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PHYSICAL MODALITIES AND THE SYSTEM E

KENNETH W. COLLIER

A sentence may be said to be physically necessary either simpliciter or relatively to another sentence. Candidates for the former are such sentences as 'the speed of light is finite' (assuming this is not analytic) or '$F = MA$'. Candidates for the latter are such sentences as 'the pencil will fall,' which could be necessary only given the truth of other sentences about the status of the pencil. Surely the use of the former, the monadic, modality is somewhat limited in scope, while the latter, the binary one, is quite widespread.

In "A New System of Modal Logic," Georg von Wright constructs a system of binary modalities called $\mathbf{M}_d$, introducing as primitive '$M($___$/$ . . .$)$' to be read '___ is possible relatively to . . .'. An interesting feature of this system is that the more familiar alethic modalities can be defined in it, as can the binary physical ones. Thus it would seem as if $\mathbf{M}_d$ is just the system to distinguish the two. Unfortunately, von Wright imbeds $\mathbf{M}_d$ in a system of material implication. Anderson and Belnap have argued at some length in [1] and [2] that material implication, like the Lord Privy Seal, who is neither the Lord nor a privy nor a seal, is neither material nor an implication relation. Instead, they offer a system called E which suffers from none of the difficulties besetting systems of material implication. In this paper I shall try to carry out in E von Wright's program with respect to the physical and alethic modalities.

The most obvious way to do this is simply to adopt von Wright's system wholesale, interpreting his implications as Anderson and Belnap entailments. In doing so, I shall make two minor changes. First, while von Wright takes relative possibility to be primitive, I take relative necessity to be primitive in order to facilitate the fit with E. (The two notions are dual in the usual way.) Second, von Wright introduces '$N($__$/$. . .$)$' as symbolizing relative necessity. This notation proved a little confusing when the 'N__' of alethic necessity is introduced to the system. Thus, I have replaced it with '$\mathcal{H}($___, . . .$)$' and '$\mathcal{M}($___, . . .$)$' for relative necessity and possibility respectively, where '$\mathcal{H}$' and '$\mathcal{M}$' are Cyrillic script for the Latin 'N' and 'M'. The von Wright axioms, then, become these:

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M_EA1 $\mathcal{M}(p, p) \rightarrow \mathcal{H}(p, p)$
M_EA2 $\mathcal{H}(p, q) \rightarrow \mathcal{M}(p, q)$
M_EA3 $\mathcal{M}(p \,\&\, q, r) \leftrightarrow \mathcal{M}(p, r) \,\&\, \mathcal{M}(q, p \,\&\, r)$
M_EA4 $\mathcal{H}(p, t) \rightarrow p$.

An immediate change to be made is that M_EA4 can be replaced. One of the interesting features of E is that neither necessity nor possibility are primitive in it. Necessity is defined as

$$Np =_{df} \mathrm{t} \rightarrow p,$$

where 't' is a propositional constant answering to the law of identity. (This is important for E, so throughout I shall use 't' to be just this. In M_d it was simply a tautology, so my usage shouldn't present any problems.) This means that we have to be careful to insure that the formula in M_E which answers to logical necessity fits neatly into this definition. Since $\mathcal{H}(p, t)$ is this formula, replacing M_EA4 with

M_EA4′ $\mathcal{H}(p, t) \leftrightarrow Np$

does the job nicely.

Before going any further, I need to say a word about the so-called *Entailment Theorem*. This theorem is the deduction theorem for E. It states that if there is a proof in E from $A_1, \ldots, A_n$ to B, then there is a proof in E of $(A_1 \,\&\, \ldots \,\&\, A_n) \rightarrow B$. I am going to need this theorem for M_E. It follows immediately from the following theorem.

