Notre Dame Journal of Formal Logic
Volume XIV, Number 2, April 1973
NDJFAM

THE COMPLETENESS OF AN INTENSIONAL LOGIC: DEFINITE TOPOLOGICAL LOGIC

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1. *Introduction* In Rescher and Garson [3] a number of so-called topological logics, or logics of "position" were developed. The systems presented were in many ways similar to Rescher's chronological logics presented in [2]. These systems govern the behavior of an indexed operator T such that TxA is read 'It is the case as of x that A'. The index x may be given a wide range of interpretations compatible with these systems. It may represent a date, a spatial position, an n-tuple of coordinates (perhaps in the physicist's four-dimensional continuum), a spatial interval, a possible world, a set of postulates, an individual, and the like. It represents, in short, an unspecified context or "point of reference."

2. *Syntax for Topological Logic* The formation rules for topological logics may be defined as follows. Let P be a set $\{p_1, p_2, \ldots, p_n, \ldots\}$ of propositional variables, and let X be the set $\{x_1, x_2, \ldots, x_n, \ldots\}$ of context variables,[1] or indices associated with the T-operator. Here and in the rest of this paper, we use italicized lower case letters p and x as metavariables ranging over the corresponding sets P and X. We will also use y as a metavariable over X.

The set F of formulas of topological logic is the smallest set such that:

(i) $P \subseteq F$.
(ii) If A and B are members of F, then so are TxA, $\sim A$, $(A \supset B)$, $(A \,\&\, B)$, $(A \vee B)$, $(A \equiv B)$, $\forall xA$, $\exists xA$.

In this article we use italicized capital letters A, B, C, D as metavariables over formulas.

Notice that although we have introduced quantifiers, we have no predicate letters. The quantifiers range only over variables, or indices

1. We might also wish to introduce into the language a set of constants ranging over contexts. We omit them to simplify our discussion, but clearly their inclusion would not affect anything that follows in a fundamental way.

Received November 7, 1970

introduced by the T-operator. The introduction of predicate letters and individual variables results in a much more complicated system which will not be discussed here.

3. *Semantics for Topological Logic* It is natural to ask whether it is possible to discover acceptable semantics for systems of topological logic, and whether the systems presented in the literature are consistent and complete. This project was tackled in [1]. The investigation has borne fruit, since it has been found that the systems so far developed require strengthening in order to be complete.

We will report here only a portion of those results. The semantics to be defined here concerns itself primarily with the case where the indices associated with the T-operator are definite. We may define definite index on analogy with the notion of a definite sentence. A *sentence* is *definite* just in case its truth value does not vary as a function of the context in which the sentence is used, otherwise it is *indefinite*. The sentence 'It is snowing at 12:00, July 26, 1943 in Pittsburgh' is a definite sentence; however the sentence it contains 'It is snowing' is indefinite, for its truth value depends on the time and place of its utterance. Similarly an *index* is *definite* if and only if its denotation is not a function of the context of its use. Dates, for instance, are definite indices. 'July 26, 1943' denotes the same time, regardless of the context of its use. Temporal expressions like 'now' and 'forty years ago' are indefinite indices,[2] for the time denoted by these expressions depends on the time of their use.

If there is to be any point in introducing the T-operator into a system of logic, then at least some of our formulas must represent indefinite sentences. An example may help show this. Consider the sentences:

A Two plus two is equal to four.
B It is the case as of x that two plus two is equal to four.

B may be symbolized as TxA. But clearly B is true just in case A is true, for any value of x. Since A is definite, its truth value cannot vary as a function of the context. Since A is true, it must be true in all contexts equally. So in general, TxA and A are equivalent if A is definite. If all formulas of a logic represent definite sentences, then we ought to have $TxA \equiv A$ as a theorem. But this would render the T-operator useless, for given a rule of replacement of equivalents, T-operators may be deleted from any formula at will. So the T-operator is otiose unless some of the formulas represent indefinite sentences. This becomes more clear when we reflect that the function of the T-operator is to make distinctions between saying that A is the case as of x, and that A is the case as of y. The truth value of some of the formulas of our system ought to vary as we go from one context to another, otherwise we need no T-operator.

