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# THE COSUBSTITUTION CONDITION

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1. *Introduction*. Let $n$ be a natural number, $n \geq 2$, and $N = \{1, 2, \ldots, n\}$. Martin [3] showed in 1954 that necessary conditions for a two-place functor to be a Sheffer function are that it should possess none of the properties of proper closure, $t$-closure, proper substitution or cosubstitution. He proved these conditions sufficient in the 3-valued case. Foxley [1] demonstrated that, in the 3-valued case, any function which possessed the cosubstitution property must also possess at least one of the properties of proper closure, proper substitution or $t$-closure. We shall establish the corresponding result for $n$-valued logic. Initially we establish a necessary and sufficient condition for a function to be $t$-closing. By investigating the conditions implied by the cosubstitution property it will follow that if $F$ possesses the cosubstitution property for a decomposition of the $n$ truth values into less than $n$ classes then it will also possess the proper substitution property for the same decomposition. In the remaining case of a decomposition of the $n$ truth values into exactly $n$ classes it will be shown that $F$ will possess at least one of the properties of proper closure, proper substitution or $t$-closure if it possesses the cosubstitution property for such a decomposition.

Before proceeding any further we will introduce definitions of these terms as given by Martin [3]. Suppose we have a decomposition of the $n$ marks into two or more disjoint, non-empty classes. If $a, b \in N$ we write $a \sim b$ to indicate that $a$ and $b$ are elements of the same class. Let $a'$, $b'$, $c'$, $d'$, $e'$, $f'$ be logical constants taking the truth values $a$, $b$, $c$, $d$, $e$, $f$ respectively ($a, b, c, d, e, f \in N$). A binary functor $F$ satisfies the *substitution law* if, for any $a, b, c, d$, whenever $a \sim c$ and $b \sim d$ then $e \sim f$ where $Fa'b' =_T e'$ and $Fc'd' =_T f'$. If $F$ is a binary functor such that whenever $e \sim f$ and $Fa'b' =_T e'$, $Fc'd' =_T f'$ then either $a \sim c$ or $b \sim d$ then $F$ satisfies the *cosubstitution law*. We say $F$ has the *proper substitution property* if there is a decomposition of the $n$ truth values into less than $n$ classes for which $F$ satisfies the substitution law. Similarly $F$ has the *cosubstitution property* if there is a decomposition of the $n$ truth values for which $F$ satisfies the cosubstitution law.

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A one-place functor $T$ is a *t-functor* if the following hold.

(I) $T^n a' =_T a'$ for all $a \in N$

(II) for every $i$ $(1 \leq i \leq n - 1)$ and $a (a \in N)$ $T^i\ a' \neq_T a'$

$F$ is said to be *t-closing* if there is some $t$-functor $F$ such that for every $i, j$ there is a $k$ such that $FT^i pT^j p =_T T^k p$. Finally $F$ has the *proper closure* property if some non-empty proper subset of the $n$ marks is closed under $F$.

2. *Notation and Definitions*. We use the notation

$$\{A_1\}, \{A_2\}, \ldots, \{A_g\} \subseteq \{B_1\}, \{B_2\}, \ldots, \{B_h\}$$

where $A_i$, $B_j$ $(1 \leq i \leq g; 1 \leq j \leq h)$ represent sequences of symbols denoting truth values, and for any sequence $C$, $\{C\}$ is the class of truth values denoted by the elements of $C$, to mean that there exists an integer $j$ $(= j(i))$ such that $\{A_i\}\ c\ \{B_j\}$ for each $i = 1, 2, \ldots, g$. If the values $j(1), j(2), \ldots, j(g)$ are all necessarily distinct we shall write

$$\{A_1;\ A_2;\ \ldots;\ A_g\} \subseteq \{B_1;\ B_2;\ \ldots;\ B_h\}.$$

If we have $g^2$ sequences $A_{i_1 i_2}$ $(1 \leq i_1, i_2 \leq g)$ then

$$\{A_{11} : A_{12} : \ldots : A_{1g}; A_{21} : A_{22} : \ldots : A_{2g}; \ldots; A_{g1} : A_{g2} : \ldots : A_{gg}\} \subseteq \{B_1; B_2; \ldots; B_h\}$$

is used to mean that there exists an integer $j$ $(= j(i_1))$ such that, for all $i_2$ $(1 \leq i_2 \leq g)$, $\{A_{i_1 i_2}\} \subseteq \{B_j\}$ with $j(1), j(2), \ldots j(g)$ necessarily all distinct.