Let E_I *be the axiomatic version of the pure entailment fragment of* E, *and* E_I* *the natural deduction version of it. Then if one adds to* E_I *any axiom of the form* $C \rightarrow D$ *he can get an equivalent natural deduction system by adding to* E_I* *the rule 'from* C_a *to infer* D_a*'.*

Proof. Call the new systems E_I' and $E_I^{*\prime}$ respectively. That $E_I^{*\prime}$ contains E_I' is trivial. Anderson and Belnap prove in [1] that E_I contains E_I*. Add to that proof the following. If D_a is a consequence of C_b, where $b = a$, we have two cases.
(i) $k \in a$. Then C_b' is $(A_1 \rightarrow C)_{a-k}$ and D_a' is $(A_1 \rightarrow D)_{a-k}$. Insert the new axiom, $C \rightarrow D$, and the appropriate instance of the axiom $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$. Then $\rightarrow$E used twice gives D_a'.
(ii) $k \notin a$. Then C_b' is C_a and D_a' is D_a. Insert $C \rightarrow D$, and $\rightarrow$E used once gives D_a'.

With this in hand, we can go on. An immediate consequence of the axioms that we will find useful is

M_ET $\mathcal{M}(p, t) \leftrightarrow Mp$

Proof A: $\mathcal{M}(p, t) \rightarrow Mp$

(1) $\mathcal{M}(p,t)$	Assump
(2) $\sim\mathcal{H}(\sim p, t)$	1, def of '$\mathcal{M}(p, q)$'
(3) $\sim N \sim p$	2, M_EA4′
(4) Mp	3, def of 'Mp'

Proof B: $Mp \rightarrow \mathcal{M}(p, t)$

(1) Mp	Assump
(2) $\sim N \sim p$	1, def of 'MP'
(3) $\sim\mathcal{H}(\sim p, t)$	2, M_EA4′
(4) $\mathcal{M}(p, t)$	3, def of '$\mathcal{M}(p, t)$'

Now, von Wright argues that to say that p is physically necessary given a contingent q is to say "(a) that p is not logically necessary, (b) that p is necessary relative to q but (c) that this necessity is *not* a necessity in all possible worlds, i.e., is not a *logical* necessity relatively to q," [3]. Thus he gives the following definition (with 'p is physically necessary given q' abbreviated '$\mathcal{N}(p, q)$'):

(1) $\mathcal{N}(p, q) =_{df} \mathcal{M}(p, t)\ \&\ \mathcal{M}(\sim q, t)\ \&\ \mathcal{M}(\sim p, t)\ \&\ \mathcal{H}(p, q)\ \&\ \mathcal{M}(\mathcal{M}(\sim p, q), \mathrm{t})$.

This comes over into our system with a little housecleaning via $M_E A4'$ and $M_E T$ as

$M_E D$ $\quad \mathcal{N}(p, q) =_{df} Mq\ \&\ M \sim q\ \&\ \sim Np\ \&\ \mathcal{H}(p, q)\ \&\ \sim N(\mathcal{H}(p, q))$.

This looks fine, especially since the monadic operators are, if we accept the E doctrine, real, honest to God logical modalities. $M_E D$ says just what von Wright thinks (not too implausibly) it ought to say. But does it? We won't know for sure until we know what '$\mathcal{H}(p, q)$' says. Aside from telling us that it is to be read 'p is necessary relatively to q', von Wright does not offer us any semantics, and I am not at all sure what is intended. So let us see if we can tell.

Suppose that $\mathcal{H}(p, q)$ is to be true just in case $Nq \rightarrow p$ is. The axioms would become:

$M_E A1'$ $\sim(Np \rightarrow \sim p) \rightarrow (Np \rightarrow p)$
$M_E A2'$ $(Nq \rightarrow p) \rightarrow \sim(Nq \rightarrow \sim p)$
$M_E A3'$ $\sim(Nr \rightarrow \sim(p\ \&\ q)) \longleftrightarrow (\sim(Nr \rightarrow \sim p)\ \&\ \sim(N(p\ \&\ r) \rightarrow \sim q))$
$M_E A4''$ $(Nt \rightarrow p) \rightarrow Np$.

$M_E D$ is going to present something of a problem. Its definiens becomes

(2) $\quad Mq\ \&\ M \sim q\ \&\ M \sim p\ \&\ (Nq \rightarrow p)\ \&\ \sim N(Nq \rightarrow p)$.