This brings out a feature of topological logics which would be an embarrassment to the "hard-headed" logician who insists that the truth of

2. Rescher refers to such expressions as pseudo-dates in [2].

a full-blooded sentence referred to by a logical formula must be context independent. The introduction of the T-operator only makes sense when at least some of our sentences are not full-blooded in this sense. This does not embarrass *us* however, for it is exactly the logical behavior of such indefinite sentences which we hope to clarify by introducing the T-operator.

Since the truth value of an indefinite sentence cannot be assessed apart from its context, the fundamental notion of our semantics[3] must be that of the value of a formula on an interpretation and valuation *in a context*, as opposed to the usual notion of the value of a formula on an interpretation and valuation alone.

A *TDQ-interpretation* $\mathfrak{I}$ is defined as an ordered pair $\langle D, I\rangle$ consisting of a domain D of contexts, and an interpretation function I defined from $D \times P$ into the set of truth values $\{\mathsf{T}, \bot\}$. $I(d, p)$ is the truth value of the propositional variable p in context d on the interpretation $\mathfrak{I}$. If we allow both definite and indefinite indices in a topological logic, then a valuation (on the variables) should be a function from $D \times X$ into D. Hence $v(d, x)$ would be the denotation of the variable x in the context d. We are assuming however that our indices are definite, so it follows that $v(d, x) = v(d', x)$ for all d and d'. Hence we may suppress the reference to the context in the valuation function and simply define a *valuation* v to be a function from X into D.

We may now define the value of a formula A in a context d, on the interpretation $\mathfrak{I}$ and valuation v, which we write $\mathsf{Val}(\mathfrak{I}, v, d)\, A$

(i) $\mathsf{Val}(\mathfrak{I}, v, d)\, p = I(d, p)$

(ii) $\mathsf{Val}(\mathfrak{I}, v, d)\, TxA = \mathsf{Val}(\mathfrak{I}, v, v(x))\, A$

(iii) $\mathsf{Val}(\mathfrak{I}, v, d) \sim A = \mathsf{T}$ iff $\mathsf{Val}(\mathfrak{I}, v, d)\, A = \bot$

(iv) $\mathsf{Val}(\mathfrak{I}, v, d)\, (A \supset B) = \mathsf{T}$ iff $\mathsf{Val}(\mathfrak{I}, v, d)\, A = \bot$ or $\mathsf{Val}(\mathfrak{I}, v, d)\, B = \mathsf{T}$ and similarly for the other connectives

(v) $\mathsf{Val}(\mathfrak{I}, v, d)\, \forall xA = \mathsf{T}$ iff $\mathsf{Val}(\mathfrak{I}, w, d)\, A = \mathsf{T}$ for all valuations w such that $w =_x v$, and similarly for the existential quantifier. ($w =_x y$ just in case $w(y) = v(y)$ for all y such that $y \neq x$.)

Clauses other than (ii) are fairly standard, and should be self-explanatory. In the general case where we have indefinite indices, clause (ii) should read $\mathsf{Val}(\mathfrak{I}, v, d)\, TxA = \mathsf{Val}(\mathfrak{I}, v, v(d, x))\, A$. For TxA is true in a context d just in case A is true in the context denoted by x in d. For example, 'It is raining as of forty years ago' is true in 1970 just in case 'It is raining' is true in the context which 'forty years ago' denotes in 1970 (i.e., in 1930). But since we assume our indices to be definite, our valuations need not take account of the context, and we arrive at the truth condition expressed in clause (ii).

A formula A is *true* in a context d on an interpretation $\mathfrak{I}$ just in case

3. More correctly, we might employ Montaque's term 'pragmatics' here, to refer to what concerns itself with the notion of *truth in a context*, and leave 'semantics' to refer to the study of notions of truth which are not context relative.

$\mathrm{Val}(\mathfrak{I}, v, d)\, A = \mathrm{T}$ on all valuations v. A formula A is valid ($\models A$) if and only if it is true on all interpretations and in all contexts.

