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ON CONSERVING POSITIVE LOGICS

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Let L^+ be a sentential logic without negation. One frequently wishes to know which classically valid negation axioms can be added *conservatively* to L^+, in the sense that the negation-free fragment of the resulting logic L is precisely L^+. The question becomes more urgent as the strength of the axioms to be added increases, for it frequently happens that one cannot add together axioms sufficient for the full classical principles of double negation, excluded middle, and contraposition conservatively.[1]

In the present paper, we shall develop a method which will enable us to prove, for several interesting systems, that their negation-free fragments are determined by their negation-free axioms. We take as the negation axioms to be added those given by Anderson and Belnap for their system E of entailment, namely

A1. $\overline{\overline{A}} \rightarrow A$
A2. $(A \rightarrow \overline{B}) \rightarrow (B \rightarrow \overline{A})$
A3. $(A \rightarrow \overline{A}) \rightarrow \overline{A}$.

We note in passing that these axioms lead in E (and in related systems) to the theoremhood of all forms of the double negation laws, the DeMorgan laws, contraposition laws, and laws of excluded middle and non-contradiction. In short they are strong axioms, raising non-trivial questions of conservative extension.[2]

1. For example, the addition of plausible axioms expressing all these principles causes the negation-free fragment J^+ of the intuitionist sentential calculus to collapse into the classical calculus K, as is well known. Cf. [7].

2. Anderson in [1] explicitly takes note of the conservative extension question for the system E^+ determined by the negation-free axioms and rules of E. He lists this as an open problem for E significant not only in its own right but on account of relations between E and J investigated in [4]; the problem is similarly interesting for the Anderson-Belnap system R, which is even more intimately related to J. Cf. [11].

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1. I assume the sentences of a *positive logic* L^+ to be built up using binary connectives $\rightarrow$, &, v, without negation but perhaps with some other connectives or constants, from a denumerable set of sentential variables. I assume as the only rules of inference *modus ponens* for $\rightarrow$ and adjunction for &; the application of these rules to instances of some definite set of axiom schemes yields as usual the set T of theorems of L^+. By the *negation completion* L of L^+, I mean the result of adding-to the formation apparatus, and taking as additional axioms all of the new sentences in-which are instances of the old axiom schemes together with all instances of A1-A3.

A *possible matrix* for a logic L is a triple $\mathfrak{M} = \langle M, O, D\rangle$, where M is a non-empty set, O is a set of operations on M corresponding to the connectives, and D is a non-empty subset of (*designated* elements of) M. An interpretation α of L in $\mathfrak{M}$ is a homomorphism from the algebra of formulas of L, in the sense of [15], into $\mathfrak{M}$. A sentence A is *true* on the interpretation α in the matrix $\mathfrak{M}$ if and only if $\alpha(A) \in D$; A is *valid* in $\mathfrak{M}$ just in case A is true on all interpretations α of L in $\mathfrak{M}$. Finally, $\mathfrak{M}$ is a *matrix for L* (for short, an *L-matrix*) if and only if (i) the axioms of L are all valid in $\mathfrak{M}$ and (ii) $\mathfrak{M}$ strongly satisfies *modus ponens* and adjunction—i.e., for all a, b in M, if both $a \in D$ and $b \in D$, a & $b \in D$, and if both $a \in D$ and $a \rightarrow b \in D$, $b \in D$. (Where in applying these concepts we wish to call attention to the fact that we are dealing with a positive logic, we suffix '+' where appropriate—e.g., we speak of the L^+-matrix $\mathfrak{M}^+ = \langle M^+, O^+, D^+\rangle$.)

Our stipulations imply, as is well-known, that the set T of theorems of what we call a sentential logic is closed under substitution for sentential variables. Accordingly, for every logic L the *canonical L-matrix* $\mathfrak{L} = \langle F, O, T\rangle$ exists, where F is the set of sentences of L, O is the set of connectives (taken as operations on F), and T is the set of theorems. The *canonical interpretation* α_L of L in $\mathfrak{L}$ is the function which assigns to each sentence of L itself.[3]

We note the following truism.

Lemma 1. *Let L be a sentential logic. The following conditions are equivalent, for each sentence A of L.*

(1) *A is a theorem of L.*
(2) *A is valid in every L-matrix.*
(3) *A is valid in the canonical L-matrix.*
(4) *A is true on the canonical interpretation* α_L.

Proof. Immediate from definitions, in the manner of [15].