But one of the central doctrines of E is that entailments, if true at all, are necessarily true, i.e., we have as a theorem in E,

(3) $\quad (A \rightarrow B) \longleftrightarrow N(A \rightarrow B)$.

Thus the last two conjuncts of (2) are contradictory.

The axioms themselves don't fare very well either. $M_E A4''$ is a thesis of E, and hence is perfectly acceptable. $M_E A1'$ and $M_E A3'$, however, run afoul of an important metatheorem of E: the following is not a theorem of E.

(4) $\quad \sim(A \rightarrow B) \rightarrow (C \rightarrow D)$,

i.e., the denial of any entailment does not entail any entailment. And finally, $M_E A2'$ can be counterexampled. Suppose we substitute $(p\ \&\ \sim p)$ for q. Then the antecedent, $N(p\ \&\ \sim p) \rightarrow p$, is true, while the consequent, $\sim[N(p\ \&\ \sim p) \rightarrow \sim p]$ is false.

Apparently, then, it won't do to simply fix up $M_E D$. Under this interpretation, three of the axioms are found to be quite unacceptable. One runs into pretty much the same problems if he takes the sense of '$\mathcal{H}(p, q)$' to be any of '$(q \rightarrow Np)$', '$Nq \rightarrow Np$', or '$q \rightarrow p$'.

I conclude from all this that von Wright's system simply cannot be taken over bodily into E. The things that are holding us up are the special features of E that allow it to avoid the paradoxes. It seems, then, as if we are going to have to cook up our own system. In the remainder of this paper, I attempt to do just that, starting with R (the unmodalized version of E) and grafting various things on.

There are basically two different ways of doing this. To see the difference, we need to back up a little and see clearly just what we're up to. What we want is a systematic treatment of the locutions 'it is physically necessary that p given q', and 'it is logically necessary that p', and we propose doing this by starting out with the primitive 'it is necessary that p given q'. By building onto this primitive in one way, we will, hopefully, get the physical modalities, and by building on another way we will get the alethic ones. So what we need first is to develop a system for 'it is necessary that p given q' ('$\mathcal{H}(p, q)$'). This is what could be done in two different ways depending on what we take to be the force of $\mathcal{H}(p, q)$. On the one hand, we may take it to be $\varphi q \rightarrow \psi p$, where φ and ψ are contexts involving modalities of some kind. I shall develop a few such systems in detail, and show that for our purposes they will not work. On the other hand, $\mathcal{H}(p, q)$ might be taken to be $\varphi(q \rightarrow p)$, where φ is some sort of context. This is the approach that I shall finally adopt.

To start with the first way of doing things, then, I shall present three systems which I shall call the BE systems. Each system begins with R and has the primitive '$\mathcal{H}(p, q)$' and the following definitions, axioms, and theorems in common.

Definitions:

BED1 $\mathcal{M}(p, q) =_{df} \sim\mathcal{H}(\sim p, q)$
BED2 $Np =_{df} \mathcal{H}(p, t)$
BED3 $Mp =_{df} \sim N \sim p$

Axioms:
BEA1 $(\mathcal{H}(p, q) \;\&\; \mathcal{H}(r, q)) \rightarrow \mathcal{H}(p \;\&\; r, q)$
BEA2 $(t \rightarrow p) \leftrightarrow Np$

Theorems: *Entailment theorem.*

Proof. Immediate from the fact that both axioms are entailments, and the theorem proved earlier.

BET1 $Mp \leftrightarrow \mathcal{M}(p, t)$

Proof A: $Mp \rightarrow \mathcal{M}(p, t)$

(1) Mp	Assump
(2) $\sim N \sim p$	1, BED3
(3) $\sim\mathcal{H}(\sim p, t)$	2, BED2
(4) $\mathcal{M}(p, t)$	3, BED1

Proof B: $\mathcal{M}(p, t) \rightarrow Mp$

(1) $\mathcal{M}(p, t)$	Assump
(2) $\sim\mathcal{H}(\sim p, t)$	1, BED1
(3) $\sim N \sim p$	2, BED2
(4) Mp	3, BED3

BET2 $\mathcal{H}[\mathcal{H}(p, q), t] \rightarrow \mathcal{H}(p, q)$

Proof. Immediate from BED2. This theorem is going to be important later on.