4. *The System* TDQ We will now present a system TDQ which is consistent and complete with respect to the semantics just defined. TDQ consists of the axioms and rules of quantificational logic (hereafter referred to as Q.L.) plus the rule

(R) $\vdash A$ *yields* $\vdash TxA$

and the axioms

(A$\sim$) $\sim TxA \supset Tx{\sim}A$
($\sim$A) $Tx{\sim}A \supset {\sim}TxA$
(A$\supset$) $Tx(A \supset B) \supset (TxA \supset TxB)$
(AT) $TxA \supset TyTxA$
(AQ) $\forall xTyA \supset Ty\forall xA$, for $y \neq x$
(A$\exists$) $\exists x(TxA \equiv A)$.

Axioms (A$\sim$) and ($\sim$A) guarantee the equivalence of 'It is not the case as of x that A' and 'It is the case as of x that not A'. (A$\supset$) asserts that 'It is the case as of x that if A then B' entails 'If it is the case as of x that A, then it is the case as of x that B'. We may justify (AT) by appeal to clause (ii) of our semantics. $TyTxA$ is true in a context d just when TxA is true in the context denoted by y, and TxA is true in the context denoted by y if and only if A is true in the context denoted by x. But this is exactly the truth condition for TxA in the context d, hence if TxA is true in a context, then so is $TyTxA$.

The rule (R) and axioms (A$\sim$), ($\sim$A), (A$\supset$) and (AT) just discussed, together with the principles of propositional logic, form an unquantified definite topological logic which we may call TD. The converses of (A$\supset$) and (AT) are provable in TD, (and so also in TDQ); hence TD may be formulated alternatively with (R) and the equivalences

(E$\sim$) $\sim TxA \equiv Tx{\sim}A$
(E$\supset$) $Tx(A \supset B) \equiv (TxA \supset TxB)$
(ET) $TxA \equiv TyTxA$.

These equivalences together with the replacement of provable equivalents within the scope of the T-operator which is guaranteed by (R) (see Theorem 1 of section **5**), yield a system where the T-operator distributes through all the connectives, and where iterated T-operators are superfluous. It is not difficult to show that TD is consistent and complete with respect to a semantics like that presented in section **3** when clause (v) is deleted.

When we introduce the quantifiers, we need the axiom (AQ) to guarantee that the T-operator distributes through the quantifiers, as long as free variables are not bound, and bound variables not freed in the process. The converse of (AQ) is derivable in TDQ, hence we may postulate equivalently

(EQ) $\forall xTyA \equiv Ty\forall xA$, where $y \neq x$.

We require that $y \neq x$ because otherwise we would have formulas of the form $\forall xTxA \supset Tx\forall xA$ as theorems. Here the variable x, bound in the antecedent, has "escaped" in the consequent. This does not force us to reject the formula in itself; however, some of its instances are clearly unwanted. For example consider the formula $\forall xTx(p \supset Txp) \supset Tx\forall x(p \supset Txp)$. Given the principles discussed so far, the antecedent is equivalent to the theorem $\forall x(Txp \supset Txp)$, hence the whole formula is equivalent to its consequent. But the consequent is equivalent to $Txp \supset \forall xTxp$ which reads 'If p is the case as of x, then p is the case in any context'. Clearly this is not a logical truth. We may avoid this restriction on (AQ) by insisting in our formation rules that no single variable letter appear both bound and free in the same formula.

(A$\exists$) is the most interesting of all the axioms of TDQ, for the others merely guarantee the distribution of the T-operators through the logical constants, and the vacuousness of iterated T-operators. (A$\exists$) is the only axiom that connects formulas (like p) which contain no T-operators with those that do. It has the effect of asserting that for every formula A we may find an index x such that in a context the truth conditions for TxA are identical to those for A. An example may help motivate this. Consider 'It is snowing', used in the context January 1, 1970. Then there is another sentence of the form TxA, namely 'It is snowing as of January 1, 1970' which is true in the context January 1, 1970 if and only if 'It is snowing' is true in this context. Alternatively, we may justify this axiom by referring to clause (ii) of our semantics. TxA is true in the context d just when A is true in the context denoted by x. If x denotes d, then A and TxA are true under exactly the same conditions; hence there is a context x such that $TxA \equiv A$.