Consider a decomposition $D$ of the $n$ truth values $1, 2, \ldots, n$ into $m$ non-empty classes $(1 \leq m \leq n)$ where the $i$th class contains $s_i$ truth values $(i = 1, 2, \ldots, m)$. If $a'$, $b'$, $c'$, $d'$, $e'$, $f'$ are logical constants assuming the truth values $a, b, c, d, e, f$ respectively $(a, b, c, d, e, f \in \{1, 2, \ldots, n\})$ such that $Fa'b' =_T e'$ and $Fc'd' =_T f'$ and the corresponding truth values $e, f$ are such that $e \sim f$ in the decomposition $D$ then we write $Fa'b' \sim Fc'd'$.

The $m$ classes of truth values will be denoted by

$$\{a_{11}, a_{12}, \ldots, a_{1s_1}\}, \{a_{21}, a_{22}, \ldots, a_{2s_2}\}, \ldots, \{a_{m1}, a_{m2}, \ldots, a_{ms_m}\}$$

where $a_{11}, a_{12}, \ldots, \ldots, a_{ms_m}$ is some rearrangement of $1, 2, \ldots, n$. $\{A_i\}$ is used to denote $\{a_{i1}, a_{i2}, \ldots, a_{is_i}\}$ $(1 \leq i \leq m)$. We shall write $\{B_{ij}\}$ $(1 \leq i, j \leq m)$ to denote

$$\{Fb_{i1}\,b_{j1}, Fb_{i1}\,b_{j2}, \ldots, Fb_{i1}\,b_{js_j}, Fb_{i2}\,b_{j1}, Fb_{i2}\,b_{j2}, \ldots, \ldots, Fb_{is_i}\,b_{js_j}\}$$

where $b_{11}$, $b_{12}, \ldots, b_{ms_m}$ are logical constants taking truth values $a_{11}$, $a_{12}, \ldots, a_{ms_m}$ respectively.

If $Fb_{ik}\ b_{jl} =_T c_{kl}^{ij}$ where $1 \leq i, j \leq m$; $1 \leq k \leq s_i$; $1 \leq l \leq s_j$ and $c_{kl}^{ij} \in \{b_{11}, b_{12}, \ldots, b_{ms_m}\}$ then we use $\{C_{ij}\}$ to denote

$$\{c_{11}^{ij}, c_{12}^{ij}, \ldots, c_{1s}^{ij}, c_{21}^{ij}, c_{22}^{ij}, \ldots, \ldots, c_{s_i s_j}^{ij}\}$$

and $\{D_{ij}\}$ to denote the class of truth values corresponding to the elements of $\{C_{ij}\}$.

We write $\{D_{ij}\} \sim \{D_{i'j'}\}$ to mean that there exists an integer $k$ $(k = k(i, j), 1 \leq k \leq m)$ such that $\{D_{ij}\}, \{D_{i'j'}\} \subseteq \{A_k\}$. A number of truth values are described as "*similar*" if they belong to the same class of the decomposition. For $i = 1, 2, \ldots, m$ the $i$th "*block row*" is defined to be $\bigcup_{j=1}^{m} \{D_{ij}\}$. Similarly for $j = 1, 2, \ldots, m$ the jth "*block column*" is $\bigcup_{i=1}^{m} \{D_{ij}\}$.

3. *Conditions for t-closure.* If $Tp$ is a $t$-function we define $T^0 p = p$ and $T^{k+1} p = TT^k p$ $(k = 0, 1, 2, \ldots)$.

The following theorem is stated in two forms, the proof of the second following from that of the first by interchanging rows and columns in the argument.

Theorem 3.1. (A) *A functor F is t-closing if and only if there exist integers* $i_0, i_1, \ldots, i_{n-1}$ *such that, for each* $j = 0, 1, \ldots, n-1$, $FpT^j p =_T T^{i_j} p$ *where* $0 \leq i_j \leq n - 1$.

(B) *A functor F is t-closing if and only if there exist integers* $i_0, i_1, \ldots, i_{n-1}$ *such that, for each* $j = 0, 1, \ldots, n-1$, $FT^j pp =_T T^{i_j} p$ *where* $0 \leq i_j \leq n - 1$.