2. By Lemma 1, every non-theorem A of a positive logic L^+ is invalid in some L^+-matrix $\mathfrak{M}^+ = \langle M^+, O^+, D^+\rangle$. To show that the negation completion

3. I mention [15] for its accessibility; underlying ideas are generally credited to Lindenbaum. Cf. also the dissertation of D. Ulrich (Wayne State University, 1967), and the work of Łoś.

L of L^+ is a conservative extension thereof, our strategy will be to *enlarge* $\mathfrak{M}^+$ to a possible L-matrix $\mathfrak{M} = \langle M, O, D\rangle$, where

(1) $M = M^+ \cup M^-$, where each element of M^- is the negation of some member of M^+;
(2) O is the set of operations of $\mathfrak{M}^+$, extended to all of M, together with negation;
(3) $D^+ \subseteq D$, and furthermore $M^+ \cap D = D^+$.

If we can work out the details of this plan in such a way that for every non-theorem A of L^+ there exists in conformity to (1)-(3) an *enlargement* $\mathfrak{M}$ of a matrix $\mathfrak{M}^+$ that rejects A, then L is a conservative extension of L^+ provided that each such $\mathfrak{M}$ is an L-matrix—i.e., satisfies the axioms and rules of L^+ together with the negation axioms A1-A3. The reason is that, if these conditions are fulfilled, there was some interpretation α^+ of L^+ in $\mathfrak{M}^+$ on which A is not true; letting α be the interpretation of L in $\mathfrak{M}$ which agrees with α^+ on sentential variables (which uniquely determines α), we see by (2) that α and α^+ agree wherever the latter is defined and hence by (3) that A is not true on α; hence by the lemma that ended section 1, A is a non-theorem of L.

So much for general strategy; how is it to be carried out in particular cases? As above, let L^+ be a definite positive logic and $\mathfrak{M}^+ = \langle M^+, O^+, D^+\rangle$ be an L^+-matrix. The plan to enlarge $\mathfrak{M}^+$ in accordance with (1)-(3) to $\mathfrak{M} = \langle M, O, D\rangle$ requires specific answers to the following questions:

(4) How is the set of new elements M^- to be determined?
(5) How is negation to be defined on M?
(6) Given that (by (2)) the operations of O are to agree with those of O^+ where the latter are defined, how are these operations to be defined when one of their arguments is in M^-?
(7) Which elements of M^- shall belong to D?

Strong negation laws, though they are a burden in attempting to carry out syntactic proofs of conservative extension (since with strong laws new negation-free theorems could have come in many ways), have their uses here by forcing us to answer (4)-(7) in a way that makes them true. To make double negation true, we stipulate the following:

(I) $M^- \cap M^+ = \phi$, and there shall be a bijection $*$ from M^+ onto M^- such that, for all $a \in M^+$, $-a = a^*$ and $-(a^*) = a$.

(I) answers (4) and (5); the need to make contraposition and the DeMorgan laws true suggests a partial answer to (6). Let $\to^+$, $\&^+$, $\vee^+$ be operations of O^+, and let $\to$, $\&$, $\vee$ be corresponding operations of O. Then, $*$ being as in (I),

(II) For all a, b in M^+,
a. $a^* \to b^* = b \to a = b \to^+ a$;
b. $a^* \;\&\; b^* = (a \vee b)^* = (a \vee^+ b)^*$;
c. $a^* \vee b^* = (a \;\&\; b)^* = (a \;\&^+ b)^*$.

Given a-c, the operations $\rightarrow$, &, v are defined whenever both arguments are in M^+ or both are in M^-. But suppose one argument is in M^+ and the other in M^-; what are we to do? A simple answer is to let everything in one of these sets intuitively to imply everything in the other; a felicitous choice turns out to be to let the +'s imply the -'s, and the desire to attend to elementary properties of conjunction and disjunction almost forces the following:

(III) For all a, b in M^+,

a. $a \,\&\, b^* = b^* \,\&\, a = a$;

b. $a \vee b^* = b^* \vee a = b^*$;

c. $a \rightarrow b^* \in D$;

d. $a^* \rightarrow b \notin D$.

(III) takes leave of the self-evident principles that inspired (I) and (II) and so must be considered one among alternate strategies; our justification is that it often works and may be neatly pictured. For if, as is often the case, $\mathfrak{M}^+$ is a lattice in which $\&^+$ and $\vee^+$ deliver respectively greatest lower and least upper bounds, and the order $\leq^+$ is defined setting $a \leq^+ b$ if and only if $a \rightarrow b \in D^+$, then the effect of (I) and (II) is to make of M^- a copy of M^+ (with the order relation reversed); (III) finishes the job of defining an extended order $\leq$ by making $a \leq b$ whenever $a \in M^+$ and $b \in M^-$; we complete the picture by observing that if $\mathfrak{M}^+$ is a (distributive) lattice, $\mathfrak{M}$ is a (distributive) lattice.