5. *Replacement, Normal Form and the Decision Problem for* TDQ Let us discuss some formal properties of TDQ which will be of help in the completeness proof to follow. First of all, we will prove that the rule (R) has the effect of allowing the replacement of provable equivalents of TDQ, included within the scope of the T-operator. (Their replacement elsewhere is guaranteed by principles of Q.L.)

Theorem 1. *The following rule of replacement* (RR) *is derivable in* TDQ: $\vdash A \equiv A'$ *yields* $\vdash B \equiv B'$, *where* B' *is like* B *save that an instance of* A *in* B *is replaced with an instance of* A' *in* B'.

The proof of Theorem 1 is by induction on the structure of B. Suppose $A \equiv A'$ is provable in TDQ. When B has one of the forms p, $\sim C$, $(C \supset D)$, $(C \& D)$, $(C \vee D)$, $(C \equiv D)$, $\exists xC$, $\forall xC$ the proof is trivial. If B has the form TxC, then $C \equiv C'$ is provable if $A \equiv A'$ is, by the hypothesis of the induction. Hence both $C \supset C'$ and $C' \supset C$ are provable in TDQ. From the first of these we may derive $TxC \supset TxC'$ by (R) and (A$\supset$), and from the second we obtain $TxC' \supset TxC$. Hence $TxC \equiv TxC'$ is provable in TDQ, and this is $B \equiv B'$.

We know that every formula of Q.L. is equivalent to a formula written with $\sim$, $\supset$, and $\forall$ as its only logical constants, such that no single variable letter appears in the formula both bound and free. Since (RR) is derivable in TDQ, it follows that every formula A of TDQ is equivalent to a formula $C(A)$ written with $\sim$, $\supset$, $\forall$, and T in which no single variable letter appears bound and free. Furthermore, we may show that every formula $C(A)$ of TDQ is equivalent to a formula $N(A)$ in *normal form* such that no connective, quantifier, or T-operator appears in the scope of any T-operator.

Let us introduce the metavariable t to range over strings $Tx_1Tx_2 \ldots Tx_n$ of T-operators and associated variables, including the null string. Then $N(A)$ may be defined for formulas of the form $C(A)$ as follows:

(i) $N(p) = p$
(ii) $N(tTxp) = Txp$
(iii) $N(t \sim A) = \sim N(tA)$
(iv) $N(t(A \supset B)) = (N(tA) \supset N(tB))$
(v) $N(t\forall xA) = \forall xN(tA)$.

$N(A)$ is essentially a formula of the monadic predicate calculus with *atoms* of the form p and Txp.

We must now prove that $C(A)$ and $N(A)$ are equivalent in TDQ.

Theorem 2. $\vdash_{\text{TDQ}} C(A) \equiv N(A)$.

The proof is by induction on the structure of $C(A)$. Every formula $C(A)$ of TDQ has one of the following forms: tp, $t \sim B$, $t(B \supset C)$, $t\forall xB$.

Case 1. $C(A)$ has the form tp. If t is null, then $N(p) = p$, and the equivalence is trivial. If t is not null, we may write $C(A)$ as $t'Txp$, where $t'Tx$ is t. $t'Txp$ is equivalent to Txp by as many applications of the equivalence (ET) and the rule of replacement (RR) as there are T-operators in t'. But $Txp = N(t'Txp)$, hence $t'Txp \equiv N(t'Txp)$ is provable in TDQ.

Case 2. $C(A)$ has the form $t \sim B$. $t \sim B \equiv \sim tB$ is provable in TDQ by applications of (E$\sim$) and (RR). By the hypothesis of the induction $tB \equiv N(tB)$ is provable in TDQ; hence, so is $\sim tB \equiv \sim N(tB)$. But we have shown $t \sim B \equiv \sim tB$ and $\sim N(tB) = N(t \sim B)$, so $t \sim B \equiv N(t \sim B)$ is provable.

Case 3. $C(A)$ has the form $t(B \supset C)$. Similar to case 2.

