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Notre Dame Journal of Formal Logic
Volume XIII, Number 4, October 1972
NDJFAM

SEMI-INTUITIONISTIC SET THEORY

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The Zermelo-Fraenkel axioms represent set theory as a kind of "constructive" enterprise involving the production of new sets from old ones by means of carefully delineated processes. This could suggest that for the purposes of formalization the usual classical first-order logic may not be altogether appropriate (see [3]). In any case it seems worthwhile to investigate what happens when this quasi-constructive attitude is reflected in the very logic of the system, and what follows is an attempt to formulate a definite formal system along such lines and to compare it with the usual formulation of Zermelo-Fraenkel set theory as a classical first-order theory.*

Our system will be "constructive" in the sense that the law of the excluded middle will be postulated only for predicates with quantifiers restricted to certain terms representing sets. To get a sufficiency of such terms, we shall introduce various term-operators corresponding to the constructions postulated in the axioms. If "ZF_c" represents the usual classical formulation of Zermelo-Fraenkel set theory (including the axioms of regularity and choice), we may represent our system by "ZF_s*", where the "s" represents the "semi-intuitionistic" logic and the asterisk represents the addition of the term-operators. In section **1** we shall present the system ZF_s*, in **2** discuss the development of set theory within ZF_s*, and in **3** add some further remarks.

1. *The System* ZF_s*. The symbols are the constants $0, \omega$, the variables $x_0, x_1, x_2, \ldots$, the operators U, $\mathcal{P}$, $\mathcal{C}$, the predicates $=, \epsilon$, the connectives $\rightarrow, \&, \vee, -, \forall, \exists$, and the punctuational symbols (,), [,], {, }, :, and ,. In the metalanguage we shall represent variables by lower-case letters a, b, $c, \ldots$, terms by early capitals $A, B, C, \ldots$, and formulas by later

*Most of this work was done in connection with a National Science Foundation program of Research Participation for College Teachers conducted at the University of Oklahoma in the summer of 1968.

Received September 6, 1970

capitals P, Q, R, ... We define terms and formulas simultaneously as follows: The primitive terms are the constants and variables. If A, B are terms and P is a formula and x, y are distinct variables, then $\{A, B\}$, $\bigcup(A)$, $\mathcal{P}(A)$, $\mathcal{C}(A)$, and $\{y: (\exists x \in A)\ [P]\}$ are terms and $(A) = (B)$ and $(A) \in (B)$ are formulas. If P, Q are formulas and x is a variable, then $-[P]$, $[P]\ \&\ [Q]$, $[P] \vee [Q]$, $[P] \rightarrow [Q]$, $(\forall x)\ [P]$, $(\exists x)\ [P]$ are formulas.

In the metalanguage we shall freely depart from the formal symbolism for greater clarity in representing formal expressions. Some particular terminology and notation follows:

(a) The notions of "free" and "bound" variables are the same as usual, but with the added stipulation that the term-operator $\{y: (\exists x \in A)\ [P]\}$ should act as a kind of double quantifier on the formula P, with each of the dummy variables x and y having P, but *not* A, within its scope.

(b) If P is a formula and x is a variable not occurring free in the term A, we shall write "$(\forall x \in A)\ [P]$" for $(\forall x)\ [x \in A \rightarrow P]$ and "$(\exists x \in A)\ [P]$" for $(\exists x)\ [x \in A\ \&\ P]$. In each case we shall say that the quantifier in question is "restricted to the term A." If all the quantifiers of a formula Q other than those occurring *within* terms are so restricted to terms, Q will be called a "restricted-quantifier formula (rqf)." Thus the formulas $x \in 0$, $(\exists x)\ [x \in \omega\ \&\ x = y]$ and $0 \in \{y: (\exists z \in \mathcal{P}(x))\ [(\exists w)\ [z \in w\ \&\ y = z]]\}$ are all rqf's.