BET3 $(Np \mathbin{\&} Nq) \rightarrow N(p \mathbin{\&} q)$

Proof.	(1) $Np \mathbin{\&} Nq$	Assump
	(2) $\mathcal{H}(p, t) \mathbin{\&} \mathcal{H}(q, t)$	1, BED2
	(3) $\mathcal{H}(p \mathbin{\&} q, t)$	2, BEA1
	(4) $N(p \mathbin{\&} q)$	3, BED2

BET4 E *is a proper subsystem of the* BE *systems.*

Proof. That the nonmodal axioms of E hold in the BE systems is trivial. Hence we need only see that the thesis corresponding to the E definition of 'Np' and the modal axiom hold in the BE systems. But BEA2 is the thesis corresponding to the E definition of 'Np', and BET3 is the modal axiom of E. The rules of inference of E are also rules of inference in the BE systems. That there are theses of the BE systems which are not theses of E is seen from the fact that E does not have a binary modal operator as each of the BE systems has.

BET5 $Np \rightarrow p$

Proof. As per the proof in E. I have included this because it will be useful to be able to refer to it by name.

The three systems, then, are as follows.

BE1: to the above apparatus add the rule of inference

BE1R *from* $\mathcal{H}(p, q)$ *and* Nq *to infer* p.

BE2: to the above apparatus, add the rule of inference

BE2R *from* $\mathcal{H}(p, q)$ *and* Nq *infer* Np.

BE2T1 $(\mathcal{H}(p, q) \mathbin{\&} Nq) \rightarrow p$

Proof.	(1) $\mathcal{H}(p, q) \mathbin{\&} Nq$	Assump
	(2) Np	1, BE2R
	(3) p	2, BET5

BE2T2 BE1 *is contained in* BE2

Proof. That the axioms of BE1 hold in BE2 is trivial. Any formula, say A, the proof in BE1 of which uses BE1R at some step, say i, can be proved in BE2 as follows. Consider the proof up through step $i - 1$. Everything up to that point will be valid in BE2. At some steps before $i - 1$, we must have had $\mathcal{H}(A, q)$ and Nq, else we couldn't have used BE1R. For our new step i', join these two by adjunction. Then at step $i' + 1$ we get A by BE2T1 and modus ponens.

BE3: to the above apparatus add the rule of inference

BE3R *from* $\mathcal{H}(p, q)$ *and* q *to infer* Np.

BE3T1 $(\mathcal{H}(p, q) \mathbin{\&} Nq) \rightarrow Np$

Proof.	(1) $\mathcal{H}(p, q) \mathbin{\&} Nq$	Assump
	(2) q	1, BET5
	(3) Np	1, 2, BE3R

BE3T2 BE2 *is contained in* BE3

Proof. That the axioms of BE2 hold in BE3 is trivial. Any formula, say A, the proof in BE2 of which uses BE2R at some step, say i, can be proved in BE3 as follows. Consider the proof up through step $i-1$. Everything up to that point will be valid in BE3. At some steps before $i-1$, we must have had $\mathcal{H}(A, q)$ and Nq, else we couldn't have used BE2R. For our new step i', join these two by adjunction. Then at step $i'+1$ we get A by BE3T1 and modus ponens.

There is one more matter to clear up before we consider the definition of physical necessity. So far, all of the axioms we have introduced are relatively innocuous, but consider BET2. It almost invites us to ask whether or not we can add its converse:

(5) $\mathcal{H}(p, q) \rightarrow \mathcal{H}(\mathcal{H}(p, q), t)$.

