Notre Dame Journal of Formal Logic
Volume XIV, Number 1, January 1973
NDJFAM

A SET-THEORETIC MODEL FOR NONASSOCIATIVE NUMBER THEORY

D. BOLLMAN and M. LAPLAZA

1 *Introduction.* To our knowledge, the first reference to nonassociative numbers as an independent concept is in a paper of Etherington [3], in which it is related to some situations in biology. Recently, it has been shown in [1] and [7] that a suitable representation of nonassociative numbers can be a useful tool to solve some problems of coherence in the sense of [8]. Moreover, the set of nonassociative numbers is one of the simplest free algebras and can be used to give descriptions of nonassociative free algebras.

Formally, the theory **N** of nonassociative numbers bear similarities to those of the theory of natural numbers. In [4], Evans characterized the nonassociative numbers by a set of "Peano-like" axioms. In [2], these axioms were formalized and following a suggestion of Evans, it was shown that **N** is incomplete and furthermore that it is essentially undecidable. It is natural to ask if and how **N** can be formalized within formal set theory, say Zermelo-Fraenkel (ZF). In the present work we do exactly this. Furthermore, by considering variations of this model, we show that the axioms of **N** are independent.

The representations of **N** by coordinates in [1] and [7] offer the possibility of constructing other models for **N**, but they would be more complicated than ours. In this connection we refer to Freyd's Adjoint Theorem [5], one of whose consequences is the existence of free algebras, which therefore also gives a way to construct a model for **N**, but this too would be quite sophisticated.

2 *A model for nonassociative number theory.* In [2], nonassociative number theory is defined to be the first-order theory with equality, **N**, having one individual constant 1, three binary function letters corresponding to addition (+), multiplication (·), and exponentiation and whose proper axioms are:

(N1) $x_1 + x_2 \neq 1$
(N2) $x_1 + x_2 = x_3 + x_4 \,.\supset.\, x_1 = x_3 \wedge x_2 = x_4$

Received October 7, 1971

(N3) $x_1 \cdot 1 = x_1$
(N4) $x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$
(N5) $x_1^1 = x_1$
(N6) $x_1^{x_2+x_3} = x_1^{x_2} \cdot x_1^{x_3}$
(N7) (Nonassociative Induction) *For any well-formed* (wff) $\mathcal{A}(x)$ *of* **N**, $\mathcal{A}(1) \wedge (x_1)(x_2)(\mathcal{A}(x_1) \wedge \mathcal{A}(x_2) \,.\supset.\, \mathcal{A}(x_1 + x_2)) \,.\supset.\, (x_1)\, \mathcal{A}(x_1)$.

Consider the set P defined as follows: Let $P_1 = \{1\}$ where $1 = \emptyset$ and for each natural number n let $P_{n+1} = P_n \cup \{\langle a, b\rangle \mid a \in P_n \wedge b \in P_n\}$. Then let $P = \bigcup_{n \in \omega} P_n$, where ω denotes the set of positive integers. Also define $a + b = \langle a, b\rangle$ for each $a \in P$ and $b \in P$.

It is easy to see that under this interpretation, axioms (N1) and (N2) is satisfied. It also follows that (N7) is satisfied. For suppose that

$$\mathcal{A}(1) \,.\wedge.\, (x_1 \in P)(x_2 \in P)(\mathcal{A}(x_1) \wedge \mathcal{A}(x_2) \,.\supset.\, \mathcal{A}(x_1 + x_2)) \,.$$

Then $\mathcal{A}(x)$ for each $x \in P_1$. Furthermore, if $\mathcal{A}(x)$ for each $x \in P_n$ and if $x \in P_{n+1}$ then $x \in P_n$ or $x = \langle a, b\rangle$ for some $a \in P_n$ and $b \in P_n$. In either case, $(x \in P_{n+1})\, \mathcal{A}(x)$. Hence $(x \in P_n)\, \mathcal{A}(x)$ for each $n \in \omega$ and so $(x \in P)\, \mathcal{A}(x)$. Thus

$$\vdash_{ZF} \mathcal{A}(1)(x_1 \in P)(x_2 \in P)(\mathcal{A}(x_1) \wedge \mathcal{A}(x_2) \,.\supset.\, \mathcal{A}(x_1 + x_2)) \,.\supset.\, (x_1 \in P)\mathcal{A}(x_1) \,.$$