(c) The notation "$P(A, B, C, \ldots)$" will be used in accordance with [2, pp. 78-79], although the stipulation that A, B, C, . . . be "free for" x, y, z, . . . respectively in $P(x, y, z, \ldots)$ is of course subject to the modification induced by (a).

(d) Write "$(E!x)\ [P(x)]$" for $(\exists x)\ [P(x)\ \&\ (\forall y)\ [P(y) \rightarrow y = x]]$, "$(E!x = A)\ [P(x)]$" for $[P(A)\ \&\ (\forall y)\ [P(y) \rightarrow y = A]]$, "$[P \leftrightarrow Q]$" for $[P \rightarrow Q]\ \&\ [Q \rightarrow P]$, "$A \subseteq B$" for $(\forall x \in A)\ [x \in B]$ (where x is any variable not occurring free in A or B), "$\{A\}$" for $\{A, A\}$, "$A \cup B$" for $\bigcup(\{A, B\})$, and "A'" for $A \cup \{A\}$.

(e) Unless the context indicates otherwise, distinct lower-case letters occurring in an expression for a formula or term should be taken to represent *definite* and distinct variables. Thus for example axiom [*E1*] cited below is to be taken as a single formula, not a schema.

In addition to the axioms and rules of the *intuitionistic* predicate calculus [2, pp. 82, 101], we postulate the following logical axioms for ZF_s*:

[*EM*] $P \vee -P$ *for every rqf* P.

[*E1*] $x = y \rightarrow [x \in z \leftrightarrow y \in z]$.

[*E2*] $x = y \rightarrow \mathcal{C}(x) = \mathcal{C}(y)$.

[*E3*] $[a = b\ \&\ (\forall x)\ (\forall y)\ [P \leftrightarrow Q]] \rightarrow \{y: (\exists x \in a) P\} = \{y: (\exists x \in b) Q\}$ *for all formulas* P, Q, *all distinct variables* a, b, x, y.

[*E4*] $\{y: (\exists x \in a)\ P(x, y)\} = \{z: (\exists w \in a)\ P(w, z)\}$ *for all formulas* P, *all variables* a, x, y, z, w *such that:* (*1*) x *is distinct from* y; z *from* w; *and* a *from* x, y, z, w; (*2*) w, z *are free for* x, y *respectively in* $P(x, y)$; *and* (*3*) w, z *do not occur free in* $P(x, y)$ *unless* w *is* x *or* z *is* y *or both.*

The following are the set-theoretical axioms of ZF_s*:

[*Ex*] $x = y \leftrightarrow [x \subseteq y\ \&\ y \subseteq x]$.

[*Em*] $x \notin 0$.
[*Pr*] $z \in \{x, y\} \leftrightarrow [z = x \vee z = y]$.
[*Sm*] $y \in \mathbf{U}(x) \leftrightarrow (\exists z \in x)[y \in z]$.
[*In*] $(E! x = \omega)[x \subseteq \omega \ \& \ 0 \in x \ \& \ (\forall y \in x)[y' \in x]]$.
[*Pw*] $y \in \mathcal{P}(x) \leftrightarrow y \subseteq x$.
[*Ch*] $x \neq 0 \rightarrow \mathcal{C}(x) \in x$.
[*Rg*] $x \neq 0 \rightarrow (\exists y \in x)(\forall z \in x)[z \notin y]$.
[*Rp*] $(\forall x \in A)(E! y) P \rightarrow [y \in \{y\colon (\exists x \in A) P\} \leftrightarrow (\exists x \in A) P]$ *for all formulas P, all distinct variables x, y and all terms A not containing x free.*

Note that apart from the addition of the term-operators and the consequent strengthening of the axiom of choice, these nonlogical axioms are essentially the same as the usual ones (as formulated, for example, in [1, pp. 51-53]). In [3] we speculated that the quasi-constructive attitude which we wish to formalize might warrant a change in the formulation of the replacement axiom, but we have since realized that this is unnecessary; the change in the logic is sufficient. A weakened form of the axiom of separation does result, however, as can be seen at the end of the following list of basic metatheorems.

