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SQUARES OF OPPOSITION: COMPARISONS BETWEEN SYLLOGISTIC AND PROPOSITIONAL LOGIC

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It has been pointed out, for example by Bocheński,[1] that the principles of propositional logic now known as DeMorgan's Laws bear a certain resemblance to the laws depicted in the traditional Square of Opposition. The analogy, however, is not as perfect as it could be. The aim of this paper is to explore some of the consequences of seeking a more exact comparison between syllogistic and propositional logic.

The propositional operator K (conjunction) may be defined as follows: $K11 = 1$, $K10 = 0$, $K01 = 0$, $K00 = 0$. We may therefore regard the resulting values, 1000, as a satisfactory definition of K. Eight further operators will be defined in the same way:

$$\begin{array}{rcl} B = 1101 & \quad & L = 0100 \\ C = 1011 & & M = 0010 \\ D = 0111 & & V = 1110 \\ J = 0110 & & X = 0001 \end{array}$$

With the exception of V (non-exclusive disjunction), then, these signs are used with the sense assigned to them by Łukasiewicz. Using this notation, the square that Bocheński and others refer to has the following form:

$$\begin{tikzcd} Kpq & D \arrow[l] \arrow[r] & Xpq \\ C \arrow[u, dash] \arrow[d] & J \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & C \arrow[u, dash] \arrow[d] \\ Vpq & V \arrow[l] \arrow[r] & Dpq \end{tikzcd}$$

The traditional square of opposition concerns the relations between four forms of "categorical" proposition: Aab ("All a's are b's"), Eab ("No a's are b's"), Iab ("Some a's are b's") and Oab ("Some a's are not b's"). These four forms are arranged in a square like the one given above:

1. J. M. Bocheński, *A Prēcis of Mathematical Logic*, Holland (1959), p. 14.

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$$\begin{tikzcd}
Aab \arrow[rr, leftrightarrow, "D"] \arrow[dd, dash, "C"'] \arrow[ddrr, leftrightarrow] & & Eab \arrow[dd, dash, "C"] \arrow[ddll, leftrightarrow] \\
& J & \\
Iab \arrow[rr, leftrightarrow, "V"'] & & Oab
\end{tikzcd}$$

How exact is the comparison between these two squares? In the case of the traditional square there is a set procedure for defining any three corners of the square in terms of the fourth. If N signifies propositional-negation and n signifies term-negation, then, according to the medieval Rules of Equipollence:

$$\begin{aligned}
Aab &\equiv Eanb \equiv NOab \equiv NIanb \\
Eab &\equiv Aanb \equiv NIab \equiv NOanb \\
Iab &\equiv Oanb \equiv NEab \equiv NAanb \\
Oab &\equiv Ianb \equiv NAab \equiv NEanb
\end{aligned}$$

Now suppose that we regard q as the 'predicate' of Kpq, Xpq, Vpq and Dqp. The rules of equipollence specify that when the predicate of Aab is negated, the resulting proposition is equivalent to Eab. We might be tempted to think, by analogy, that when the 'predicate' of Kpq is negated, the resulting proposition is equivalent to Xpq. Of course this is not the case: Xpq is equivalent to $KNpNq$, not $KpNq$. Similarly, Iab is equivalent to $NAanb$; but Vpq is equivalent to $NKNpNq$, not $NKpNq$.

These discrepancies are inevitably reflected elsewhere. For example, only two of the four traditional propositions can be converted: Eba can be inferred from Eab and Iba can be inferred from Iab, but Aba cannot be inferred from Aab and Oba cannot be inferred from Oab. In the modern parallel, however, all of the constituent propositions can be 'converted': $Kpq \equiv Kqp$, $Xpq \equiv Xqp$, $Vpq \equiv Vqp$ and $Dpq \equiv Dqp$.[2]

Granted these disparities, it is natural to ask whether a more precise analogy between syllogistic and propositional logic can be formulated. As it happens, there are *two* squares of opposition in the propositional calculus which correspond in a very exact sense to the traditional square. Let us first consider one of these squares:

$$\begin{tikzcd}
Lpq \arrow[rr, leftrightarrow, "D"] \arrow[dd, "C"'] \arrow[ddrr, leftrightarrow] & & Kpq \arrow[dd, "C"] \arrow[ddll, leftrightarrow] \\
& J & \\
Dpq \arrow[rr, leftrightarrow, "V"'] & & Cpq
\end{tikzcd}$$

It may first be remarked that the constituents of this square are interdefinable in a way that provides a perfect analogue for the traditional rules of equipollence:

2. In order to make plain the similarities involved, I use "≡" as the sign of equivalence, even though equivalence is differently represented in the Łukasiewicz notation used elsewhere. For the same reason "→" is sometimes used as an implication sign, even though it is indistinguishable in meaning from "C".