To prove that the other axioms are satisfied under the given interpretation we need only prove a theorem of nonassociative recursion in ZF. Informally, we must prove that if $f\colon x \times x \to x$ and if $a \in x$ then there exists a unique function $\mathbf{t}\colon P \to x$ such that $\mathbf{t}(1) = a$ and $\mathbf{t}(m + n) = f(\mathbf{t}(m), \mathbf{t}(n))$. Formally, we use the notation of Hatcher [6] (*cf.*, in particular, page 186): $\mathcal{J}(x)$ is the wff of ZF which says that x is a function, $\mathsf{D}(x)$ is the term of ZF which denotes the domain of a function x, $\mathsf{I}(x)$ is the term of ZF which denotes the range of a function x, and $x_1{}'' x_2$ is the term of ZF which denotes the image of x_2 under the function x_1.

Theorem of Nonassociative Recursion:

$$\vdash_{ZF}(x_1)(x_2)(\mathcal{J}(x_1) \,.\wedge.\, \mathsf{I}(x_1) \times \mathsf{I}(x_1) \subset \mathsf{D}(x_1) \,.\wedge.\, \{x_2\} \times \mathsf{I}(x_1) \subset \mathsf{D}(x_1) \,.\wedge.\, \mathsf{I}(x_1) \times \{x_2\} \subset \mathsf{D}(x_1) \,.\wedge.\, \langle x_2, x_2\rangle \in \mathsf{D}(x_1) .:\supset.\, (\mathrm{E}!x_3)(\mathcal{J}(x_3) \,.\wedge.\, P = \mathsf{D}(x_3) \,.\wedge.\, \mathsf{I}(x_3) \times \mathsf{I}(x_3) \subset \mathsf{D}(x_1) \,. \wedge.\, x_3{}''\, 1 = x_2 \,.\wedge.\, (x_4)(x_5)(\langle x_4, x_5\rangle \in P \,.\supset.\, x_3{}''\langle x_4, x_5\rangle \langle x_4, x_5\rangle = x_1{}''\langle x_3{}'' x_4, x_3{}'' x_5\rangle))) \,.$$

Proof. Suppose

$$\mathcal{J}(x_1) \,.\wedge.\, \mathsf{I}(x_1) \times \mathsf{I}(x_1) \subset \mathsf{D}(x_1) \,.\wedge.\, \{x_2\} \times \mathsf{I}(x_1) \subset \mathsf{D}(x_1) \,.\wedge.\, \mathsf{I}(x_1) \times \{x_2\} \subset \mathsf{D}(x_1) \,. \wedge.\, \langle x_2, x_2\rangle \in \mathsf{D}(x_1)$$

and let

$$\mathcal{E} = \{x_5 \mid x_5 \in \mathcal{P}(P \times \mathsf{D}(x_1)) \,.\wedge.\, \langle 1, x_2\rangle \in x_5 \,.\wedge.\, (x_6)(x_7)(x_8)(x_9)(\langle x_6, x_7\rangle \in x_5 \,. \wedge.\, \langle x_8, x_9\rangle \in x_5 :\supset.\, \langle\langle x_6, x_8\rangle,\ x_1{}''\langle x_7, x_9\rangle\rangle \in x_5)\}$$

and

$$\mathbf{t} = \bigcap \mathcal{E} \,.$$

We first show that $(u)(u \epsilon P \supset (\mathrm{E}!v)(\langle u, v\rangle \epsilon \mathbf{t})$. First suppose that $\langle 1, v\rangle \epsilon \mathbf{t} \wedge v \neq x_2$. Let $\mathsf{L} = \{w | w \epsilon \mathbf{t} \wedge w \neq \langle 1, v\rangle\}$. Then $\mathsf{L} \subset \mathbf{t} \subset P \times \mathsf{D}(x_1)$. Also, since $v \neq x_2$, $\langle 1, x_2\rangle \epsilon \mathsf{L}$. Furthermore, if $\langle x_6, x_7\rangle \epsilon \mathsf{L} .\wedge. \langle x_8, x_9\rangle \epsilon \mathsf{L}$ then $\langle\langle x_6, x_8\rangle, x_1{}''\langle x_7, x_9\rangle\rangle \epsilon \mathsf{L}$ since $\langle x_6, x_8\rangle \neq 1$. Hence $\mathsf{L} \epsilon \mathcal{E}$ and so $\mathbf{t} \subset \mathsf{L}$, which is a contradiction, since $\langle 1, v\rangle \epsilon \mathbf{t}$ and $\langle 1, v\rangle \notin \mathsf{L}$. Hence $\langle 1, v\rangle \epsilon \mathbf{t}$ implies that $v = x_2$ and we have thus proved that $(\mathrm{E}!v)(\langle 1, v\rangle \epsilon \mathbf{t})$.