I have not found a way to prove it (and I doubt that a way can be found), so if it is to be added it will have to be as an axiom. Now consider $\mathcal{H}(p, q)$. We have taken that its force is that something or other involving q entails something or other involving p. But as pointed out above, one of the central doctrines of E is that entailments, if true at all, are necessarily true; and this ought to hold whether the entailment is explicit as in (3) or implicit as in $\mathcal{H}(p, q)$. Thus it seems to me that if we are to adopt the spirit of E wholeheartedly, we must adopt (5) and dignify it with the title BEA4. This immediately gives us a new, S4-ish theorem:

BET6 $\mathcal{H}(p, q) \leftrightarrow \mathcal{H}(\mathcal{H}(p, q), \mathrm{t})$.

Now, finally, we are in a position to evaluate the BE systems with respect to their adequacy for the physical modalities. We can see immediately from BET6 that we cannot introduce physical necessity via von Wright's definition, containing as it does the clause $\mathcal{H}(p, q)$ & $\sim N(\mathcal{H}(p, q))$. Furthermore, it seems to me that each of the BE systems contains, inherently, a feature which vitiates it. Before we see that, we must be sure we know just what we're looking for. We need a definition of 'p is physically necessary given q' ('$\mathcal{N}(p, q)$') in terms of our primitive '$\mathcal{H}(p, q)$'. But however we introduce '$\mathcal{N}(p, q)$', it seems to me that at the very minimum we will want it to turn out that $(\mathcal{N}(p, q)$ & $q) \rightarrow p$, and we want to reject $(\mathcal{N}(p, q)$ & $q) \rightarrow Np$. With this in mind, let us look at the various systems.

Both BE1 and BE2 suffer from the same difficulty. Surely somewhere in our definition we will want the clause '$\mathcal{H}(p, q)$', that being the whole enterprise. But the rules of inference in both of these systems require Nq in order to infer p. This proves a grave difficulty in interpreting q. Just what is its status to be? Surely it cannot be a law of nature or some initial condition or other since neither of these is a logical necessity. But if we are faithful to the guiding insight behind E, we must admit that no logical necessity ever entails (or is entailed by) a contingent truth, and thus p cannot be such. In what sense, then, could we be said to have captured the notion of *physical*, as opposed to *logical*, necessity?

BE3 fares just as poorly. As noted above, we want to be careful to reject $(\mathcal{N}(p, q) \,\&\, q) \rightarrow Np$. But if the definition is to involve '$\mathcal{H}(p, q)$', BE3R gives us just that. And again, it seems, we have not captured the notion of physical necessity.

At this point, perhaps, we would do well to stop and take stock for a moment. Each of the BE systems does half of the work we want done quite admirably: each gives a systematic account of the alethic modalities. But each fails miserably as an account of the physical modalities. As noted, we have another line of attack open to us. We might take the force of '$\mathcal{H}(p, q)$' to be $\varphi(q \rightarrow p)$ where φ is some context or other. So let us think a minute about what one might intend when he says that p is necessary relative to q.

It seems to me that there are essentially two possibilities. On the one hand, one may intend that under certain conditions, the truth of q insures the truth of p. But this does not seem to be the way we actually use the physical modalities. For example, it is perfectly natural to say "given that I drop the pencil, it is physically necessary that it fall to the floor." But I cannot, for the life of me think under what conditions the truth of the proposition that I drop the pencil *insures* the truth of the proposition that it falls to the floor. Rather, it seems to me that what is being said is that in the light of what we already know, from the proposition that I drop the pencil, it follows that it falls to the floor. And this is the second possibility, that q enthymematically entails p. This "enthymematic context" is part of what I want φ to reflect.

In addition, one would intuitively expect, surely, that the theorems of a theory are necessary relative to the axioms. My intuitions are not too clear, however, about whether or not a proposition is necessary relative to itself. I am accepting the thesis, though, on the grounds that sometimes it seems plausible and I have not found any untoward consequences. If the reader finds any, it can be excised easily enough. The system I shall call BEE.