Theorem (Substitutivity of Equality [SE]). *If y is a variable free for x in A(x) and P(x), we have:*

$$\vdash x = y \rightarrow [A(x) = A(y) \ \& \ [P(x) \leftrightarrow P(y)]].$$

Theorem (Equivalence [EQ]). *Let R_Q and A_Q be the result of replacing certain specified occurrences of P in R and A by Q. If P and Q do not contain free variables belonging to quantifiers (in the extended sense of (a) above) of R or A having the specified occurrences of P within their scope, then* $\vdash [P \leftrightarrow Q] \rightarrow [A = A_Q \ \& \ [R \leftrightarrow R_Q]]$.

Theorem (Change of Bound Variables [BV]). *Let P and Q be congruent with respect to their bound variables in the sense of [2, p. 153] and likewise for A and B. Then* $\vdash [A = B \ \& \ [P \leftrightarrow Q]]$.

Theorem (Separation). *If A is any term not containing x free and P is any formula, write "$\{x \in A\colon [P]\}$" for the term* $\mathbf{U}\{y\colon (\exists x \in A)[(P \ \& \ y = \{x\}) \vee (-P \ \& \ y = 0)]\}$. *Then:* $[Sp] \vdash (\forall x \in A)[P \vee -P] \rightarrow [x \in \{x \in A\colon [P]\} \leftrightarrow [x \in A \ \& \ P]]$.

2. *The Development of Set Theory within* ZF_s*. Rather than list a multitude of formal theorems, we shall compare the results derivable in ZF_s* with those listed for ZF_c in [4]. We begin with a theorem of a general nature which makes use of Gödel's method [2, pp. 492-7] for interpreting classical number theory within the intuitionistic version. We cannot quite get an interpretation of classical ZF within intuitionistic ZF, but the method does provide a handy means of comparing the relative strength of ZF_s* to that of its classical counterpart. We begin with some definitions and lemmas.

Let "ZF_c*" be obtained from ZF_s* by extending [*EM*] to include [$P \vee -P$] for *all* formulas *P*. For any term *A* and any formula *P* we define corresponding expressions A° and P° as follows. If *A* is a constant or variable, let A° be *A*. If *A* is $\mathbf{U}(B)$, $\{B, C\}$, $\mathcal{P}(B)$, $\mathcal{C}(B)$, or $\{y\colon (\exists x \in B)[P]\}$, let A° be $\mathbf{U}(B^\circ)$, $\{B^\circ, C^\circ\}$, $\mathcal{P}(B^\circ)$, $\mathcal{C}(B^\circ)$, and $\{y: (\exists x \epsilon B^\circ)[P^\circ]\}$ respectively. If P is $(A) = (B)$ or $(A) \epsilon (B)$, let P° be $(A^\circ) = (B^\circ)$ and $(A^\circ) \epsilon (B^\circ)$ respectively. If P is $Q \rightarrow R$, $Q \,\&\, R$, $Q \vee R$, $-Q$, $\forall x Q$, or $\exists x Q$, define P° as indicated in [2, p. 494]. Note that in the following results the symbol $\vdash$ continues to refer to provability in ZF_s*.

Lemma 1. *For any formula* P, $\vdash -(-P^\circ) \rightarrow P^\circ$.

Proof. By induction on the complexity of P° using the same argument as given for Lemma 43a in [2, p. 495] except that for the basis we note that every formula of the form $(A) = (B)$ or $(A) \epsilon (B)$ is an rqf, so that [*EM*] (and *49c in [2, p. 119]) applies.

Lemma 2. *If P is an rqf and A is a term containing only rqf's as subformulas, then* $\vdash [A = A^\circ \,\&\, [P \leftrightarrow P^\circ]]$.

Proof. By induction on the conplexity of terms and formulas simultaously, using [*SE*], [*E3*], [*EQ*], [*EM*] and basic facts about intuitionistic logic like *86 in [2, p. 162].