Case 4. $C(A)$ has the form $t\forall xB$. $C(A)$ contains no variable both bound and free, therefore x does not appear in t. Using (EQ) and (RR) we may derive $t\forall xB \equiv \forall xtB$. By the hypothesis of the induction $tB \equiv N(tB)$ is provable in TDQ; hence so is $\forall xtB \equiv \forall xN(tB)$. But $\forall xN(tB) = N(t\forall xB)$, and so $t\forall xB \equiv N(t\forall xB)$ is provable in TDQ.

Theorem 2 entails the following theorem which will be of use in proving the completeness of TDQ:

Theorem 3. *If A is not provable in* TDQ, *then $N(A)$ is not provable in* Q.L. *on the hypothesis $\exists y(N(TyA) \equiv N(A))$, where y is a variable not appearing in $N(A)$.*

Notice that the hypothesis mentioned is in normal form, and so it is, in essence, a formula of the monadic predicate calculus. We may prove Theorem 3 as follows. If A is not provable in TDQ then neither is $C(A)$, since A and $C(A)$ are equivalent. By Theorem 2, if $C(A)$ is not provable in

TDQ, then neither is $N(A)$. But if $N(A)$ is not provable in TDQ, then it could not be provable in Q.L. on the hypothesis $\exists y(N(TyA) \equiv N(A))$ because the principles of Q.L. and the hypothesis mentioned are present in TDQ.

The converse of Theorem 3 holds as well. (The proof is by induction of the proof of A, and is straightforward.) It follows that we may check for provability of a formula A of TDQ by asking whether $N(A)$ follows from $\exists y(N(TyA) \equiv N(A))$ in the monadic predicate calculus. So TDQ has a decision procedure.

6. *The Consistency and Completeness of* TDQ We will now prove that TDQ is consistent and complete with respect to the semantics of section **3**. To show consistency we prove

Theorem 4. *If* $\vdash A$ *in* TDQ *then* $\vDash A$.

The proof is by induction on the proof of A in TDQ.

Case 1. A is derived by a principle of Q.L. That the axioms of Q.L. are valid, and that the rules of Q.L. preserve validity is a consequence of the standard nature of clauses (iii)-(v) of the definition of $\mathsf{Val}(\mathfrak{I}, v, d)\ A$ and may be easily checked by the reader.

Case 2. A is derived using (R). We show that if A is valid, then so is TxA. By hypothesis $\mathsf{Val}(\mathfrak{I}, v, d)\ A = \mathsf{T}$ for arbitrary v, $\mathfrak{I}$, and d. It follows that $\mathsf{Val}(\mathfrak{I}, v, v(x))\ A = \mathsf{T}$ for all v and $\mathfrak{I}$. But $\mathsf{Val}(\mathfrak{I}, v, v(x))\ A = \mathsf{Val}(\mathfrak{I}, v, d)\ TxA$ for arbitrary d; hence $\mathsf{Val}(\mathfrak{I}, v, d)\ TxA = \mathsf{T}$ for all $\mathfrak{I}$, v and d.

Case 3. A is an instance of (A$\sim$), ($\sim$A), (A$\supset$), (AT). These cases are easily verified. We will present the proof for (AT) and leave the proof of the others to the reader.

$\mathsf{Val}(\mathfrak{I}, v, d)\ (TyTxB \supset TxB) = \mathsf{T}$
iff $\mathsf{Val}(\mathfrak{I}, v, d)\ TyTxB = \bot$ or $\mathsf{Val}(\mathfrak{I}, v, d)\ TxB = \mathsf{T}$
iff $\mathsf{Val}(\mathfrak{I}, v, v(y))\ TxB = \bot$ or $\mathsf{Val}(\mathfrak{I}, v, v(x))\ B = \mathsf{T}$
iff $\mathsf{Val}(\mathfrak{I}, v, v(x))\ B = \bot$ or $\mathsf{Val}(\mathfrak{I}, v, v(x))\ B = \mathsf{T}$.

But the value of B must either be T or $\bot$; hence $\mathsf{Val}(\mathfrak{I}, v, d)\ (TyTxB \supset TxB) = \mathsf{T}$.

Case 4. A is an instance of (AQ).

