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ANALOGOUS CHARACTERIZATIONS OF FINITE AND ISOLATED SETS

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Introduction. Let $E = \{0, 1, 2, \ldots\}$. Members of E will be called *numbers*. A *set* shall mean a subset of E, and a *function* shall mean a function from a set into E. For a function f, then δf will denote its domain. Post [2] introduced simple sets; i.e., recursively enumberable (r.e.) sets with infinite isolated complements. Dekker [1] observed that if Dedekind's definition of finiteness (α is finite iff α is not equivalent to any proper subset of α) is made effective in a natural way, then exactly the class of isolated sets is obtained. The purpose of this note is to characterize when a set α is finite, by giving a condition $\mathrm{C}(\alpha)$ that involves partial orderings of α, and proving

Theorem A. *α is finite* $\Longleftrightarrow \mathrm{C}(\alpha)$.

In addition, we effectivize, in the spirit of Dekker, the condition $\mathrm{C}(\alpha)$ obtaining $\mathrm{C}^e(\alpha)$, and prove

Theorem B. *α is isolated* $\Longleftrightarrow \mathrm{C}^e(\alpha)$.

1. $\mathrm{C}(\alpha)$ *and* $\mathrm{C}^e(\alpha)$. We write P_α if α is a set and P is a binary relation that partially orders α. If P_α and Q_β, then we write $P_\alpha \leq Q_\beta$ if there is a function f such that

$$(1) \quad \begin{cases} \alpha \subseteq \delta f,\ f \text{ is one-to-one on } \alpha, f(\alpha) \subseteq \beta, \\ \text{and } (\forall x, y \in \alpha)\ [xPy \Rightarrow f(x)\, Qf(y)]. \end{cases}$$

The condition $\mathrm{C}(\alpha)$ is defined by,

$$(\forall P, Q)\ [P_\alpha \leq Q_\alpha \text{ and } Q_\alpha \leq P_\alpha \Rightarrow P_\alpha \sim Q_\alpha],$$

where $P_\alpha \sim Q_\alpha$ means that there is a function f such that,

$$(2) \quad \begin{cases} \alpha \subseteq \delta f, f \text{ is one-to-one on } \alpha, f(\alpha) = \alpha \\ \text{and } (\forall x, y \in \alpha)\ [xPy \Longleftrightarrow f(x)\, Qf(y)]. \end{cases}$$

If P_α and Q_β then we write $P_\alpha \leq^* Q_\beta$ if there is a partial recursive function f that satisfies (1). The condition $\mathrm{C}^e(\alpha)$ is defined by

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$$(\forall P, Q)\ [P_\alpha \leq^* Q_\alpha \text{ and } Q_\alpha \leq^* P_\alpha \Rightarrow P_\alpha \simeq Q_\alpha],$$

where $P_\alpha \simeq Q_\alpha$ means that there is a partial recursive function f that satisfies (2).

2. *Proof of Theorem* B. ($\Longrightarrow$). Assume that α is an isolated set and that P and Q are partial orders that satisfy $P_\alpha \leq^* Q_\alpha$ and $Q_\alpha \leq^* P_\alpha$. We wish to prove that $P_\alpha \simeq Q_\alpha$. Let f and g be partial recursive functions that effect the conditions $P_\alpha \leq^* Q_\alpha$ and $Q_\alpha \leq^* P_\alpha$ respectively. Since $P_\alpha \leq^* Q_\alpha$, we know that for any numbers $x, y \in \alpha$,

(3) $xPy \Longrightarrow f(x)\,Qf(y)$.

Now, suppose there are numbers $x, y \in \alpha$ such that $f(x)\,Qf(y)$. We want to show that in this case that xPy. Because $x \in \alpha$ and $gf(\alpha) \subseteq \alpha$, it follows that $\{x, gf(x), (gf)^2(x), \ldots\}$ will be an r.e. subset of α. Since α is an isolated set, this particular set must be finite. Hence there will exist numbers n and $k \geq 1$, such that $(gf)^n(x) = (gf)^{n+k}(x)$; and since the function gf will be one-to-one on α, it follows that also,

(4) $x = (gf)^k(x)$.

In a similar manner one can show that there will be a number $m \geq 1$, such that

(5) $y = (gf)^m(y)$.

Combining (4) and (5) we obtain,

(6) $x = (gf)^{mk}(x)$ and $y = (gf)^{mk}(y)$.

Because $f(x)\,Qf(y)$ and g effects $Q_\alpha \leq^* P_\alpha$, it follows that $gf(x)\,Pgf(y)$, and therefore also,

(7) $(gf)^{mk}(x)\,P(gf)^{mk}(y)$.

In view of (6) and (7), we obtain that xPy. We can conclude from these remarks that for all numbers $x, y \in \alpha$,

(8) $f(x)\,Qf(y) \Longrightarrow xPy$.