We prove the result in form A. Necessity of the condition follows from the definition of $t$-closure. For sufficiency we must show, for all values of $k (0 \leq k \leq n-1)$ and for all values of $l (0 \leq l \leq n-1)$, that for some value of $j = j(k, l)$ $FT^k pT^l p =_T T^j p$.

(a) We have $FT^k pT^k p = T^{i_0} T^k p =_T T^{i_0 + k} p$ for each value of $k$, $0 \leq k \leq n - 1$.

(b) Consider $FT^k pT^l p$, with $k \neq l$ and $0 \leq k, l \leq n-1$; define $j \equiv n + l - k$ (modulo $n$), $0 \leq j \leq n-1$. Then either (I) $j + k = l$ or (II) $j + k = n + l$.

If (I) then
$$\begin{aligned} FT^k pT^l p &=_T FT^k pT^{j+k} p \\ &=_T T^{i_j} T^k p \\ &=_T T^{i_j + k} p. \end{aligned}$$

If (II) then
$$\begin{aligned} FT^k pT^l p &=_T FT^k pT^n T^l p \\ &=_T FT^k pT^{j+k} p \\ &=_T T^{i_j + k} p \text{ as above.} \end{aligned}$$

(c) For any general value of $k$ we note that $T^k p =_T T^{k'} p$ where $k' \equiv k \pmod n$, $0 \leq k' \leq n - 1$.

Consequently we have that for all $a, b$ there exists an integer $c$ such that $FT^a pT^b p =_T T^c p$, and the result follows.

4. *The Cosubstitution Property.*

Lemma 4.1. *If F possesses the cosubstitution property then* $\{D_{11}; D_{22}; \ldots; D_{mm}\} \subseteq \{A_1; A_2; \ldots; A_m\}$.

*Proof.* If for some $i, j$ $(1 \leq i, j \leq m)$ $Fb_{iu} b_{iv} \sim Fb_{jw} b_{jx}$ for any $u, v, w, x$ such that $1 \leq u, v \leq s_i$ and $1 \leq w, x \leq s_j$ then the cosubstitution property implies that either $a_{iu} \sim a_{jw}$ or $a_{iv} \sim a_{jx}$ and, in both cases, this implies $i = j$. Consequently, since there are exactly $m$ classes in the decomposition, the lemma follows.

Theorem 4.2. *If F possesses the cosubstitution property then either all the entries in each block row are similar or all the entries in each block column are similar.*

To establish this result it will be shown that either one or the other of the following hold.

I $\{D_{11}:D_{12}:\ldots:D_{1m};D_{21}:D_{22}:\ldots:D_{2m};\ldots;D_{m1}:D_{m2}:\ldots:D_{mm}\}\subseteq\{A_1;A_2;\ldots;A_m\}$
II $\{D_{11}:D_{21}:\ldots:D_{m1};D_{12}:D_{22}:\ldots:D_{m2};\ldots;D_{1m}:D_{2m}:\ldots:D_{mm}\}\subseteq\{A_1;A_2;\ldots;A_m\}$.

In the truth table of $F$ consider the $a_{iu_i}$-th row (for some $u_i$, $1 \le u_i \le s_i$, $1 \le i \le m$). Consider the entry in the $a_{ju_j}$-th column ($j \ne i$, some $u_j$, $1 \le u_j \le s_j$, $1 \le j \le m$). We have $Fb_{iu_i}b_{ju_j} =_T c^{ij}_{u_iu_j}$ and let the truth value corresponding to $c^{ij}_{u_iu_j}$ be denoted by $\alpha$.

Then, in view of the above lemma, there exist $\theta$, $k$ such that $\alpha \sim \theta$ and $\theta \in \{D_{kk}\}$, $(1 \le k \le m)$. Hence, we have $Fb_{iu_i}b_{ju_j} \sim Fb_{ku_k}b_{kv_k}$ for all $u_k$, $v_k$ such that $1 \le u_k$, $v_k \le s_k$. Then the cosubstitution property implies that either $a_{iu_i} \sim a_{ku_k}$ or $a_{ju_j} \sim a_{kv_k}$ giving either $i = k$ or $j = k$. Consequently if $\alpha \in \{D_{ij}\}$ then there exists $\theta$ such that $\alpha \sim \theta$ and

$$\theta \in \{D_{ii}\} \text{ or } \theta \in \{D_{jj}\}. \tag{1}$$

We now show that if $\alpha$, $\beta \in \{D_{ij}\}$ $(i \ne j)$ then it is not possible for $\alpha \sim \gamma$ and $\beta \sim \delta$ where $\gamma$, $\delta$ are the truth values corresponding to the values of $Fb_{iw_i}b_{ix_i}$, $Fb_{jw_j}b_{jx_j}$ respectively for any $w_i$, $x_i$, $w_j$, $x_j$ $(1 \le w_i, x_i \le s_i;\ 1 \le w_j, x_j \le s_j)$.