$$\begin{aligned} Lpq &\equiv KpNq \equiv NCpq \equiv NDpNq \\ Kpq &\equiv LpNq \equiv NDpq \equiv NCpNq \\ Dpq &\equiv CpNq \equiv NKpq \equiv NLpNq \\ Cpq &\equiv DpNq \equiv NLpq \equiv NKpNq \end{aligned}$$

Furthermore, if we take Lpq as the analogue of Aab, Kpq as the analogue of Eab, Dpq as the analogue of Iab and Cpq as the analogue of Oab, all of the traditional laws of immediate inference are preserved:

$$\left.\begin{aligned} Kpq &\to Kqp \\ Dpq &\to Dqp \end{aligned}\right\} \text{ `simple conversion'}$$

$$\left.\begin{aligned} Lpq &\to Dqp \\ Kpq &\to Cqp \end{aligned}\right\} \text{ `conversion } \textit{per accidens}\text{'}$$

$$\left.\begin{aligned} Lpq &\to KpNq \\ Kpq &\to LpNq \\ Dpq &\to CpNq \\ Cpq &\to DpNq \end{aligned}\right\} \text{ `obversion'}$$

$$\left.\begin{aligned} Lpq &\to LNqNp \\ Cpq &\to CNqNp \end{aligned}\right\} \text{ `contraposition'}$$

$$\left.\begin{aligned} Lpq &\to CNpq \\ Kpq &\to DNpq \end{aligned}\right\} \text{ `inversion'}$$

There is in the propositional calculus a second square with exactly the same properties, namely:

$$\begin{tikzcd} Mpq \arrow[r, leftrightarrow, "D"] \arrow[d, "C"'] \arrow[dr, leftrightarrow, "J" description] & Xpq \arrow[d, "C"] \arrow[dl, leftrightarrow] \\ Vpq \arrow[r, leftrightarrow, "V"'] & Bpq \end{tikzcd}$$

For the constituents of this square too, the laws of equipollence and immediate inference hold.

It we return to the original definitions, it will be noticed that the operators capable of forming an exact analogue for the traditional square are the ones in which three and only three of the defining values are the same: 1000, 0100, 0010, 0001, 0111, 1011, 1101 and 1110. Moreover, the operators corresponding to the "universals" of syllogistic are those in which false values predominate, while the operators corresponding to the "particulars" of syllogistic are those in which true values predominate. There are at this point certain significant similarities with the traditional doctrine of distribution, but these will not be discussed here.

It may seem to show a regrettable lack of symmetry that the propositional calculus has two squares of opposition, whereas traditional logic has only one. The fact is, however, that there are—or ought to be—two such squares in traditional logic also. Syllogistic, as DeMorgan was perhaps the first clearly to realize, deals with *eight* varieties of logically independent propositions. Suppose that, in addition to *Aab*, *Eab*, *Iab* and *Oab*, we introduce four new expressions, *Rab*, *Sab*, *Tab* and *Uab*. These four new forms may be defined as follows:

$$
\begin{aligned}
Rab &\equiv Ananb \equiv Aba \\
Sab &\equiv Enanb \\
Tab &\equiv Inanb \\
Uab &\equiv Onanb \equiv Oba
\end{aligned}
$$

These four propositions form a square exactly like the old one:

$$
\begin{tikzcd}
Rab & D \arrow[l] \arrow[r] & Sab \\
C \arrow[u, dash] \arrow[d] & J \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & C \arrow[u, dash] \arrow[d] \\
Tab & V \arrow[l] \arrow[r] & Uab
\end{tikzcd}
$$

Now, the connection between these two squares of syllogistic is exactly like that between the two squares of propositional logic: *Rab* is the analogue of *Mpq*, *Sab* is the analogue of *Xpq*, *Tab* is the analogue of *Vpq* and *Uab* is the analogue of *Bpq*. The comparison is most clearly brought out in, for example, the following equivalences:

$$
\begin{aligned}
Mpq &\equiv LNpNq \equiv Lqp \\
Xpq &\equiv KNpNq \\
Vpq &\equiv DNpNq \\
Bpq &\equiv CNpNq \equiv Cqp
\end{aligned}
$$

There is, then, a most precise analogy between the two squares of traditional logic and the two squares of propositional logic.

These results cast some light on a certain kind of connection between syllogistic and propositional logic. It has been stressed, especially by Łukasiewicz, that the procedures of traditional logic presuppose laws of propositional calculus. The analogies described above, however, rest on a *direct* comparison of the logic of terms and the logic of propositions; and they appear to suggest that syllogistic and propositional logic express, at some level, a common structure of reasoning.

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