(II) and (III) constituted a partial answer to (6), but we have by the way answered (7) also. For since all +'s imply all -'s, we can only close D under *modus ponens* as follows:

(IV) $D = D^+ \cup M^-$.

(IV) has the advantage of being simple, but it also makes $\mathfrak{M}$ inconsistent in the sense that, for some $a \in M$, $a \in D$ and $-a \in D$, since D^+ was non-empty to begin with. This would have cost us some pain had we included the implicational paradox $A \,\&\, \overline{A} \rightarrow B$ among the negation axioms; as it is, all that is shown is that our simple-minded approach won't work for systems simple-minded enough to have implicational paradoxes of that sort; since in particular the relevant logics (for which the questions we are attempting to answer are open) don't have such paradoxes, there is no reason yet to quit.

(I-IV) already enable us to get conservative extension results for a number of positive logics L^+. We shall not pause to derive them, however, since for logics as strong as E the vagueness of (IIIc) and (IIId) won't do; for unlike our other specifications, (IIIc) and (IIId) do not yield specific values for specific arguments. It turns out, however, that if the matrix $\mathfrak{M}^+$ from which we began has a certain simple property, we can specify (IIIc) and (IIId) satisfactorily for E also. As it turns out, for a wide class of logics each non-theorem can be rejected in a matrix with this property, here called *rigorous compactness*. Accordingly we devote the next section to the study of rigorously compact matrices, proving a pretty lemma of some interest on its own; the use of this lemma lies in this paper in the area of technical maneuver, however; the reader who wishes to get on with the argument may skip the section and refer back as needed.

3. Let all of the operations of O be among &, v, $\rightarrow$, -, o, $\cup$, where & and v may be thought of as standard conjunction and disjunction and o and $\cup$, if present, as intensional analogues thereof.[4] A matrix $\mathfrak{M} = \langle M, O, D\rangle$ is rigorously compact provided that there exist elements $I \in D$ and $0 \in M - D$ with the following properties:

(V) For all $a \in M$, whenever the relevant operation is in O,

a. $0 \rightarrow a = a \rightarrow I = a \cup I = I \cup a = a \vee I = I \vee a = I$;

b. $0 \,\&\, a = a \,\&\, 0 = 0 \circ a = a \circ 0 = 0$;

c. $I \,\&\, a = a \,\&\, I = 0 \vee a = a \vee 0 = a$;

d. If $a \neq 0$, $a \rightarrow 0 = 0$ and $a \circ I = I \circ a = I$;

e. If $a \neq I$, $I \rightarrow a = 0 \cup a = a \cup 0 = 0$;

f. $-0 = I$ and $-I = 0$.

(a-f) give I and 0 many of the properties of truth-table T and F; in particular, if $M = \{0, I\}$, what (a-f) determine are classical truth-tables (for intensional as well as standard connectives).

A somewhat more interesting case arises when $M = \{0, N, I\}$ and $D = \{N, I\}$, where N is a (neuter) element distinct from 0 and I. If we specify

(VI) $N = N \,\&\, N = N \vee N = N \rightarrow N = N \circ N = N \cup N$, and $N = -N$,

and otherwise let operations on M be determined by (a-f) above, the result is a matrix of some importance. It is, in fact, the first distinctive *Sugihara matrix*; we shall call it $\mathbf{S}_3$ and note that it has a natural representation in the integers $\{-1, 0, +1\}$, with & going to min, v to max, - to inverse.[5]

Apart from truth-tables, most familiar matrices are not rigorously compact. The reason lies in (Vd) and (Ve); in, e.g., Boolean or pseudo-Boolean algebras (cf. [15]), for a such that $0 \neq a \neq I$, $I \rightarrow a = a$, violating

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4. o and $\cup$ may be interpreted as intensional analogues of conjunction and disjunction respectively. They were introduced into the relevant logics by Church and were studied in a number of dissertations, including Belnap's, mine, and Dunn's. (They are the *sort* of connectives one studies in dissertations, though I shall give reasons here for the postgraduate utility of o, as Fisk has given reasons for that of $\cup$.)