Suppose for some $u_i$, $u_j$, $v_i$, $v_j$ $(1 \le u_i,\ v_i \le s_i;\ 1 \le u_j,\ v_j \le s_j)$

that $$Fb_{iu_i}b_{ju_j} \sim Fb_{iw_i}b_{ix_i} \tag{2}$$
and $$Fb_{iv_i}b_{jv_j} \sim Fb_{jw_j}b_{jx_j}. \tag{3}$$

Consider $Fb_{ju_j}b_{iv_i}$: by (1) either (a) $Fb_{ju_j}b_{iv_i} \sim Fb_{iy_i}b_{iz_i}$ for some $y_i$, $z_i$ $(1 \le y_i, z_i \le s_i)$ or (b) $Fb_{ju_j}b_{iv_i} \sim Fb_{iy_j}b_{iz_j}$ for some $y_j$, $z_j$ $(1 \le y_j, z_j \le s_j)$.

In case (a), since $Fb_{iy_i}b_{iz_i} \sim Fb_{iw_i}b_{ix_i}$ (by lemma 4.1), we have, by (2), that $Fb_{ju_j}b_{iv_i} \sim Fb_{iu_i}b_{ju_j}$ and the cosubstitution property implies that either $a_{ju_j} \sim a_{iu_i}$ or $a_{iv_i} \sim a_{ju_j}$ both of which lead to $i = j$. Case (b) follows similarly to (a), by interchanging rows and columns and using (3) instead of (2).

Consequently we must have either

$$\{D_{ij}\} \sim \{D_{ii}\} \text{ or } \{D_{ij}\} \sim \{D_{jj}\}. \tag{4}$$

In particular either $\{D_{12}\} \sim \{D_{11}\}$ or $\{D_{12}\} \sim \{D_{22}\}$. We assume $\{D_{12}\} \sim \{D_{11}\}$ and deduce alternative I of the result. Alternative II follows similarly using the other assumption. The proof will be completed in three stages, namely by showing:

(a) $\{D_{1j}\} \sim \{D_{11}\}$ for each $j$, $3 \le j \le m$.
(b) $\{D_{i1}\} \sim \{D_{ii}\}$ for each $i$, $2 \le i \le m$.
(c) $\{D_{ij}\} \sim \{D_{ii}\}$ for $i$, $j$ such that $2 \le j \le m$; $2 \le i \le m$.

(a) Consider $\{D_{1j}\}$, $3 \le j \le m$. By (4) either $\{D_{1j}\} \sim \{D_{11}\}$ or $\{D_{1j}\} \sim \{D_{jj}\}$. If $\{D_{1j}\} \sim \{D_{jj}\}$ then either $\{D_{j1}\} \sim \{D_{jj}\} \sim \{D_{1j}\}$, i.e., $\{D_{j1}\} \sim \{D_{1j}\}$ which contradicts the cosubstitution property for $j \ne 1$; or $\{D_{j1}\} \sim \{D_{11}\} \sim \{D_{12}\}$, i.e., $\{D_{j1}\} \sim \{D_{12}\}$ which contradicts the cosubstitution property for $j \ne 1$. Hence $\{D_{1j}\} \sim \{D_{11}\}$.

(b) By (4) either $\{D_{i1}\} \sim \{D_{ii}\}$ or $\{D_{i1}\} \sim \{D_{11}\}$ $(2 \le i \le m)$. If $\{D_{i1}\} \sim \{D_{11}\}$

then, since $\{D_{11}\} \sim \{D_{12}\}$ we have $\{D_{i1}\} \sim \{D_{12}\}$ which contradicts the cosubstitution property for $i \neq 1$. Hence $\{D_{i1}\} \sim \{D_{ii}\}$ and consequently $\{D_{11}; D_{21}; \ldots ; D_{m1}\} \subseteq \{A_1; A_2; \ldots ; A_m\}$.