Combining (3) and (8) we obtain,

$$(\forall x, y \in \alpha)\ [xPy \Longleftrightarrow f(x)\,Qf(y)].$$

Let f^* be the restriction of the function f to the r.e. set

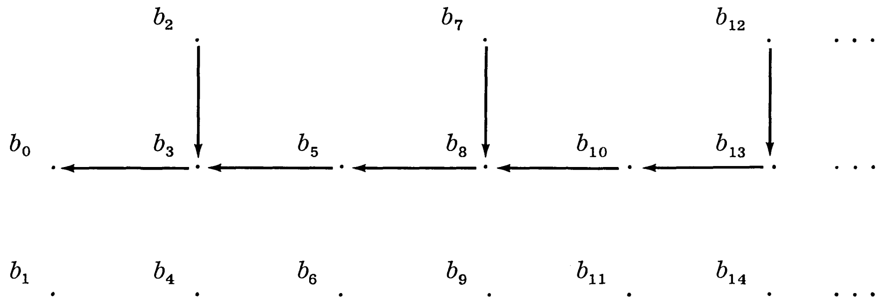
$$\eta = \{x \in \delta f \mid (\exists m)\ [m \geq 1 \text{ and } (gf)^m(x) = x]\}.$$

In view of our remarks above it is easy to verify that $\alpha \subseteq \eta$, and that f^* will be a partial recursive function which satisfies the conditions in (2), i.e., f^* will effect $P_\alpha \simeq Q_\alpha$.

($\Longleftarrow$). (Indirect) Assume that α is not an isolated set. Let β be an infinite recursive subset of α and let $\beta = \{b_0, b_1, b_2, \ldots\}$, with b_n denoting a one-to-one recursive function. We desire to construct two partial orders P_α and Q_α such that

(9) $P_\alpha \leq^* Q_\alpha$ and $Q_\alpha \leq^* P_\alpha$ and yet not $P_\alpha \simeq Q_\alpha$.

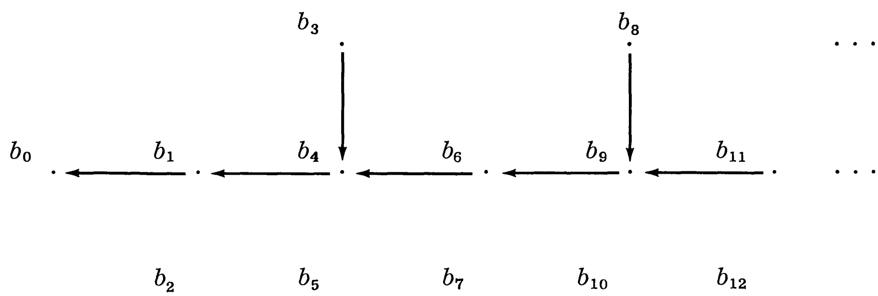
Consider the following array of all the elements of β, here identified by points.



We define on β a binary relation $\leqslant_1$ in the following way, $b_m \leqslant_1 b_n$ if either $b_m = b_n$ or else $m < n$ and there is a route via consecutive arrows in the preceding array that leads from the point b_n to the point b_m. It is clear that the relation $\leqslant_1$ partially orders the set β. Also we observe that different points appearing in the list $b_1, b_4, b_6, \ldots$, will not be $\leqslant_1$ related. We define a binary relation P on α in the following manner,

$$xPy \text{ iff } \begin{cases} x = y, \text{ or} \\ x,\ y \in \beta \text{ and } x \leqslant_1 y. \end{cases}$$

We now define by a similar approach a binary relation Q on α. Consider the following array of all the elements of β, here also identified by points.



We define on β the relation $\leqslant_2$ in the following way, $b_m \leqslant_2 b_n$ if either $b_m = b_n$ or else $m < n$ and there is a route via consecutive arrows in the preceding array that leads from the point b_n to the point b_m. It is easy to see that the relation $\leqslant_2$ partially orders the set β. Also we note that different points appearing in the list $b_2, b_5, b_7, \ldots$, will not be $\leqslant_2$ related. We define a binary relation Q on α in the following manner,

$$xQy \text{ iff } \begin{cases} x = y, \text{ or} \\ x,\ y \in \beta \text{ and } x \leqslant_2 y. \end{cases}$$

It is easy to see that each of the relations P and Q partially orders α, i.e., P_α and Q_α. Moreover, it is clear that the relation $P_\alpha \simeq Q_\alpha$ will not be true, for even the relation $P_\alpha \sim Q_\alpha$ is not true. On the other hand, we now verify that each of the relations $P_\alpha \leq^* Q_\alpha$ and $Q_\alpha \leq^* P_\alpha$ does hold.

