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EFFECTIVE EXTENDABILITY AND FIXED POINTS

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Let α be any sequence and let $\varphi_1, \varphi_2, \ldots$ be a standard enumeration of the partial recursive functions. A p.r.f. δ is said to be a *fixed-point algorithm* for α if and only if $\delta(n)$ is an α-fixed point for φ_n (i.e., $n \in \mathrm{Dom}\, \delta$ and $\alpha(\delta(n)) = \alpha(\varphi_n(\delta(n)))$ whenever φ_n is total). α has the *effective fixed-point property* if and only if α has a total fixed-point algorithm. The purpose of this paper is to show that the effective fixed-point property is more properly viewed as an extendability property since:

(1) *α has the* e.f.p.p. *if and only if every partial recursive function ψ has a total recursive α-extension f* (i.e., $\alpha(f(n)) = \alpha(\psi(n))$ *for all* $n \in \mathrm{Dom}\, \psi$).
(2) *There is a sequence having a fixed-point algorithm but not the* e.f.p.p. (Hence totalness of the fixed-point algorithm is crucial to the e.f.p.p.)
(3) *If there is a total recursive function f such that $f(x)$ is an α-fixed point of φ_x whenever φ_x is total and constant, then α has the* e.f.p.p. (Hence the fixed points are somewhat incidental to the e.f.p.p. since every sequence has a nontotal algorithm which finds fixed points for constant functions, for example, $\lambda x[\varphi_x(1)]$.)

Proof of 1. See [3], Lemma 1.1.
Proof of 2. We let α be the canonical sequence of equivalence classes associated with the equivalence relation $\approx$ constructed below. Along with $\approx$ we construct a partial recursive function ψ having no total recursive α-extension. Thus α lacks the e.f.p.p. by (1).

Let $T_1, T_2, \ldots$ be a recursive sequence of disjoint infinite recursive sets. Members of T_x are called *test values* for φ_x. Let f be a one to one recursive enumeration of $\{\langle x, y\rangle \mid y \in \mathrm{Dom}\, \varphi_x\}$. We suppose that $\varphi_1, \varphi_2, \ldots$ are being constructed in stages so that $\varphi_x(y)$ becomes defined at stage $f^{-1}(\langle x, y\rangle)$ and at this stage we perform the following three steps in the construction of $\approx$ and ψ:

(Step 1) If φ_x does not already have an α-fixed point we give it one by letting $y \approx \varphi_x(y)$ provided that we do not thereby cause the violation of a prohibition of order x or less.

(Step 2) If y is a test value of φ_x and φ_x agrees, modulo $\approx$, with ψ

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wherever both are currently defined, then we force them to disagree at y by letting $\psi(y)$ be the smallest number not currently equivalent to $\varphi_x(y)$ under $\approx$ and place a prohibition of order x against the situation that $\varphi_x(y) \approx \psi(y)$.

(Step 3) Make sure there are x distinct numbers whose equivalence (pair wise) is prohibited by a currently unviolated prohibition of order x, adding new prohibitions when and only when necessary.

Notice that a prohibition of order x is violated only when a predecessor of φ_x in the sequence $\varphi_1, \varphi_2, \ldots$ acquires an α-fixed point—a situation which can occur only a finite number of times. Thus step 2 insures that no total φ_x can be an α extension of ψ, and step 3 guarantees that $\approx$ has an infinite number of equivalence classes. It follows that every total φ_x eventually gets an α-fixed point since y is prevented from becoming a fixed point of φ_x only if $y \approx z$ for some z involved in a prohibition or order x or less. But there are only finitely many such prohibitions while there are infinitely many $\approx$-equivalence classes. Obviously this construction can be done so that ψ is partial recursive and a fixed-point algorithm can be found for α. Q.E.D.

Proof of 3. Let f be a recursive function such that $f(n)$ is an α-fixed point of φ_n whenever φ_n is total and constant. Using the recursion theorem we obtain a number m such that

$$\varphi_m = \lambda n[f(g(n, m))]$$

where g is any total recursive function such that

$$\varphi_{g(n,m)} = \lambda x[\varphi_n(\varphi_m(n))].$$

Now if φ_n is total we have that

$$\alpha(\varphi_m(n)) = \alpha(f(g(n, m))) = \alpha(\varphi_{g(n,m)}(f(g(n,m)))) = \alpha(\varphi_n(\varphi_m(n))).$$

Thus φ_m is a total fixed-point algorithm for α. Q.E.D.

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