(c) Suppose there exist $i, j$ $(2 \leq i, j \leq m, i \neq j)$ such that $\{D_{ij}\} \sim \{D_{jj}\}$. By (b) $\{D_{j1}\} \sim \{D_{jj}\}$. Hence $\{D_{ij}\} \sim \{D_{j1}\}$ which contradicts the cosubstitution property if $j \neq 1, j \neq i$. This completes the proof of the theorem.

It is now necessary to distinguish between a decomposition of the $n$ truth values into less than $n$ classes and one into $n$ classes. The former case is easily disposed of and this is done in theorem 4.3. In the latter case, however, it will be seen that a considerably more detailed examination is required.

Theorem 4.3. *If $F$ possesses the cosubstitution property for any decomposition of the $n$ truth values into less than $n$ classes then $F$ also possesses the proper substitution property.*

The conditions which must be satisfied if $F$ is to possess the proper substitution property with respect to the same decomposition as that for which it possesses the cosubstitution property are that for each value of $i, j$ there exists some $k$ $(k = k(i, j), 1 \leq k \leq m)$ such that $\{D_{ij}\} \subseteq \{A_k\}$. However these conditions are immediately satisfied: following by the results proved in the previous theorem. Hence $F$ possesses the proper substitution property.

In the following work we shall be considering a decomposition of the $n$ truth values into $n$ classes. Consequently we can omit the second subscript when writing both truth values and logical constants, since it is always 1. This is done throughout the following theorem.

Initially from Theorem 4.2 the truth table of $Fpq$ will either contain $n$ identical entries in each row and $n$ different entries in each column or vice versa. We assume the former, i.e., for $1 \leq j \leq n$

$$Fb_j b_k =_T b_{j'} \text{ where } j' = j'(j),\ 1 \leq j' \leq n \text{ and } b_{j'} \neq_T b_{l'} \text{ unless } j = l. \tag{5}$$

The result will then be proved, making use of Theorem 3.1(A). It will follow for the other case by interchanging rows and columns and using Theorem 3.1(B).

We define the one-place functor $G$ by $Gb_j =_T b_{j'}$ for $j = 1, 2, \ldots, n$ and we note that, for all $k = 1, 2, \ldots, n$ and $1 \leq j \leq n$

$$Fb_j b_k =_T Gb_j. \tag{6}$$

Theorem 4.4. *If $F$ possesses the cosubstitution property for any decomposition of the $n$ truth values into $n$ classes then $F$ possesses at least one of the properties of proper closure, proper substitution or $t$-closure.*

The result will be established in three parts:

I. *If $Gb_j =_T b_j$ for some $j$, $1 \leq j \leq n$, then $F$ has the proper closure property.*

II. *If I is not satisfied and $G^p b_j =_T b_j$. for some $j$, $1 \leq j \leq n$, and some $p$, $1 \leq p \leq n$, then $F$ has the proper substitution property.*

III. *If* I *and* II *are not satisfied then we show that $F$ possesses the $t$-closure property.*

I. If $Gb_j =_T b_j$ for some $j, 1 \leq j \leq n$, then by (6) $Fb_jb_j =_T b_j$ and consequently possesses the proper closure property.

II. If $G^pb_j =_T b_j$ for some $j$, $1 \leq j \leq n$ and some $p, 1 < p < n$. Let $k$ be the least such $p$, $1 < k \leq p < n$ so that $G^kb_j =_T b_j$ $1 < k < n$, but $G^lb_j \neq_T b_j$ for any $l$, $1 \leq l < k$. Initially we have, for $1 \leq i \leq k - 1$,

$$G^kG^ib_j =_T G^ib_j, \text{ but } G^lG^ib_j \neq_T G^ib_j \text{ for } 1 \leq l < k. \tag{7}$$

This follows since $G^kG^ib_j =_T G^iG^kb_j =_T G^ib_j$, but if $G^lG^ib_j =_T G^ib_j$ for some $l$, $1 \leq l < k$, then $G^{k-i}G^lG^ib_j =_T G^{k-i}G^ib_j =_T G^kb_j =_T b_j$ and $G^{k-i}G^lG^ib_j =_T G^lG^kb_j =_T G^lb_j$ so that $G^lb_j =_T b_j$, $1 \leq l < k$, which contradicts our assumption of $k$ being the least such value.