5. On this labeling 0 stands for N, -1 for 0, 1 for I. The original Sugihara matrix $\mathbf{S}_\omega$ contained all the non-zero integers; a variant appears in [17] as a matrix plausible for the Sugihara system of strict implication, though not characteristic. That $\mathbf{S}_\omega$ itself is important for relevant logic became evident through my proof that it is characteristic for the Dunn-McCall system R-mingle. (That proof is unpublished, but an algebraic counterpart which links extensions of R-mingle to finite Sugihara matrices (i.e., sub-matrices of $\mathbf{S}_\omega$, sometimes with 0 added) is to be found in Dunn's [8].) The matrix $\mathbf{S}_3$ originates in [16].

(Ve), and $a \to 0 = -a$, violating (Vd) in the Boolean case. It is again those rigorously compact matrices that are lattices that are most easily pictured (cf. remarks following (III) above); it is obvious from definitions that 0 and I are then the lattice zero and unit; moreover 0 and I are *isolated* with respect to negation and the intensional operations $\to$, $\circ$, $\cup$, in the sense that if *one* argument to an intensional operation is 0 or I, the value will be 0 or I whatever the other argument. Not surprisingly, some rigorously compact matrices have been useful for the semantics of relevant logics—e.g., in [5]; it turns out, as I prove below, that any matrix for a relevant logic may be trivially embedded in a rigorously compact matrix.

In fact, where $\mathfrak{M} = \langle M, O, D\rangle$ is an L-matrix, let the *rigorously compact extension of* $\mathfrak{M}$ be the matrix $\mathfrak{M}^* = \langle M^*, O^*, D^*\rangle$, where M^* is got by adding 0 and I to M; D^*, by adding I to D; O^*, by extending the operations of O to M^* by (Va-Vf). $\mathfrak{M}^*$ is a possible L-matrix, though it is not necessarily an L-matrix; e.g., if L is classical sentential logic and $\mathfrak{M}$ is truth-tables, $\mathfrak{M}^*$ is a 4-element matrix (the Sugihara matrix $\mathbf{S}_4$, in fact) in which the classical theorem scheme $A \to (B \to A)$ is invalid; so $\mathfrak{M}^*$ is not in this case an L-matrix.[6]

It would be interesting, accordingly, to characterize the class of sentential logics L such that rigorously compact extensions of L-matrices are themselves invariably L-matrices. My attempts to solve this problem have all foundered on counter-examples, but a necessary condition toward its solution is found in the following lemma.

Lemma 2. *Let L be a sentential logic whose connectives are among* $\to$, $\&$, $\vee$, $-$, $\circ$, $\cup$. *Suppose that for every L-matrix* $\mathfrak{M}$ *the rigorously compact extension* $\mathfrak{M}^*$ *just defined for* $\mathfrak{M}$ *is also an L-matrix. Then all theorems of L are valid in the Sugihara matrix* $\mathbf{S}_3$.

Proof. It suffices to note that the 1-point matrix $\mathbf{N} = \langle\{N\}, O, \{N\}\rangle$, with operations defined trivially by (VI), is an L-matrix for every L with the connectives above, and that the rigorously compact extension of $\mathbf{N}$ is $\mathbf{S}_3$. So trivially $\mathbf{S}_3$ is by hypothesis an L-matrix, which was to be proved.

Lest the reader feel that we have cheated by dragging in the trivial matrix $\mathbf{N}$, we present him with a corollary: Change 'every' to 'some' in the second sentence of Lemma 2; the lemma still holds. (Essentially the idea is that if the rigorously compact extension of any L-matrix $\mathfrak{M}$ is an L-matrix, so is its image under the function that takes I into I, 0 into 0, and everything else into N. For that function is readily observed to be a *matrix homomorphism* from $\mathfrak{M}^*$ into $\mathbf{S}_3$, whence the validity of the axioms of L in S_3 follows from their validity in $\mathfrak{M}^*$.)

Every sub-logic L of the classical sentential calculus admits a

6. A particular matrix $\mathfrak{M}_0$ plays a central role in [5] and in many other Anderson-Belnap investigations into the semantics of relevant logics. $\mathfrak{M}_0$ and the products and implicative extensions thereof presented in [5] are all rigorously compact; so is any Sugihara matrix (cf. last footnote) with least and greatest elements.

rigorously compact L-matrix—truth-tables, as we have noted, will do. But what Lemma 2 and its corollary show is that we cannot in general preserve L-matrixhood by tacking on I and 0 satisfying (Va-Vf); in particular, such tacking on is never successful if any of $(p \& -p) \to q$, $p \to (q \to p)$, $p \to (q \vee -q)$ and their ilk is a theorem of L, since all such are invalid in $\mathbf{S}_3$. On the other hand, for the logics classified as relevant such tacking on always works, an observation to be confirmed below where needed for present purposes.