For: $P_\alpha \leq^* Q_\alpha$. Consider the two given arrays of the elements of β; call the first $\mathcal{A}_1$ and the second $\mathcal{A}_2$. Alter the array $\mathcal{A}_2$ by removing the point b_0 together with the arrow that appears to its right, and call this new array $\mathcal{A}_2^!$. Then the first two rows of both $\mathcal{A}_1$ and $\mathcal{A}_2^!$ together with their respective arrows are the same except for the indexing of the points. Let the mapping $g: \beta \to \beta$ be characterized by the property that it maps the first two rows of $\mathcal{A}_1$ onto the first two rows of $\mathcal{A}_2^!$ in the natural way, i.e.,

$$g(b_0) = b_1, g(b_2) = b_3, g(b_3) = b_4, g(b_5) = b_6, \ldots,$$

and define

$$g(b_1) = b_0, g(b_4) = b_2, g(b_6) = b_5, g(b_9) = b_7, \ldots.$$

Then it is easy to verify that g maps β onto β in a one-to-one manner, and in addition,

(10) $x \leqslant_1 y \Longrightarrow g(x) \leqslant_2 g(y)$.

Let $g^*: E \to E$ denote the function that is equal to g on numbers in β, and is the identity map on the complement of β. In view of the recursive property of the function b_n, the definition of the mapping g, and the fact that β is a recursive set it follows that the function g^* is recursive. We also note that $g^*(\alpha) = \alpha$. It is an easy consequence of (10) and the definitions of the partial orderings P_α and Q_α and of the function g^*, that

$$(\forall x, y \in \alpha)\ [xPy \Longrightarrow g^*(x)\, Qg^*(y)].$$

We can conclude from this property that $P_\alpha \leq^* Q_\alpha$.

For: $Q_\alpha \leq^* P_\alpha$. Our proof here is similar to the one given above. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the first and second given arrays respectively. Alter the array $\mathcal{A}_1$ by removing the two points b_0 and b_2 together with their arrows adjacent to them and call this new array $\mathcal{A}_1^!$. Then the first two rows of both $\mathcal{A}_2$ and $\mathcal{A}_1^!$ together with their respective arrows are the same except for the indexing of the points. Let the mapping $h: \beta \to \beta$ be characterized by the property that it maps the first two rows of $\mathcal{A}_2$ onto the first two rows of $\mathcal{A}_1$ in the natural way, i.e.,

$$h(b_0) = b_3,\ h(b_1) = b_5,\ h(b_4) = b_8,\ h(b_3) = b_7,\ h(b_6) = b_{10}, \ldots,$$

and define

$$h(b_2) = b_0,\ h(b_5) = b_2,\ h(b_7) = b_1,\ h(b_{10}) = b_4, \ldots.$$

Then it is easy to verify that h maps β onto β in a one-to-one manner, and also, such that $x \leqslant_2 y \Longrightarrow h(x) \leqslant_1 h(y)$. Let $h^*: E \to E$ denote the function that is equal to h on numbers in β, and is the identity map on the complement of β. As in the previous case with the function g^*, one can readily verify that here also the function h^* is recursive and will satisfy

$$(\forall x, y \in \alpha)\ [xQy \Longrightarrow h^*(x)\, Ph^*(y)].$$

This will establish the property $Q_\alpha \leq^* P_\alpha$ and complete the proof.

3. *Proof of Theorem* A. Because the proof of Theorem A is essentially identical to the proof of Theorem B, we will confine ourselves to a few remarks.

($\Rightarrow$). The essential step in the proof of Theorem B ($\Rightarrow$) was based on the fact that any r.e. subset of an isolated set is finite. Here, the essential step is based on the fact that any subset of a finite set is finite.

($\Leftarrow$). In the proof of Theorem B ($\Leftarrow$), the main step was, if α is not an isolated set, then there will be a one-to-one recursive function b_n that ranges over an infinite recursive subset of α. Here, the main step is, if α is not a finite set, then there is a one-to-one function b_n that will range over α.

If the modifications indicated by these remarks are made to the proof of Theorem B, then a proof of Theorem A can be readily obtained.

4. *Concluding remarks.* We wish to mention that there are finiteness conditions whose effective versions (as in the spirit of the paper) have the property that the class of sets which satisfy them properly contains the finite sets and is properly contained in the collection of isolated sets. An example of a finiteness condition of this kind is given by

$$\mathrm{D}(\alpha) = (\forall f)\ [\alpha \subseteq \delta f \text{ and } f(\alpha) = \alpha \Rightarrow f \text{ is one-to-one on } \alpha],$$

whose effective version, $\mathrm{D}^{\mathrm{e}}(\alpha)$, we would define as,

$$(\forall f)\ [f \text{ partial recursive and } \alpha \subseteq \delta f \text{ and } f(\alpha) = \alpha \Rightarrow f \text{ is one-to-one on } \alpha].$$

In addition, it can be shown that there is a finiteness condition $\mathrm{H}(\alpha)$ whose effective version $\mathrm{H}^{\mathrm{e}}(\alpha)$ is satisfied by every set.

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