We shall denote by $a_i^j$ the truth value corresponding to $G^jb_i$ $(1 \leq i \leq n$, $j = 0, 1, 2, \ldots)$. If

$$a_i^0 \notin \{a_j^0, a_j^1, \ldots, a_j^{k-1}\} \tag{8}$$

for some $i$, $1 \leq i \leq n$ (at least one such $a_i^0$ must exist since $k < n$) then we prove, by induction on $r$, that for any $q$, $0 \leq q \leq k - 1$

$$G^rb_i \neq_T G^qb_j. \tag{9}$$

For $r = 0$ the result follows immediately from (8). Assume the result for some nonnegative integral $r$.

Consider $G^{r+1}b_i =_T G^qb_j$ for some $q, 0 \leq q \leq k - 1$, then $GG^rb_i =_T GG^{q-1}b_j$ (if $q = 0$ then $GG^rb_i =_T GG^{k-1}b_j$). If $G^rb_i =_T b_x$ (say) and $G^{q-1}b_j =_T b_y$ (say) then by the induction hypothesis $b_x \neq_T b_y$. However $Gb_x =_T Gb_y$, which by (6) implies that $b_{x'} =_T b_{y'}$, which, by (5), is only true if $x = y$, which contradicts the induction hypothesis. Consequently $G^rb_i \neq_T G^qb_j$ $q = 0, 1, \ldots, k - 1$; $r = 0, 1, 2, \ldots$ We now show that, for any $i$, $1 \leq i \leq n$, $b_i =_T G^{k_i}b_i$ for some $k_i$, $1 \leq k_i \leq n$, and if $k_i$ is the least possible such value, then

$$a_i^0, a_i^1, \ldots, a_i^{k_i-1} \text{ are all distinct.} \tag{10}$$

Consider $\{a_i^0, a_i^1, \ldots, a_i^n\}$. Since there are only $n$ different truth values we must have, for some $p, q$, $0 \leq p < q \leq n$, $G^pb_i =_T G^qb_i$. Suppose $p$ is the least possible such value and assume $p \neq 0$. Then we have $G^{p-1}b_i \neq_T G^{q-1}b_i$. If $G^{p-1}b_i =_T b_x$ (say) and $G^{q-1}b_i =_T b_y$ (say) then $Gb_x =_T Gb_y$, but $b_x \neq_T b_y$ which contradicts (5). Hence the least possible value for $p$ is $p = 0$, i.e., $b_i =_T G^{k_i}b_i$ for some $k_i$, $1 \leq k_i \leq n$. Further if $k_i$ is the least possible such value then $a_i^0, a_i^1, \ldots, a_i^{k_i-1}$ are all distinct, since otherwise $G^pb_i =_T G^qb_i$ for some $p, q$, $0 \leq p < q < k_i$ and this implies either $b_i =_T G^{q-p}b_i$ with $0 < q - p < k_i$ which contradicts $k_i$ being the least such value, or $G^xb_i =_T G^yb_i$ for some $x, y$ such that $p - x = q - y$ and $0 < x < p$; $0 < y < q$, and $G^{x-1}b_i \neq_T G^{y-1}b_i$, which, as above, contradicts (5). We are now able to start defining a decomposition of the truth values with respect to which it will be shown that $F$ possesses the proper substitution property.

Define $E_1 = \{a_1^0, a_1^1, \ldots, a_1^{k_1-1}\}$, where $k_1$ is the least positive value such that $G^{k_1}b_1 =_T b_1$.

We have either $b_1 =_T G^q b_j$ for some $q$, $0 \leq q < k$, in which case $k_1 = k$ by (7), or $b_1 \neq_T G^q b_j$ for any $q$, $0 \leq q < k$, in which case since $a_1^0, a_1^1, \ldots, a_1^{k_1-1}$ are all distinct (by (10)) and there are a maximum of $n - k$ truth values which can be taken by $G^q b_1$ (for any $q$, $0 \leq q < k$) (by (9)) we have $k_1 \leq n - k$. Consequently in both cases $k_1 < n$.

Define $r_1 = 1$, $R_1 = 1$, and for each value of $j = 2, 3, \ldots, n$:

(I) if $a_j^0 \notin \bigcup_{s=1}^{R_{j-1}} E_s$ then $r_j = R_{j-1} + 1$ and $E_{r_j} = \{a_j^0, a_j^1, \ldots, a_j^{k_j-1}\}$ where $k_j$ is the least positive integer for which $G^{k_j} b_j =_T b_j$; otherwise, i.e., if $a_j^0 \in \bigcup_{s=1}^{R_{j-1}} E_s$, then there exists a unique integer $s$, $1 \leq s \leq R_{j-1}$ such that $a_j^0 \in E_s$ and we define $r_j = s$, ($s$ is unique by note (a) below).
(II) $R_j = max\,(r_i)$ $(1 \leq i \leq j)$.