4. That interlude being over, we can return to (IIIc) and (IIId). Let L^+ be a positive logic; $\mathfrak{M}^+$, a *rigorously compact* L^+*-matrix.*

(VII) For all $a, b \in M^+$, $*$ being as in (I),
a. $a \to b^* = I$;
b. $a^* \to b = 0$.

Having cleared that up, let $\mathfrak{M} = \langle M, O, D\rangle$ be the enlargement of $\mathfrak{M}^+$ got by applying stipulations (I-IV) and (VII), which we call the *rigorous enlargement* of $\mathfrak{M}^+$.

We shall now characterize certain logics as *rigorous*. By the *basic positive rigorous logic* BR^+, we mean the logic formulated with $\to$, &, $\vee$ primitive, with rules of *modus ponens* for $\to$ and adjunction for &, and with the following axiom schemes:

B1. $A \to A$
B2. $(A \to B) \to ((B \to C) \to (A \to C))$
B3. $(B \to C) \to ((A \to B) \to (A \to C))$
B4. $(A \& B) \to A$
B5. $(A \& B) \to B$
B6. $((A \to B) \& (A \to C)) \to (A \to (B \& C))$
B7. $A \to (A \vee B)$
B8. $B \to (A \vee B)$
B9. $((A \to C) \& (B \to C)) \to ((A \vee B) \to C)$[7]

A logic L^+ is a *positive rigorous logic* if it can be formulated with the same connectives, axiom schemes, and rules of inference as BR^+, with

7. All of the axioms of BR^+ are theorems of E^+, but BR^+ is a much weaker system, lacking in particular the E-valid principles of distribution of & over $\vee$, the contraction principle C1, and the E-theory of modality. Nevertheless BR^+ (and the negation-completion BR one gets by adding A1-A3) is of some interest as a minimal relevant logic; it has a deduction theorem (of sorts) and familiar replacement properties hold; B4-B9, one notes, are just lattice properties. (Cf. Anderson [2] and Curry [6].) So far as minimality is concerned, Belnap has made a fascinating conjecture about the pure implicational system BR_I determined by B1-B3 and *modus ponens*—namely that for this system both $A \to B$ and $B \to A$ are theorems only if A and B are identically the same sentence. I add that though a disgustingly large number of persons have tried Belnap's problem, it remains disgustingly open, though some clarification has been provided by L. Powers in unpublished work.

perhaps one or more of C1-C7 below as additional axiom schemes. A logic L is a *rigorous logic* if it can be formulated as the negation completion, in the sense of section 1 of a positive rigorous logic.

C1. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$
C2. *Where B is in apodictic form,*[8] $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$
C3. $((A \rightarrow A) \rightarrow B) \rightarrow B$
C4. $(NA \,\&\, NB) \rightarrow N(A \,\&\, B)$[9]
C5. $(A \,\&\, (B \vee C)) \rightarrow ((A \,\&\, B) \vee C)$
C6. $(((A \,\&\, B) \rightarrow C) \,\&\, (A \rightarrow (B \vee C))) \rightarrow (A \rightarrow C)$
C7. $(A \rightarrow B) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow B))$

We have chosen C1-C7 because their addition to BR^+ produces logics in which people have taken an independent interest. For example, add C1 and C5 to get the positive fragment P+ of the Anderson-Belnap system P of ticket entailment (cf. [2] and [3] for motivation). Adding C2 to P+ (alternatively, adding C3 and C4) produces E+. C6 is a strengthened version of C5, producing a system MD+.[10] Adding C7 produces mingle-systems of the kind investigated by Ohnishi and Matsumoto in [14]; in particular, adding C7 to E produces the unpublished McCall-Dunn system E-mingle.

5. We can now prove our first principal result.

Theorem 1. *Let L^+ be a positive rigorous logic; then the negation completion L of L^+ which results from taking* A1-A3 *as additional axiom schemes is a conservative extension of L^+.*

Proof. We must show, for a given non-theorem A of L^+, that A is a non-theorem of L. Suppose then that A is unprovable in L^+. Consider first the canonical L^+-matrix $\mathfrak{L}^+$. By Lemma 1, A is not true in $\mathfrak{L}^+$ on the canonical interpretation α_{L^+}.