Theorem (Gödel's Method applied to ZF_s* [GM]). *Let Q be a formula provable in* ZF_c* *by a proof which uses* [*Rp*] *only with respect to rqf's P and terms A containing only rqf's such that the hypothesis* $(\forall x \epsilon A)(E!\, y)P$ *is provable in* ZF_s* *and actually occurs as a separate formula in the proof of Q. Then* $\vdash Q^\circ$ *by a proof which uses the same nonlogical axioms as the proof of Q.*

Corollary. (By Lemma 2): *If Q is as described in the theorem and is an rqf, then* $\vdash Q$ *by a proof which uses the same nonlogical axioms as its proof in* ZF_c* *(and possibly also* [*Ex*], [*Pr*], [*Sm*], *and* [*Pw*] *by way of an application of* [*SE*]).

Proof of the Theorem. By induction on the length of proofs in ZF_c* which satisfy the stated conditions, using basically the same arguments as given for Theorem 60(c) of [2, pp. 495-6], but using Lemma 2 in the consideration of [*Pr*], [*Sm*], [*Rg*] and [*Rp*], using our Lemma 1 instead of Kleene's Lemma 43a, and noting that his Theorem 59(b) also holds for ZF_s*.

What [*GM*] provides is a means of transforming proofs in ZF_c* into proofs in ZF_s*. As a result, by rephrasing definitions so that they involve only rqf's and adjusting proofs here and there so that they conform to the conditions described in [*GM*] one can verify that *all the main set-theoretical theorems listed in* [4] *as provable in* ZF_c* (using the axiom of choice) *are actually provable in* ZF_s* *as well.* Here of course we are neglecting Suppes' distinction between individuals and sets and are overlooking the fact that most of his terms are, strictly speaking, not a part of his initial symbolism. But with this reinterpretation all of his main formal theorems become theorems of ZF_s* also, with terms in his theorems representing specific terms of ZF_s*. If we take "M.N" to mean "Theorem N of Chapter M in [4]," the only exceptions are the following:

(1) We have no counterpart of Suppes' notation $\{x: \varphi(x)\}$ and so cannot obtain 2.47-54.

(2) In 4.32, 4.86, 5.22, 7.1, 7.4-5 and 7.61-62 we need to stipulate that the formula $\varphi(x)$ be an rqf.

The details of this development, though complicated, are relatively straightforward and shall not be presented here.[1]

To round off this section we note that the following axiom schema, which seems intuitively acceptable from the point of view which we are trying to formalize, would enable us to remove the restriction on $\varphi(x)$ mentioned in (2) as regards the theorems 4.32, 4.86, 5.22, 7.1 and 7.4 (but not 7.5 or 7.61-2):

Axiom of set induction: $(\forall x)\ [(\forall y \in x)\ [P(y)] \to P(x)] \to (\forall x)\ [P(x)]$, *for all formulas P.*

This schema, which is obtainable in ZF_c using the notion of rank, would also make $[Rg]$ superfluous.

3. *Further Remarks.* One may think of ZF_c* as an extension of ZF_c obtained by adding the term-operators and axioms of ZF_s*. The question arises whether it is a "conservative" extension in the sense that every formula of ZF_c which is provable in ZF_c* is also provable in ZF_c. It turns out that this is actually the case, although we shall not present a proof here. If we omit the choice operator, standard methods such as those discussed in [4, pp. 14-19] give the desired result. And we have been informed that certain unpublished results involving more sophisticated methods provide ways of handling even the choice operator.

It follows from this in particular that the consistency of ZF_c is equivalent to that of ZF_c*, which in turn implies that of ZF_s*. An interesting question is whether in some sense the latter implication goes the other way. We have hopes of formulating some sort of constructive or quasi-constructive proof in that direction but so far have not been successful.

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1. A more ample treatment in mimeographed form may be obtained from the author.