$\mathsf{Val}(\mathfrak{I}, v, d)\ (\forall xTyB \supset Ty\forall xB) = \mathsf{T}$
iff $\mathsf{Val}(\mathfrak{I}, v, d)\ \forall xTyB = \bot$ or $\mathsf{Val}(\mathfrak{I}, v, d)\ Ty\forall xB = \mathsf{T}$
iff $\mathsf{Val}(\mathfrak{I}, w, d)\ TyB = \bot$ on some valuation w such that $w =_x v$ or $\mathsf{Val}(\mathfrak{I}, v, v(y))\ \forall xB = \mathsf{T}$
iff $\mathsf{Val}(\mathfrak{I}, w, w(y))\ B = \bot$ on some valuation w such that $w =_x v$ or $\mathsf{Val}(\mathfrak{I}, w, v(y))\ B = \mathsf{T}$ on all w such that $w =_x v$.

Now the restriction on (AQ) is that $y \neq x$. Since $w =_x v$, it follows that $v(y) = w(y)$, hence the last statement in the above derivation is necessarily the case, and it follows that $\mathsf{Val}(\mathfrak{I}, v, d)\ (\forall xTyB \supset Ty\forall xB) = \mathsf{T}$. The latter part of this proof illustrates the necessity of the restriction on (AQ).

Case 5. A is an instance of (A$\exists$).

$\mathsf{Val}(\mathfrak{I}, v, d)\ \exists x(TxA \equiv A) = \mathsf{T}$

iff $\mathrm{Val}(\mathfrak{I}, w, d)\ (TxA \equiv A) = \top$ for some w such that $w =_x v$
iff $\mathrm{Val}(\mathfrak{I}, w, d)\ TxA = \mathrm{Val}(\mathfrak{I}, w, d)\ A$ for some w such that $w =_x v$
iff $\mathrm{Val}(\mathfrak{I}, w, w(x))\ A = \mathrm{Val}(\mathfrak{I}, w, d)\ A$ for some w such that $w =_x v$.

Let w be a valuation identical to v save that $w(x) = d$. Then on w $\mathrm{Val}(\mathfrak{I}, w, w(x))\ A = \mathrm{Val}(\mathfrak{I}, w, d)\ A$, and so $\mathrm{Val}(\mathfrak{I}, v, d)\ \exists x(TxA \equiv A) = \top$.

To prove TDQ complete we must demonstrate

Theorem 5. *If* $\vDash A$, *then* $\vdash A$ *in* TDQ.

This is proved by demonstrating the contrapositive: If not $\vdash A$ in TDQ then there is a TDQ-interpretation $\mathfrak{I}$, a valuation v, and a context d such that $\mathrm{Val}(\mathfrak{I}, v, d)A = \bot$. From Theorem 3 we know that if A is not provable in TDQ, then $N(A)$ is not provable in Q.L. on the hypothesis $\exists y(N(TyA) \equiv N(A))$, where y is not in $N(A)$. From this it follows that there is a monadic predicate logic interpretation $\mathfrak{M}$ and a valuation a on which $N(A)$ is false and on which $\exists y(N(TyA) \equiv N(A)$ is true. We may define the notion of a predicate logic interpretation for formulas in normal form in a fairly standard way as follows: an interpretation $\mathfrak{M} = \langle D, M_1, M_2\rangle$ where $M_1(p) \in \{\top, \bot\}$ and $M_2(Tp) \subseteq D$. Here we treat atoms of the form Txp as composed of a variable x and a predicate $T\ p$ (or Tp). The interpretation function M_2 ranges over the set of entities that satisfy the predicate $T\ p$, that is, over the contexts at which p is true. As before, a valuation a is a function from the set X into D. The value of a formula A in normal form on an interpretation $\mathfrak{M}$ and a valuation a (which we write $\mathrm{val}(\mathfrak{M}, a)A$) may be defined as follows:

(i) $\mathrm{val}(\mathfrak{M}, a)\ p = M_1(p)$
(ii) $\mathrm{val}(\mathfrak{M}, a)\ Txp = \top$ iff $a(x) \in M_2(Tp)$
(iii) $\mathrm{val}(\mathfrak{M}, a) \sim A = \top$ iff $\mathrm{val}(\mathfrak{M}, a)\ A = \bot$, and similarly for the other connectives
(iv) $\mathrm{val}(\mathfrak{M}, a)\ \forall xA = \top$ iff $\mathrm{val}(\mathfrak{M}, b)\ A = \top$ for all b such that $b =_x a$, and similarly for the existential quantifier.