Let $\mathfrak{M}^+ = \langle M^+, O^+, D^+\rangle$ be the rigorously compact extension of $\mathfrak{L}^+$ defined in section **3**. We remark that by Lemma 2 and its incofmal corollary at least a necessary condition that $\mathfrak{M}^+$ be an L^+-matrix is fulfilled; the sufficient condition is that the axioms of L^+ are valid in $\mathfrak{M}^+$ and that the rules are strongly satisfied; since $\mathfrak{M}^+$ is got from $\mathfrak{L}^+$ by adding 0 and I, and since $\mathfrak{L}^+$ is known to be an L^+-matrix, this is exhaustive but easy; we do two cases and leave the rest to the reader.

Ad B4. Show for all $a, b \in M^+$, $(a \,\&\, b) \rightarrow a \in D^+$. Cases. (1) $a = 0$. By (V), $(0 \,\&\, b) \rightarrow 0 = 0 \rightarrow 0 = I \in D^+$. (2) $a = I$. By (V), $(I \,\&\, b) \rightarrow I = I \in D^+$.

8. As in [10], B is in apodictic form if B is of the form $D \rightarrow E$ or is a conjunction of sentences in apodictic form.

9. As in [5], NA is defined as $(A \rightarrow A) \rightarrow A$.

10. When added to E+. Until Dunn in [9] found a way of motivating R+ which did not motivate C6, I thought MD more likely to prove semantically stable than E itself; since Dunn has at least temporarily grounded that view, I have named the system in his honor.

(3) $b = 0$. By (V), $(a \,\&\, 0) \to a = 0 \to a = I$. (4) $a \neq 0$ and $a \neq I$ and $b = I$. By (V), $(a \,\&\, I) \to a = a \to a$, which is an element of $\mathfrak{L}^+$ and which is designated therein by the validity in $\mathfrak{L}^+$ of B1. (5) a and b are both distinct from 0 and I. Then $(a \,\&\, b) \to a$ is by the validity of B4 in $\mathfrak{L}^+$ a designated element of $\mathfrak{L}^+$ and hence belongs to D^+. Cases exhausted.

Ad adjunction. Show for all $a, b \in D^+$, $a \,\&\, b \in D^+$. Cases. (1) $a = I$. By (Vc), $I \,\&\, b = b \in D^+$ on assumption. (2) $b = I$. Similar. (3) $a \neq I$, $b \neq I$. Adjunction holds on assumption for $\mathfrak{L}^+$. Cases closed.

So $\mathfrak{M}^+$ is an L^+-matrix. Let $\mathfrak{M} = \langle M, O, D\rangle$ be the rigorous enlargement of $\mathfrak{M}^+$ defined at the beginning of section **4**. Clearly A is not true on the interpretation α_M which agrees with α_{L^+} on sentential variables, for since A is negation-free $\alpha_M(A) = \alpha_{L^+}(A) \notin D$ by definition.

Since as just noted A is invalid in $\mathfrak{M}$, A is by Lemma 1 a non-theorem of L provided that $\mathfrak{M}$ is an L-matrix. We end the proof accordingly by showing $\mathfrak{M}$ an L-matrix, given that $\mathfrak{M}^+$ is an L^+-matrix. That D is closed under *modus ponens* and adjunction is clear from (II) and (III). Show axioms valid by cases. Example—let * be as in (I) and let a, b, c belong to M^+; then as part of the verification of B2 note that $(a^* \to b^*) \to ((b^* \to c^*) \to (a^* \to c^*))$ = (by IIa) $(b \to a) \to ((c \to b) \to (c \to a))$; but the latter belongs to D^+ because B3 is valid in $\mathfrak{M}^+$. Similar moves validate the negation axioms A1-A3, bearing in mind that (I-IV) and (VII) were chosen with the validation of those axioms in mind; example—for $a \in M^+$ note that $-a \to a = -(a^*) \to a$ (by I) $= a \to a$, which belongs to D^+ since B1 is valid in $\mathfrak{M}^+$; this partially confirms the validity of A1 in $\mathfrak{M}$. So it goes, and the interested reader may amuse himself by checking all the computational possibilities in like manner.[11]

6. Theorem 1 answers Anderson's question affirmatively for E and for a number of related logics. It does not, however, answer the question for R, since the proof of Theorem 1 breaks down when one attempts to verify the R-theorem $A \to ((A \to A) \to A)$. The result of this failure is that we shall leave open the question whether R is a conservative extension of the system R′ determined by the negation-free axioms of [5].[12] If we take the connective $\circ$ of intensional conjunction as an additional negation-free primitive, however, a modification of the proof of Theorem 1 will work.[13]

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11. A similar, though less general, argument is used to prove Lemma 3 in [11].