Now suppose that $u_1 \epsilon P .\supset. (\mathrm{E}!v_1)(\langle u_1, v_1\rangle \epsilon \mathbf{t})$, $u_2 \epsilon P .\supset. (\mathrm{E}!v_2)(\langle u_2, v_2\rangle \epsilon \mathbf{t})$, and that $u_1 \epsilon P .\wedge. u_2 \epsilon P$. Then $(\mathrm{E}!v_1)(\langle u_1, v_1\rangle \epsilon \mathbf{t})$ and $(\mathrm{E}!v_2)(\langle u_2, v_2\rangle \epsilon \mathbf{t})$ and so $\langle\langle u_1, u_2\rangle, x_1{}''\langle v_1, v_2\rangle\rangle \epsilon \mathbf{t}$. Suppose $\langle\langle u_1, u_2\rangle, v\rangle \epsilon \mathbf{t}$, where $v \neq x_1{}''\langle v_1, v_2\rangle$. Let $\mathsf{K} = \{u | u \epsilon \mathbf{t} .\wedge. u \neq \langle\langle u_1, u_2\rangle, v\rangle\}$. Then $\langle 1, x_2\rangle \neq \langle\langle u_1, u_2\rangle, v\rangle$ and since $\langle 1, x_2\rangle \epsilon \mathbf{t}$, $\langle 1, x_2\rangle \epsilon \mathsf{K}$. Suppose $\langle x_6, x_7\rangle \epsilon \mathsf{K}$ and $\langle x_8, x_9\rangle \epsilon \mathsf{K}$. Then $\langle\langle x_6, x_8\rangle, x_1{}''\langle x_7, x_9\rangle\rangle \epsilon \mathbf{t}$. Futhermore, $\langle\langle x_6, x_8\rangle, x_1{}''\langle x_7, x_9\rangle\rangle = \langle\langle u_1, u_2\rangle, v\rangle \Rightarrow \langle x_6, x_8\rangle = \langle u_1, u_2\rangle$ and $x_1{}''\langle x_7, x_9\rangle = v \Rightarrow x_6 = u_1$ and $x_8 = u_2 \Rightarrow \langle u_1, x_7\rangle \epsilon \mathsf{K}$ and $\langle u_2, x_9\rangle \epsilon \mathsf{K} \Rightarrow x_7 = v_1$ and $x_9 = v_2 \Rightarrow x_1{}''\langle v_1, v_2\rangle = v$, a contradiction. Hence:

$\langle\langle x_6, x_8\rangle, x_1{}''\langle x_7, x_9\rangle\rangle \neq \langle\langle u_1, u_2\rangle, v\rangle$ and thus $\langle\langle x_6, x_8\rangle, x_1{}''\langle x_7, x_9\rangle\rangle \epsilon \mathsf{K}$. Also $\mathsf{K} \subset \mathbf{t} \subset P \times \mathsf{D}(x_1)$ and so $\mathbf{t} \subset \mathsf{K}$. But this is a contradiction since $\langle\langle u_1, u_2\rangle, v\rangle \epsilon \mathbf{t}$ but $\langle\langle u_1, u_2\rangle, v\rangle \notin \mathsf{K}$. Hence $\langle\langle u_1, u_2\rangle, v\rangle \epsilon \mathbf{t} \Rightarrow v = x_1{}''\langle v_1, v_2\rangle$ and we have thus proved $(\mathrm{E}!v)(\langle\langle u_1, u_2\rangle, v\rangle \epsilon \mathbf{t})$. Hence by nonassociative induction, $(u)(u \epsilon P .\supset. (\mathrm{E}!v)(\langle u, v\rangle \epsilon \mathbf{t})$ and so $\mathcal{F}(\mathbf{t})$.