We start with a little notation. Let S be a set of accepted propositions, and S_i be some subset of S. Then to R we add the following. (N.B. Proofs of some of the theorems are, for simplicity, carried out in the natural deduction system BEE* which is just like E* with one exception. The exception is the rule reit. This is changed to: either $(A \rightarrow B)_a$ or $\mathcal{H}(A, B)_a$ may be reiterated, retaining a. The original restriction on reiteration is to insure that only necessaries are reiterated, thus avoiding fallacies of necessity. In our new system, we have expanded the class of necessaries to include anything of the form $\mathcal{H}(p, q)$, so I do not think that this change in rules will occasion any difficulties. Note that the theorem on page 186 does not insure that BEE* and BEE are equivalent. While I am sure that they are, I have no proof of it.)

Definitions:

BEED1 $\mathcal{H}(p, q) =_{df} (\exists s_1) \ldots (\exists s_n)\, (s_1, \ldots, s_n \in S_i \,\&\, (s_1 \,\&\ldots\&\, s_n \,\&\, q) \rightarrow p)$
BEED2 $\mathcal{M}(p, q) =_{df} \sim\mathcal{H}(\sim p, q)$
BEED3 $Np =_{df} \mathcal{H}(p, t)$
BEED4 $Mp =_{df} \sim N \sim p$

Axiom:

BEEA $s \in S \rightarrow \vdash s$

Theorems:

BEET1 $Mp \longleftrightarrow \mathcal{M}(p, t)$

Proof. as per BET1.

BEET2 $(\mathcal{H}(p, q) \& q) \rightarrow p$

Proof. (1) $\mathcal{H}(p, q)$ Assump

(2) $(\exists s_1), \ldots, (\exists s_n)\ (s_1, \ldots, s_n \in S_i \ \& \ (s_1 \ \& \ldots \& \ s_n \ \& \ q) \rightarrow p)$ 1, BEED1

(3) $s_1, \ldots, s_n \in S$ 2, set theory

(4) $s_1, \ldots, s_n$ 3, BEEA

(5) $s_1 \ \& \ldots \& \ s_n \ \& \ q$ 1, 4 adjunction

(6) p 2, 5 modus ponens

BEET2 $(t \rightarrow p) \longleftrightarrow \mathcal{H}(p, t)$

Proof A: $(t \rightarrow p) \rightarrow \mathcal{H}(p, t)$

Assume $t \rightarrow p$. Let $S_i = \{\text{Axioms of E}\}$. Then we have some s_i such that

(1) $s_1, \ldots, s_n \in S_i$.

But since $(A \rightarrow B) \rightarrow .(A \ \& \ C) \rightarrow B$ in E, and we have $t \rightarrow p$ by hypothesis,

(2) $(s_1 \ \& \ldots \& \ s_n \ \& \ t) \rightarrow p$,

i.e., putting (1) and (2) together we have by BEED1

(3) $\mathcal{H}(p, t)$.

Proof B: $\mathcal{H}(p, t) \rightarrow (t \rightarrow p)$

(1)	$\mathcal{H}(p, t)_{\{1\}}$	Hyp
(2)	$t_{\{2\}}$	Hyp
(3)	$\mathcal{H}(p, t)_{\{1\}}$	1, reit
(4)	$(\mathcal{H}(p, \mathrm{t}) \ \& \ \mathrm{t}) \rightarrow p$	theorem introduction
(5)	$\mathcal{H}(p, \mathrm{t}) \ \& \ \mathrm{t}_{\{1,2\}}$	2, 3 & I
(6)	$p_{\{1,2\}}$	4, 5 → E
(7)	$\mathrm{t} \rightarrow p_{\{1\}}$	2, 6 → I

BEET4 $(Np \ \& \ Nq) \rightarrow N(p \ \& \ q)$

Proof. Assume $Np \ \& \ Nq$. Unpacking this via BEED3 and BEED1, we have

(1) $(\exists s_1) \ldots (\exists s_n)\ (s_1, \ldots, s_n \in S_i \ \& \ ((s_1 \ \& \ldots \& \ s_n \ \& \ \mathrm{t}) \rightarrow p)) \ \&$
$(\exists s_{n+1}) \ldots (\exists s_m)\ (s_{n+1}, \ldots, s_m \in S_j \ \& \ ((s_{n+1} \ \& \ldots \& \ s_m \ \& \ t) \rightarrow q))$.