12. Counting v as primitive. (Added in proof: The answer is yes.)

13. There are good reasons to formulate R^+ with $\circ$ quite independent of the desire to extend Theorem 1 to R. First, Dunn has showed in his dissertation (University of Pittsburgh, 1966) that $\circ$ has a natural interpretation in the algebraic semantics of R, which will be presented by him, Leblanc, and me in a forthcoming paper; cf. also [12], which provides a simple realization of R^+ in the natural numbers. Second, Dunn has also provided (cf. the abstract [9]) a Gentzen-style consecution calculus for the negation-free part of R, the adequacy of whose interpretation depends upon the presence of $\circ$. (Indeed, at writing, that which Dunn has axiomatized *is* the negation-free part of R depends on Theorem 2 below, a dependence which hopefully will cease when his methods are extended to account for negation.)

Eschewing generality in what follows,[14] we formulate the system R^+ of positive relevant implication with $\rightarrow$, &, v, o primitive, rules of adjunction and *modus ponens*, axiom schemes B1-B9, C1, C5, and the following:

R1. $A \rightarrow (B \rightarrow (A \circ B))$
R2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \circ B) \rightarrow C)$
R3. $A \rightarrow ((A \rightarrow B) \rightarrow B)$

The negation completion R of R^+ results when A1-A3 are added.[15]

To motivate R1 and R2, we note that the biconditional $((A \,\&\, B) \rightarrow C) \leftrightarrow (A \rightarrow (B \rightarrow C))$ is unacceptable in R, since it leads immediately from the theorem $(q \,\&\, p) \rightarrow p$ to the fallacy of relevance (cf. [3]) $q \rightarrow (p \rightarrow p)$. It is convenient, however, to have a kind of conjunction which allows unrestricted exportation and importation.[16] R1 and R2 do so for o.

We truck now with the appropriate modifications in the proof of theorem 1. Let $\mathfrak{M}^+ = \langle M^+, O^+, D^+\rangle$ be an R^+-matrix. Let M^- and * be as in (I). Let 0 and I be distinct non-members of $M^+ \cup M^-$. We define the *relevant enlargement* $\mathfrak{M} = \langle M, O, D\rangle$ of $\mathfrak{M}^+$ as follows:

(VIII) 1. $M = M^+ \cup M^- \cup \{0, I\}$.
2. $D = D^+ \cup M^- \cup \{I\}$.
3. $0 = \{\rightarrow, \&, \vee, \circ, -\}$, where
   a. $\mathfrak{M}$ shall be rigorously compact—i.e., when at least one argument to an operation is 0 or I, the value shall be given by (Va-Vf).
   b. On M^+ operations of 0 shall agree with corresponding operations of O^+.
   c. For all $a \in M^+$, $-a = a^*$ and $--a = a$.
   d. For all $a, b \in M^+$, (IIa-IIc) and (IIIa-IIId) shall hold; furthermore $a \rightarrow b^* = (a \circ b)^*$[17]; $a^* \rightarrow b = 0$; $a \circ b^* = b^* \circ a = (a \rightarrow b)^*$; $a^* \circ b^* = I$.

The specifications (VIII) suffice to define $\mathfrak{M}$. The reader should note that the strategy of relevant enlargement is in a sense opposite to that of rigorous enlargement which appeared in the proof of Theorem 1, for what

14. Some generality is still attainable; the theorem to be proved still holds if we eschew the distribution axiom C5, for example. But the situation is delicate—we cannot eschew C1, since it is provable from the other axioms of R^+ in the presence of A1-A3. Likewise we lose the conservative extension property if we add C7 alone to R^+, getting Dunn's system R-mingle; for all the negation-free axioms of R-mingle thus formulated are intuitionistically valid, but A1-A3 enable one to prove the un-Brouwerian formula $(p \rightarrow q) \vee (q \rightarrow p)$ in R-mingle.

15. o becomes redundant in R, since $A \circ B$ is definable therein as $-(A \rightarrow -B)$, whence R1 and R2 become provable.