We also have that $\mathsf{D}(\mathbf{t}) \subset P$; by definition of $\mathbf{t}$ and that $P \subset \mathsf{D}(\mathbf{t})$, by definition of $\mathbf{t}$ and by nonassociative induction. Hence $P = \mathsf{D}(x_3)$. Furthermore, if $u \epsilon \mathsf{I}(\mathbf{t})$ then $u = x_2$ or $u = x_1{}''\langle \mathbf{t}''x_6, \mathbf{t}''x_8\rangle$ for some $\langle x_6, x_8\rangle \epsilon P \times P$. Hence

$$\mathsf{I}(\mathbf{t}) \times \mathsf{I}(\mathbf{t}) .\subset. \{\langle x_2, x_2\rangle\} \cup \{x_2\} \times \mathsf{I}(x_1) \cup \mathsf{I}(x_1) \times \{x_2\} \cup \mathsf{I}(x_1) \times \mathsf{I}(x_1)$$

and so $\mathsf{I}(\mathbf{t}) \times \mathsf{I}(\mathbf{t}) \subset \mathsf{D}(x_1)$. Finally,

$$\mathbf{t}''\,1 = x_2 .\wedge. (x_4)(x_5)(\langle x_4, x_5\rangle \epsilon P .\supset. \mathbf{t}''\langle x_4, x_5\rangle = x_1{}''\langle \mathbf{t}''x_4, \mathbf{t}''x_5\rangle)$$

by definition of $\mathbf{t}$ and so the theorem is proved.

Now, on the basis of this theorem, we can define operations, $\cdot$ and exp, where $\exp(x, y) = x^y$, on P, by means of (N3)-(N4) and (N5)-(N6) respectively. Consequently, we have

Proposition 1: *$\langle P, +, \cdot, \exp, 1\rangle$ is a model for* **N**.

3 *Independence.*

Proposition 2. *The axioms of* **N** *are independent.*

Proof. For each of the seven axioms of **N**, we exhibit an interpretation of **N** in which all of the axioms except the given one hold:

(N1) Let the domain of interpretation be $\{1\}$ and define $1 + 1 = 1 \cdot 1 = 1^1 = 1$.
(N2) Let ω be the domain and define $+$, $\cdot$, exp, and 1 as usual.
(N3) Let P be the domain. Define $\mathbf{t}$, exp, and 1 as before and define $x_1 \cdot 1 = 1$ and $x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$.
(N4) Let P be the domain. Define $+$, exp, and 1 as before and define $x_1 \cdot x_2 = x_1$.

(N5) Let P be the domain. Define $+$, $\cdot$, and 1 as before and define $x_1^1 = 1$ and $x_1^{x_2+x_3} = x_1^{x_2} \cdot x_1^{x_3}$.

(N6) Let P be the domain. Define $+$, $\cdot$, and 1 as before and define $x_1^{x_2} = x_1$.

(N7) Let $P \times P$ be the domain, define 1 as $\langle 1, 1\rangle$, $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle$, $\langle x_1, y_1\rangle \cdot \langle x_2, y_2\rangle = \langle x_1 \cdot x_2, y_1 \cdot y_2\rangle$, and $\langle x_1, y_1\rangle^{\langle x_2, y_2\rangle} = \langle x_1^{x_2}, y_1^{y_2}\rangle$,

where the operations indicated within the ordered pairs are those defined in the model $\langle P, +, \cdot, \exp, 1\rangle$ of $\mathbf{N}$. Let $\mathcal{A}(x)$ be the wff $x = 1 \vee (\mathrm{E}x_1)(\mathrm{E}x_2)(x = x_1 + x_2)$. Then $\mathcal{A}(1)$ and $(x_1)(x_2)(\mathcal{A}(x_1) \wedge \mathcal{A}(x_2) \supset \mathcal{A}(x_1 + x_2))$ but $\sim (x)\mathcal{A}(x)$ since $\sim\mathcal{A}(\langle 1, 2\rangle)$.

REFERENCES

[1] Bénabou, J., "Structures Algébriques dans les Catégories," *Cahiers de Topologie et Géomêtrie Différentielle*, vol. X (1968), pp. 1-126.

[2] Bollman, D., "Formal Nonassociative Number Theory," *Notre Dame Journal of Formal Logic*, vol. VIII (1967), pp. 9-16.

[3] Etherington, I. M. H., "Non-Associative Arithmetics," *Proceedings of the Royal Society of Edinburg*, vol. 62 (1949), pp. 442-453.

[4] Evans, T., "Nonassociative Number Theory," *American Mathematical Monthly*, vol. 64 (1957), pp. 299-309.

[5] Freyd, P., *Abelian Categories*, Harper & Row (1964).

[6] Hatcher, W., *Foundations of Mathematics*, W. B. Saunders (1968).

[7] Laplaza, M., "Coherence for Associativity not an Isomorphism" (to appear).

[8] MacLane, S., "Coherence and Canonical Maps," *Symposia Mathematica*, vol. IV (1970), pp. 231-242.

University of Puerto Rico
Mayagüez, Puerto Rico