursive functions with recursive ranges. Local recursive theory on $\langle N, \mathfrak{C}\rangle$ will be called *maximal*. Let $\mathfrak{M}$ be the atlas on N consisting of all increasing recursive functions. Local recursive theory on $\langle N, \mathfrak{M}\rangle$ will be called *semi-maximal*. Finally, let $\mathfrak{F}$ be the atlas on N, consisting of all increasing recursive functions whose ranges have finite complements. Local recursive theory on $\langle N, \mathfrak{F}\rangle$ will be called *Frechet*. Obviously:

$$\{I\} \subset \mathfrak{F} \subset \mathfrak{M} \subset \mathfrak{C}$$

and all four atlases are weakly equivalent on N. A set $A \subseteq N$ will be called *maximally immune* iff, for every $\alpha \in \mathfrak{C}$, $\alpha^{-1}(A)$ is classically immune, *semi-maximally immune* iff, for every $\alpha \in \mathfrak{M}$, $\alpha^{-1}(A)$ is classically immune, and *Frechet-immune* iff, for every $\alpha \in \mathfrak{F}$, $\alpha^{-1}(A)$ is classically immune. It is obvious how to define corresponding versions of "simple."

Theorem 5.1. *A set $A \subset N$ is maximally immune iff both A and is complement $\complement A$ are classically immune.*

Proof. If both A and $\complement A$ are classically immune take any $\alpha \in \mathfrak{C}$ and consider $U_\alpha \cap A$, where U_α is the range of α. If $U_\alpha \cap A$ is finite, let m_α be its maximal member. Then $\complement A$ contains the infinite recursive set $\{n \in U_\alpha | n > m_\alpha\}$. Contradiction. Thus, $U_\alpha \cap A$ is infinite and $\alpha^{-1}(A)$ is an infinite subset of N. It must be immune. Thus, A is maximally immune. Conversely, suppose that A is maximally immune. Then A is classically immune as well. If $\complement A$ (which is infinite) would contain an infinite recursive set, U_α $(\alpha \in \mathfrak{C})$ then $\alpha^{-1}(A)$ would be empty, contradicting the supposition that A is maximally immune. Therefore, both A and $\complement A$ are classically immune.

Corollary 5.1.1. *If $S \subseteq N$ is classically simple, its complement $\complement S$ cannot be maximally immune. Thus, there does not exist any maximally simple set.*

The situation is similar with semi-maximal immunity.

Theorem 5.2. *A set $A \subset N$ is Frechet immune iff it is classically immune.*

Proof. If A is classically immune take any U_α for $\alpha \in \mathfrak{F}$. *If $A \cap U_\alpha$ is finite, A is finite. Thus $\alpha^{-1}(A)$ is infinite and classically immune.* Converse obvious.

Corollary 5.2.1. *A set S is Frechet simple iff it is classically simple.*

Thus, different atlases on N appear as sieves through which fewer and fewer notions can penetrate; formerly absolute notions become relativized and we discover that on N one can have more than one recursive theory.

University of Notre Dame
Notre Dame, Indiana