16. These motivating remarks owe much to Dunn and to Belnap.

17. This may be thought of as an answer to the question posed by (IIIc) alternative to (VIIa), which doesn't work here.

we essentially did there was to take a matrix $\mathfrak{L}^+$ and to add 0 and I to get $\mathfrak{M}^+$; by copying $\mathfrak{M}^+$ we got $\mathfrak{M}$. The technique of the present construction may be viewed the other way round—first we copy, and then we add 0 and I.[18]

We now apply the proof of Theorem 1 to R, *mutatis mutandis.*

Theorem 2. R *is a conservative extension of* R^+.

Proof. Strategy is as above, so we shall be brief. If A is a non-theorem of R^+, it is not true on an interpretation α^+ in some R^+-matrix $\mathfrak{M}^+$—e.g., the canonical one. Form the relevant enlargement $\mathfrak{M}$ of $\mathfrak{M}^+$ and show, by verifying the axioms and rules of R, that $\mathfrak{M}$ is an R-matrix; the interpretation α which agrees with α^+ on sentential variables agrees with α^+ on all negation-free sentences of R, so in particular A is not true on α; hence A is a non-theorem of R. So all negation-free theorems of R are already theorems of R^+, which was to be proved.

There are two interesting corollaries to our results, which we shall draw in conclusion. First, where L is a logic, we mean by an L-theory any set T of sentences of L which contains all axioms of L and which is closed under the rules of L; we write $\vdash_T A$ if $A \in T$, and we call T complete (consistent) if for every sentence A of L at least one (not both) of A, $\overline{A}$ is in T. Then

Corollary 1. *Let L be one of the rigorous logics, or* R. *Then there is a complete L-theory T such that, for all negation-free sentences A of L, $\vdash_T A$ if and only if $\vdash_L A$. Furthermore, where L is* E, R, *or* P, *there is a consistent and complete L-theory T^* with this property.*

Proof. We prove the corollary for the rigorous logics, leaving the reader to handle R in like manner. Given L^+, construct the matrix $\mathfrak{M}$ and the interpretation α_M as in the proof of Theorem 1, and consider the set T of all sentences of L which are true on α_M. Since $\mathfrak{M}$ is an L-matrix, it is easy to show that T is an L-theory; furthermore T is complete, since by the construction of M at least one of a, $-a \in D$ for all $a \in M$. Finally, the restriction of α_M to sentences of L^+ is the canonical interpretation of L^+; hence by Lemma 1, $\vdash_T A$ iff and only if $\vdash_L A$, for all negation-free sentences of L.

Though by the construction of T that theory is complete, it is nevertheless woefully inconsistent; in fact $\vdash_T \overline{B}$ whenever B is negation-free. Suppose, however, that L is E, R, or P. Then application of the methods of [13] yields the result that T has a consistent and complete sub-L-theory

18. The question arises of unifying our two techniques by adding intensional conjunction to E^+ and the other rigorous logics. But I leave open the problem of finding the right axioms for $\circ$ in E^+ (even better, in BR^+), noting only that R1-R2 won't do.

T^*.[19] Since T^* must at any rate contain all theorems of L (since it's an L-theory) and since it cannot contain any non-theorems of its extension T, when A is negation-free $\vdash_{T^*} A$ if and only if $\vdash_L A$. This ends the proof of Corollary 1.

Corollary 1 sheds interesting light on the relevant logics. First, the construction of T shows that the means of blocking the so-called "implicational" paradoxes really work; the philosophical point, worked out nicely by Dunn in his dissertation (*op. cit.*) is that a sentence is not necessarily relevant to its negation. Second, the corollary shows that all of the negation-free theorems of one of the relevant logics L may be rejected together in a single consistent and complete L-theory T^*. It would be nice to find a recursive axiomatization of such a T^*, since that would imply a positive solution to the decision problem, not yet solved for any of the relevant logics, for at least the negation-free fragment L^+ of L.

Our final corollary slightly improves a result of [11].

Corollary 2. *Let L be* R *or a rigorous logic. Then all negation-free theorems of L are intuitionistically valid.*

Proof. It suffices to note that all negation-free axioms and rules of L (including those for o) are intuitionistically valid, whence the corollary follows by the conservative extension results.[20]

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19. A minor generalization of the technique of [13] is required, as is pointed out in my forthcoming, "On relevantly derivable disjunctions." (Since results from the present paper are cited in the latter, it is important to note that there is no circularity; for [13] is independent of both.)

20. My thanks are due to the National Science Foundation for partial support of this research through grant GS-2648. I am indebted also to a number of persons for having read the manuscript carelessly, which allowed Lemma 2 for a while to take a truly beautiful form. Oh, that Keats had been right!

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