Now then, let $S_k = S_i \cup S_j$. This gives us

(2) $s_1, \ldots, s_n, s_{n+1}, \ldots, s_m \in S_k$.

But since $((A \ \& \ B) \rightarrow C) \ \& \ ((A \ \& \ D) \rightarrow E) \rightarrow .(A \ \& \ B \ \& \ D) \rightarrow (C \ \& \ E)$ in E, and by (1) we have $(s_1 \ \& \ldots \& \ s_n \ \& \ \mathrm{t}) \rightarrow p$ and $(s_{n+1} \ \& \ldots \& \ s_m \ \& \ \mathrm{t}) \rightarrow q$, we have

(3) $(s_1 \& \ldots \& s_n \& s_{n+1} \& \ldots \& s_m \& \mathrm{t}) \rightarrow (p \& q)$.

Putting (2) and (3) together, we have by BEED1

(4) $\mathcal{H}((p \& q), \mathrm{t})$,

which is to say

(5) $N(p \& q)$.

BEET5 E *is a subsystem of* BEE.

Proof. That the nonmodal axioms of E hold in BEE is trivial. BEET3 answers to the E definition of 'Np', and BEET4 is the modal axiom of E. The rules of inference in E are rules in BEE.

Again, BEE gives a systematic account of the alethic modalities, namely the same one that E gives. It remains to give an account of the physical modalities. Letting '$\mathcal{N}$' and '$\mathcal{M}$' be read as 'physical necessity' and 'physical possibility' respectively, let us define

BEED5 $\mathcal{N}(p, q) =_{df} \mathcal{H}(p, q) \& S_i = \{s\colon\ s \in S \& s \text{ is a law of nature}\}$
BEED6 $\mathcal{M}(p, q) =_{df} \mathcal{M}(p, q) \& S_i = \{s\colon\ s \in S \& s \text{ is a law of nature}\}$.

Perhaps BEED6 needs a word of explanation. We could not define $\mathcal{M}(p,q)$ as $\sim\mathcal{N}(\sim p, q)$, because expanding $\mathcal{M}(p,q)$ we would get $\sim \mathcal{H}(\sim p, q) \vee S_i \neq \{s\colon\ s \in S \& s$ is a law of nature$\}$. But suppose we let $S_i = \{s\colon\ s \in S \& s$ is an axiom of E$\}$, and it turns out that $\mathcal{H}(\sim p, t)$. Then trivially we would have

(6) $\sim\mathcal{H}(\sim p, \mathrm{t}) \vee S_i \neq \{s\colon\ s \in S \& s \text{ is a law of nature}\}$.

This means that we would have $\mathcal{M}(p, \mathrm{t})$ and $\mathcal{H}(\sim p, \mathrm{t})$. This is, of course, intolerable. What we want is that p be physically possible relative to q only when q and the laws of nature fail to entail its denial. Hence BEED6.

At long last, we have what appears to me to be a formally adequate treatment of the two kinds of modalities which takes the insights of E seriously, and, at the same time does not do any obvious violence to our pre-analytic intuitions concerning the physical modalities. BEE has the further advantage of providing a neat fit with the Hypothetico-Deductive model of explanation. One may not like the H-D model, but if it is rejected, one must, in giving an account of the physical modalities, do so in a way that fits into his own pet model. It is, after all, only in the light of some explanation or other that one can claim physical necessity for a statement.

Our treatment is not, however, without some unsolved puzzles. As noted, I do not have proof of the equivalence of BE and BEE*. By the same token, I have every confidence in the world that the BE systems and BEE are conservative extensions of E, but alas I have no proof of this either. Next, BEE uses a little bit of set theory, and while I am certain that an adequate set theory can be worked out based on E, the fact remains that to date no one has done so. And finally, my account is of *de dicto* modalities.

I do not know how an account would go for *de re* modalities, but I should think that such an account would be extremely interesting. Indeed, I suspect that the most important *de re* modalities are the physical ones rather than the logical ones.*

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University of Pittsburgh
Pittsburgh, Pennsylvania
and
Southern Illinois University
Edwardsville, Illinois

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