We know that there is an interpretation $\mathfrak{M}$ and a valuation a on which $\mathrm{val}(\mathfrak{M}, a)\ N(A) = \bot$ and $\mathrm{val}(\mathfrak{M}, a)\ \exists y(N(TyA) \equiv N(A)) = \top$. From the latter it follows that there is a valuation b such that $b =_y a$ and such that $\mathrm{val}(\mathfrak{M}, b)$ $N(TyA) = \mathrm{val}(\mathfrak{M}, b)\ N(A)$. Now $\mathrm{val}(\mathfrak{M}, b)\ N(A) = \mathrm{val}(\mathfrak{M}, a)\ N(A)$, because $b =_y a$ and no y appears in $N(A)$. Therefore $\mathrm{val}(\mathfrak{M}, a)\ N(A) = \mathrm{val}(\mathfrak{M}, b)\ N(A) = \mathrm{val}(\mathfrak{M}, b)\ N(TyA) = \bot$. Now we may make use of the interpretation $\mathfrak{M}$ to define a TDQ-interpretation $\mathfrak{I}$ which has the property

P1. $\mathrm{val}(\mathfrak{M}, b)\ N(TyA) = \mathrm{Val}(\mathfrak{I}, b, b(y))\ N(A)$.

If we are successful, then $\mathrm{val}(\mathfrak{M}, a)\ N(A) = \mathrm{val}(\mathfrak{I}, b)\ N(TyA) = \mathrm{Val}(\mathfrak{I}, b, b(y))$ $N(A) = \bot$, hence we know that there is an interpretation, valuation and context on which $N(A)$ is false, hence $N(A)$ is invalid. Since $N(A) \equiv A$ is provable in TDQ (Theorem 2), and since TDQ is consistent (Theorem 4), it follows that A is false on exactly the interpretation, valuation and context on which $N(A)$ was false, and so A is not valid.

All that remains for the proof of Theorem 5 is to show that it is possible to define a TDQ-interpretation $\mathfrak{I}$ from $\mathfrak{M}$ so that P1 is the case. $\mathfrak{I}$ may be defined in terms of $\mathfrak{M}$ as follows: $\mathfrak{I} = \langle D, I\rangle$, where $I(d, p) = \mathsf{T}$ iff $d \in M_2(Tp)$. Now we must prove that $\mathfrak{I}$ satisfies P1.

Lemma 1. $\mathsf{val}(\mathfrak{M}, a)\ N(TyA) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(A)$ *for all valuations* a.

P1 follows as a special case of Lemma 1. The proof of this lemma is by induction on the structure of $N(A)$.

Case 1. $N(A)$ has the form p. Then $N(TyA) = N(Typ) = Typ$.
$\mathsf{val}(\mathfrak{M}, a)\ Typ = \mathsf{T}$ iff $a(y) \in M_2(Tp)$ iff $I(a(y), p) = \mathsf{T}$ iff $\mathsf{Val}(\mathfrak{I}, a, a(y))\ p = \mathsf{T}$. Since we have made no assumptions about a, it follows that $\mathsf{val}(\mathfrak{M}, a)$ $N(TyA) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(A)$ for arbitrary a in this case.

Case 2. $N(A)$ has the form Txp. Then $N(TyA) = N(TyTxp) = Txp$.
$\mathsf{val}(\mathfrak{M}, a)\ Txp = \mathsf{T}$ iff $a(x) \in M_2(Tp)$ iff $I(a(x), p) = \mathsf{T}$ iff $\mathsf{Val}(\mathfrak{I}, a, a(x))\ p = \mathsf{T}$ iff $\mathsf{Val}(\mathfrak{I}, a, a(y))\ p = \mathsf{T}$.
We have made no assumptions about a so $\mathsf{val}(\mathfrak{M}, a)\ N(TyA) = \mathsf{Val}(\mathfrak{I}, a, a(y))$ $N(A)$ for all a in this case.