We note that

(a) By (9) we cannot have $a_i^0 \in E_s$ and $a_i^0 \in E_u$, $s \neq u$, for any $i$, $1 \leq i \leq n$.
(b) For any $i$, $1 \leq i \leq n$, there exists an $s$ such that $a_i^0 \in E_s$, viz. $s = r_i$.

Consequently we have a decomposition of the $n$ truth values $a_1^0, a_2^0, \ldots, a_n^0$ into $m$ disjoint classes where $m = R_n$. It remains to prove that $F$ possesses the proper substitution property with respect to this decomposition.

Suppose $a_j \sim a_l$, $j \neq l$, $1 \leq j, l \leq n$. Then for some $r_s$, $1 \leq r_s \leq R_n$, $a_j$, $a_l \in E_{r_s}$. Consequently for some $p$, $1 \leq p \leq k_s$

$$b_j =_T G^p b_s \tag{11}$$

and for some $q$, $1 \leq q \leq k_s$

$$b_l =_T G^q b_s \tag{12}$$

and suppose $p < q$. Also, from the definition of $k_s$ ((i) above),

$$b_s =_T G^{k_s} b_s. \tag{13}$$

Hence, by (11), (12), $b_l =_T G^q b_s =_T G^{q-p} G^p b_s =_T G^{q-p} b_j$ and $1 \leq q - p \leq k_s$ and by (11), (12), (13) $b_j =_T G^p b_s =_T G^p G^{k_s} b_s =_T G^{k_s - q} G^p G^q b_s =_T G^{k_s - q + p} b_l$ and $1 \leq k_s - q + p \leq k_s$. Hence $b_l =_T G^v b_j$ and $b_j =_T G^u b_l$ for some $u, v$, $1 \leq u, v \leq k_s$.

We now show that if $a_j \sim a_l$ and $a_x \sim a_y$ then $Fb_j b_x \sim Fb_l b_y$. Suppose $a_j$, $a_l \in E_{r_s}$ and $a_x, a_y \in E_{r_z}$ $(1 \leq r_s, r_z \leq R_n)$.

We have that there exist $p, q$ $(1 \leq p \leq k_s;\ 1 \leq q \leq k_z)$ such that $Fb_l b_y =_T FG^p b_j G^q b_x =_T G^{p+1} b_j$ by (6) and $a_j^{p+1} \in E_{r_s}$.

But $Fb_j b_x =_T Gb_j$ by (6) and $a_j^1 \in E_{r_s}$. Hence $Fb_j b_x \sim Fb_l b_y$ and $F$ possesses the proper substitution property for this decomposition. This concludes II.

III. If

$$G^p b_j \neq_T b_j \tag{14}$$

for any $j$, $1 \leq j \leq n$, and any $p$, $1 \leq p \leq n - 1$, then we show initially that $G^n b_j =_T b_j$ for each $j$, $1 \leq j \leq n$. Consider $\{a_j^0, a_j^1, \ldots, a_j^n\}$. Since there are only $n$ different truth values we must have either $G^n b_j =_T b_j$ or $G^q b_j =_T G^p b_j$ for some $p, q$, $1 \leq p < q \leq n$, in which case $G^{q-p}(G^p b_j) =_T G^p b_j$, $0 < q - p < n$ which contradicts (14). Consequently $G^n b_j =_T b_j$, $1 \leq j \leq n$, and so $G$ is a $t$-function.

Similarly $a_j^0, a_j^1, \ldots, a_j^{n-1}$ must all be distinct to avoid contradicting (14). Hence $a_j^0, a_j^1, \ldots, a_j^{n-1}$ is just a rearrangement of $a_1^0, a_2^0, \ldots, a_n^0$. Now, by (6), $Fb_j G^p b_j =_T Gb_j$ for each $j = 1, 2, \ldots, n$ and all $p$, $0 \leq p \leq n - 1$. Hence, by Theorem 3.1(A), $F$ possesses $t$-closure. This concludes III and consequently completes the theorem.

Combining the last two theorems we have established the following result.

Theorem. *If $F$ possesses the cosubstitution property then it also possesses at least one of the properties of proper substitution, proper closure or $t$-closure.*

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