Case 3. $N(A)$ has the form $N(\sim B)$. By the hypothesis of the induction $\mathsf{val}(\mathfrak{M}, a)\ N(TyB) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(B)$; hence $\mathsf{val}(\mathfrak{M}, a) \sim N(TyB) = \mathsf{Val}(\mathfrak{I}, a, a(y)) \sim N(B)$. But $\sim N(TyB) = N(Ty \sim B)$, and $\sim N(B) = N(\sim B)$ (for t null), so it follows that $\mathsf{val}(\mathfrak{M}, a)\ N(Ty \sim B) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(\sim B)$ for arbitrary a in this case.

Case. 4. $N(A)$ has the form $N((B \supset C))$. Similar to case 3.

Case 5. $N(A)$ has the form $N(\forall xB)$. By the hypothesis of the induction we are given that $\mathsf{val}(\mathfrak{M}, a)\ N(TyB) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(B)$ for all a. So $\mathsf{val}(\mathfrak{M}, c)\ N(TyB) = \mathsf{T}$ on all valuations c such that $c =_x a$ iff $\mathsf{Val}(\mathfrak{I}, c, c(y))$ $N(B) = \mathsf{T}$ on all c such that $c =_x a$. Since y is not in $N(A)$, $y \neq x$; hence $c(y) = a(y)$. So $\mathsf{val}(\mathfrak{M}, a)\ \forall xN(TyB) = \mathsf{Val}(\mathfrak{I}, a, c(y))\ \forall xN(B) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ \forall xN(B)$. But $\forall xN(TyB) = N(Ty\forall xB)$, and $\forall xN(B) = N(\forall xB)$. So $\mathsf{val}(\mathfrak{M}, a)\ N(Ty\forall xB) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(\forall xB)$; hence $\mathsf{val}(\mathfrak{M}, a)\ N(TyA) = \mathsf{Val}(\mathfrak{I}, a, a(y))\ N(A)$ for all a in this case.

7. *Relationships between* TDQ *and Systems Presented in "Topological Logic"* Both quantified axioms (AQ) and (A$\exists$) of TDQ are significantly stronger than their counterparts in [3]. It follows that the systems presented there are not complete with respect to the semantics of this paper, and it seem doubtful whether they are complete with respect to any reasonable semantics for topological logic.

The axiom of "Topological Logic" which corresponds to (AQ) is the weaker

(P3) $\forall xTyTxA \supset Ty\forall xTxA$ ([3], p. 538)

The axiom (AQ) allows the exchange of quantifiers and T-operators in contexts where this seems desirable, but where it is impossible given (P3) alone. For instance, we cannot prove $\forall xTy(p \supset Txp) \supset Ty\forall x(p \supset Txp)$ with (P3) because it demands that there be another T-operator immediately to the right of the point at which the quantifier and T-operator are exchanged.

For this reason Theorem 2 does not hold in a system which replaces (AQ) with (P3), and such a logic will not be complete.

The counterpart of (A$\exists$) in "Topological Logic" is

(P4) $\forall x Txp \supset p$.

(P4) entails both $\exists x(Txp \supset p)$ and $\exists x(p \supset Txp)$. But these together do not entail $\exists x(Txp \equiv p)$,[4] for we have no reason to suppose that the context y such that $Typ \supset p$ is identical to the context z such that $p \supset Tzp$. For this reason a system containing (P4) instead of (A$\exists$) is incomplete. Even if we were to postulate $\exists x(Txp \equiv p)$ for propositional variables p, we would still not gain the full effect of (A$\exists$). An instance of (A$\exists$) is $\exists x(Tx(p \supset q) \equiv (p \supset q))$ which is equivalent to $\exists x((Txp \supset Txq) \equiv (p \supset q))$. True, we may derive $\exists x(Txp \equiv p)$ and $\exists x(Txq \equiv q)$, but again we have no reason to suppose that the y such that $Typ \equiv p$ and the z such that $Tzq \equiv q$ are identical, which we must know if we are to prove $\exists x((Txp \supset Txq) \equiv (p \supset q))$.

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4. It is not difficult to produce an interpretation on which (P4) is true and (A$\exists$) is false.